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EXISTENCE RESULTS FOR A CLASS OF FRACTIONAL q-DIFFERENCE EQUATIONS WITH MIXED CONDITIONS

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Abstract. In this paper, we investigate a new class of boundary value problems for fractional q-difference equations involving q-Caputo and q-Hilfer derivatives. The existence results are based on the Schauder fixed point theorem and Banach contraction mapping principle. An illustrative example is presented to illustrate the effectiveness of the theoretical results.

1 Introduction

Fractional calculus, which deals with differentiation and integration of real or complex orders, has seen a surge of interest in recent years. see [6, 10, 14, 18, 19]. This mathematical framework can better capture the memory and hereditary properties of various physical, chemical, and biological processes. In literature, the origins of fractional calculus can be traced back to the 17th century, with early contributions from renowned mathematicians such as Leibniz, l'Hôpital, Euler, Bernoulli, and Riemann. Over time, fractional calculus has found applications in diverse fields, including medicine, bioengineering, viscoelasticity, and dynamical systems, we refer to [3, 5, 8, 17, 26]. and the references therein.

q-calculus is an extension of classical calculus that introduces a parameter q, which acts as a deformation of standard operations such as differentiation, integration, and summation. It plays an important role in fields like combinatorics, special functions, and quantum mechanics. Central to q-calculus is the q-difference operator, which generalizes the usual difference and derivative operators depending on the value of q, for more information, the reader is invited to consult [9, 13, 11, 15, 24]. For it's various and interesting applications, the study of q-calculs became more and more a very attractive area of research (see [4, 7] and so on). Many works have recently been published in this direction see [21, 22, 23]. The q-Hilfer fractional derivative extends the classical Hilfer derivative to the quantum calculus setting, while the q-Caputo fractional derivative is a quantum analog of the well-known Caputo derivative, see [20].

In [1], the authors studied the existence, uniqueness, and Ulam stability of solutions for the following implicit fractional q-difference equation

$$\begin{aligned} \mathcal{D}_q^{\xi} u(s) &= \mathfrak{f}(t, u(s), \mathcal{D}_q^{\xi} u(s)),\ 0 \leq s \leq b, \\ u(0) &= u_0 \in \mathbb{R}, \end{aligned}$$

where $0 < q < 1$, $0 < \xi \leq 1$, $b > 0$, and $\mathfrak{f} : [0, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Lachouri et al [12] investigated the following problem

$$\begin{cases} {}^{C}\mathcal{D}^{\delta}\left[{}_{H}^{C}\mathfrak{D}^{\xi}\varkappa(s)\right] = \mathfrak{f}(s, \varkappa(s)),\ s \in (a, b)\,,\ a > 0, \\ \varkappa(a) = \delta,\ \varkappa(b) = \lambda\varkappa(\rho)\,,\ a < \rho < b, \end{cases}$$

where ${}^{C}\mathcal{D}^{\delta}$ and ${}_{H}^{C}\mathfrak{D}^{\xi}$ are the Caputo and Hadamard fractional derivatives of orders δ and ξ, respectively, $0 < \delta, \xi \leq 1$. The function $\mathfrak{f}$ is continuous and satisfies certain specified assumptions.

Recently, in [2], Allouch et al have arrived at proving the existence of solutions for the given problem

$$
\begin{gathered}
{}^{C}\mathcal{D}_{q}^{\delta}\,\varkappa(s)=\mathfrak{f}(s,\varkappa(s)),\ s\in[0,b],\ \delta\in(1,2]\,,\\
\varkappa(0)+\varkappa'(0)=\int_{0}^{b}g_{1}\left(t,\varkappa\left(t\right)\right),\ \varkappa(b)+\varkappa'(b)=\int_{0}^{b}g_{2}\left(t,\varkappa\left(t\right)\right),
\end{gathered}
$$

where ${}^{C}\mathcal{D}_{q}^{\delta}$ is the Caputo fractional q-derivative and $\mathfrak{f}, g_1, g_2$, are given functions defined from $[0,b]\times\mathbb{E}$ to $\mathbb{E}$, where E is a Banach space.

Building upon the works mentioned previously, we investigate a novel class of boundary value problems for fractional q-difference equations of the form

$$
\begin{gathered}
{}^{C}\mathcal{D}_{q}^{\delta}({}^{H}\mathcal{D}_{q}^{\xi,\eta}\varkappa(s))=\mathfrak{f}(s,\varkappa(s)),\ 0\leq s\leq b,\\
\varkappa\left(b\right)=0,\ {}^{C}\mathcal{D}_{q}^{\xi}\varkappa\left(b\right)=0,
\end{gathered}
\tag{1.1}
$$

where ${}^{H}\mathcal{D}_{q}^{\xi,\eta}$ is the Hilfer fractional q-derivative of order ξ and type η, $0<\xi,\delta,\eta,q<1$, and $\mathfrak{f}$ is continuous function defined from $[0,b]\times\mathbb{R}$ to $\mathbb{R}$.

The novelty of our research is the introduction of a general operator, formed by the composition of the q-Caputo and q-Hilfer operators, which encompasses several notable cases, including the following: if $\eta=0$, our problem reduces to q-sequential Riemann-Liouville and Caputo derivatives, and if $\eta=1$, (1.1) becomes the composition of the q-Caputo derivative of order δ and the q-Caputo derivative of order ξ.

The rest of the paper is structured as follows. Some fundamental ideas of fractional calculus and fixed point theorem are presented in Section 2. The main result about the existence of solutions for the (1.1) is obtained in Section 3. In the final section, an illustrative example is provided.

2 Preliminary concepts

In this part, we give some essential ideas of fractional calculus and fixed point theorems, which are used throughout this paper.

Let $\mathbb{J}=[0,b]$. We denote by $C(\mathbb{J},\mathbb{R})$ the Banach space of all continuous functions from $\mathbb{J}$ to $\mathbb{R}$, equipped with the norm $\|\varkappa\|_{\infty}=\sup\{|\varkappa(s)|:s\in\mathbb{J}\}$. Let $\gamma>0$. We consider the weighted space of continuous functions

$$
C_{\gamma}(\mathbb{J};\mathbb{R})=\{x:(0,T]\to\mathbb{R}\mid s^{\gamma}x\in C(\mathbb{J};\mathbb{R})\}\,,
$$

equipped with the norm

$$
\|x\|_{C_{\gamma}}=\sup_{s\in J}|s^{\gamma}x(s)|\,.
$$

It is clear that $C_{\gamma}(J;\mathbb{R})$ is a Banach space.

In the theory of q-calculus (see [9]), for a real parameter $q\in\mathbb{R}_{+}\setminus\{1\}$, we define the q-real number $[a]_q$ as

$$
[a]_{q}:=\frac{1-q^{a}}{1-q}\ \text{for}\,a\in\mathbb{R}.
$$

The q-analog of the Pochhammer symbol (also known as the q-shifted factorial) is defined as

$$
(a;q)_{0}=1,\ (a;q)_{k}=\prod_{i=0}^{k-1}(1-aq^{i})\ \text{for}\,k\in\mathbb{N}\cup\{\infty\}.
$$

Its natural extension to the real numbers is

$$
(a;q)_{k}=\frac{(a;q)_{\infty}}{(aq^{\xi};q)_{\infty}}\ \text{for}\,k\in\mathbb{R}.
$$

The q-gamma function is defined as

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty} \cdot \frac{(1-q)}{1-a}, \quad a \in \mathbb{R} \setminus \{0, -1, -2, \dots\}.$$

It follows that

$$\Gamma_q(a+1) = [a]_q \Gamma_q(a), \quad \Gamma_q(a) = (q;q)_{a-1} \cdot \frac{(1-q)}{1-a}.$$

The 1st-q-derivative of a function $\varkappa : \mathbb{J} \to \mathbb{R}$ is given by

$$\mathcal{D}_q \varkappa(s) = \frac{\varkappa(s) - \varkappa(qs)}{(1-q)s}, \ s \neq 0, \ \ \mathcal{D}_q \varkappa(0) = \lim_{s \to 0} \mathcal{D}_q \varkappa(s).$$

For $\mathbb{J}_s = \{sq^n : n \in \mathbb{N}, s \geq 0\} \cup \{0\}$, the q-integral of a function $\varkappa : \mathbb{J}_s \to \mathbb{R}$ is given by

$$I_q \varkappa(s) = \int_0^s \varkappa(\rho)\, d_q\rho = \sum_{n=0}^{\infty} s(1-q)q^n \varkappa(sq^n),$$

provided that the series on the right-hand side converges.

Notably, the q-integral and q-derivative satisfy the following properties:

$$\mathcal{D}_q\big(I_q \varkappa\big)(s) = \varkappa(s),$$

and if $\varkappa$ is continuous at $s = 0$, then

$$I_q\big(\mathcal{D}_q \varkappa\big)(s) = \varkappa(s) - \varkappa(0).$$

Definition 2.1 ([20]). Let $\varkappa$ be a function defined on $\mathbb{J}$ with values in $\mathbb{R}$ and $s \in [0, b]$. The q-fractional integral and derivative operator in Riemann-Liouville sense for the function $\varkappa$ of order ξ are defined respectively as

$$I_q^\xi \varkappa(s) = \frac{1}{\Gamma_q(\xi)} \int_0^s (s - q\rho)_q^{\xi-1} \varkappa(\rho)\, d_q\rho, \ \xi > 0,$$

and

$${}^{RL}\mathcal{D}_q^\xi \varkappa(s) = \frac{1}{\Gamma_q(1-\xi)} \mathcal{D}_q \int_0^s (s - q\rho)_q^{-\xi} \varkappa(\rho)\, d_q\rho, \ \ 0 < \xi < 1.$$

Additionally, for $\xi = 0$, it holds that $I_q^0 \varkappa(s) = \varkappa(\vartheta)$.

Lemma 2.2 ([20]). *For $0 \leq \xi < \infty$ and $t \in (-1, +\infty)$, let $\varkappa(s) = (s-a)_q^{(t)}$. Then, the q-fractional integral of order ξ is given by*

$$I_q^\xi \varkappa(s) = \frac{\Gamma_q(t+1)}{\Gamma_q(\xi + t + 1)} (s-a)_q^{(\xi+t)}, \quad 0 < a < s < T.$$

Additionally, for the constant function $w(s) = 1$, the integral becomes:

$$\big(I_q^\xi 1\big)(s) = \frac{1}{\Gamma_q(\xi+1)} (s-a)_q^{(\xi)}.$$

Definition 2.3 ([20]). The Caputo q-fractional derivative of order $\xi > 0$ is defined by

$${}^{C}\mathcal{D}_q^\xi \varkappa(s) = \frac{1}{\Gamma_q(1-\xi)} \int_0^s (s - q\rho)_q^{-\xi} \mathcal{D}_q \varkappa(\rho)\, d_q\rho.$$

Lemma 2.4 ([20]). *Let x be a function defined on $\mathbb{J}$ with values in $\mathbb{R}$ and $\xi, t \geq 0$. Thus, we have the following properties:*

(i) $I_q^{\xi_1}\left(I_q^{\xi_2} \varkappa\right)(s) = I_q^{\xi_1+\xi_2} \varkappa(s)$,

(ii) ${}^{C}\mathcal{D}_q^\xi \left(I_q^\xi \varkappa\right)(s) = \varkappa(s)$.

Definition 2.5 ([16]). Let $\varkappa \in C_q^1([0,b],\mathbb{R})$, $0<\xi<1$ and $0\le\eta\le 1$. The operator

$$ {}^{H}\mathcal{D}_q^{\xi,\eta}\varkappa(s) = I_q^{\eta(1-\xi)}\big[\mathcal{D}_q\big(I_q^{(1-\eta)(1-\xi)}\varkappa\big)\big](s), $$

is named the q-Hilfer fractional derivative of order ξ and type η. Furthermore, the operator ${}^{H}\mathcal{D}_q^{\xi,\eta}$ can also be expressed as:

$$ {}^{H}\mathcal{D}_q^{\xi,\eta}w = I_q^{\eta(1-\xi)}\mathcal{D}_q\big(I_q^{(1-\eta)(1-\xi)}w\big) = I_q^{\eta(1-\xi)}\mathcal{D}_q\big(I_q^{1-\zeta}w\big), $$

where $\zeta=\xi+\eta(1-\xi)$.

Theorem 2.6 (Schauder FPT [25]). *A continuous compact operator $A:\mathcal{B}\to\mathcal{B}$ on a nonempty closed convex subset $\mathcal{B}$ of a Banach space has at least one fixed point.*

Theorem 2.7 (Banach FPT [25]). *In a Banach space $(E,\|\cdot\|)$, if $\mathcal{B}$ is a nonempty closed subset and $\Phi:\mathcal{B}\to\mathcal{B}$ is a contraction mapping, then Φ has precisely one fixed point in $\mathcal{B}$.*

3 Main results

We begin by introducing a lemma that deals with the linear form of problem (1.1) and plays a crucial role in converting the problem as a fixed-point problem.

For the sake of convenience, we put

$$\begin{aligned}
\Lambda_1 &= \frac{\Gamma_q(\xi)}{\Gamma_q(\zeta)b^{\xi-1}},\ \Lambda_2 = \frac{b}{\xi\Gamma_q(\zeta)},\\
\Lambda_3 &= \frac{b^{\zeta-2\xi}\Gamma_q(\xi)}{\Gamma_q(\xi+1)\Gamma_q(\zeta-\xi)},\ \Lambda_4 = \frac{b^{\zeta-\xi}}{\xi\Gamma_q(\zeta-\xi)\Gamma_q(\xi+1)},\\
\Theta &= 1-\frac{\Gamma_q(\xi)b^{\zeta-2\xi}}{\Gamma_q(\zeta-\xi)}\neq 0,\\
\Delta &= \left(\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\xi+\delta+\zeta)}+\left(\left(1+\frac{\Lambda_4}{|\Theta|}\right)\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\delta+\zeta)}+\frac{b^{2\xi+\delta}\Lambda_3\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)}\right)\right.\\
&\quad\left.+\frac{\Lambda_1 b^{\zeta+\xi+\delta-1}\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)}+\frac{\Lambda_2\Gamma_q(\zeta)b^{\zeta+\delta-1}}{|\Theta|\,\Gamma_q(\delta+\zeta)}\right).
\end{aligned}\tag{3.1}$$

Lemma 3.1. *Let $g\in C(\mathbb{J},\mathbb{R})$ be a given function. The BVP*

$$\begin{aligned}
{}^{c}\mathcal{D}_q^{\delta}({}^{H}\mathcal{D}_q^{\xi,\eta}\varkappa(s)) &= g(s),\ 0\le s\le b,\\
\varkappa(b) &= 0,\ {}^{C}\mathcal{D}_q^{\xi}\varkappa(b)=0,
\end{aligned}\tag{3.2}$$

is equivalent to integral equation

$$\begin{aligned}
\varkappa(s) &= \frac{1}{\Gamma_q(\xi+\delta)}\int_0^s (s-qt)_q^{\xi+\delta-1}g(t)\,d_qt\\
&\quad+s^{\xi}\left(-\left(1+\frac{\Lambda_4}{\Theta}\right)\frac{1}{\Gamma_q(\delta)}\int_0^b (b-qt)_q^{\delta-1}g(t)\,d_qt\right.\\
&\quad\left.+\frac{\Lambda_3}{\Theta\Gamma_q(\xi+\delta)}\int_0^b (b-qt)_q^{\xi+\delta-1}g(t)\,d_qt\right)\\
&\quad+s^{\zeta-1}\left(-\frac{\Lambda_1}{\Theta}\frac{1}{\Gamma_q(\xi+\delta)}\int_0^b (b-qt)_q^{\xi+\delta-1}g(t)\,d_qt\right.\\
&\quad\left.+\frac{\Lambda_2}{\Theta\Gamma_q(\delta)}\int_0^b (b-qt)_q^{\delta-1}g(t)\,d_qt\right),
\end{aligned}\tag{3.3}$$

where Λ_i, $i=\overline{1,4}$ and Θ are given in (3.1)

Proof. By applying I_q^{δ} to the first equation in (3.2), we arrive at ${}^{H}\mathcal{D}_q^{\xi,\eta}\varkappa(s) = I_q^{\delta}g(s) + k$. Namely

$$I_q^{\eta 1-\xi}\mathcal{D}_q I_q^{(1-\eta)(1-\xi)}\varkappa(s) = I_q^{\delta}g(s) + k.$$

We apply I_q^{ξ} to the previous equation once again, yielding:

$$I_q^{\xi+\eta(1-\xi)}\left({}^{RL}\mathcal{D}_q^{\zeta}\right)\varkappa(s) = I_q^{\xi+\delta}g(s) + I_q^{\xi}(k),$$

this gives

$$\begin{aligned}\varkappa(s) &= I_q^{\xi+\delta}g(s) + I_q^{\xi}(k) + A\frac{1}{\Gamma_q(\zeta)}s^{\zeta-1}\\ &= I_q^{\xi+\delta}g(s) + k\frac{1}{\Gamma_q(\xi+1)}s^{\xi} + A\frac{1}{\Gamma_q(\zeta)}s^{\zeta-1},\end{aligned} \tag{3.4}$$

where $\zeta = \xi + \eta(1-\xi)$, using the boundary conditions, we find

$$A = \Gamma_q(\zeta)\left(-\frac{\Lambda_1}{\Theta}I_q^{\xi+\delta}g(b) + \frac{\Lambda_2}{\Theta}I_q^{\delta}g(b)\right),$$

and

$$K = \Gamma_q(\xi+1)\left(-\left(1+\frac{\Lambda_4}{\Theta}\right)I_q^{\delta}g(b) + \frac{\Lambda_3}{\Theta}I_q^{\xi+\delta}g(b)\right),$$

where Θ is defined in (3.1). Substituting these constants in (3.4), this yields the desired expression (3.3). □

In the sequel, we apply some fixed piont theorems to demonstrate the existence and uniqueness results for problem (1.1).

To obtain our outcomes, We need the following assumptions:

(Hyp1) There is a constant $t_{\mathfrak{f}} > 0$ such that

$$|\mathfrak{f}(s,\varkappa_1) - \mathfrak{f}(s,\varkappa_2)| \leq t_{\mathfrak{f}}|\varkappa_1 - \varkappa_2|, \ \forall s \in [0,b] \text{ and } \varkappa_1, \varkappa_2 \in \mathbb{R}.$$

(Hyp2) There exists two function $q_1 \in C_{1-\zeta}(\mathbb{J},\mathbb{R}^+)$ and $q_2 \in C(\mathbb{J},\mathbb{R}^+)$, such that

$$|\mathfrak{f}(s,\varkappa)| \leq q_1(s) + q_2(s)|\varkappa|, \ \forall (s,\varkappa_1) \in [0,b]\times\mathbb{R}.$$

3.1 Existence and uniqueness results via Banach's fixed piont theorem

Theorem 3.2. *Let (Hyp1) be valid. If*

$$t_{\mathfrak{f}}\Delta < 1, \tag{3.5}$$

then, (1.1) has a unique solution on $\mathbb{J}$, where Δ is given by (3.1).

Proof. We reinterpret problem (1.1) as a fixed point problem by introducing an operator $A : C_{1-\zeta}(\mathbb{J},\mathbb{R}) \to C_{1-\zeta}(\mathbb{J},\mathbb{R})$ as

$$\begin{aligned}A\varkappa(s) &= \frac{1}{\Gamma_q(\xi+\delta)}\int_0^s (s-qt)_q^{\xi+\delta-1}\mathfrak{f}(t,x(t))d_qt\\ &+ s^{\xi}\left(-\left(1+\frac{\Lambda_4}{\Theta}\right)\frac{1}{\Gamma_q(\delta)}\int_0^b (b-qt)_q^{\delta-1}\mathfrak{f}(t,x(t))\,d_qt\right.\\ &\left.+\frac{\Lambda_3}{\Theta\Gamma_q(\xi+\delta)}\int_0^b (b-qt)_q^{\xi+\delta-1}\mathfrak{f}(t,x(t))\,d_qt\right)\\ &+ s^{\zeta-1}\left(-\frac{\Lambda_1}{\Theta}\frac{1}{\Gamma_q(\xi+\delta)}\int_0^b (b-qt)_q^{\xi+\delta-1}\mathfrak{f}(t,x(t))\,d_qt\right.\\ &\left.+\frac{\Lambda_2}{\Theta\Gamma_q(\delta)}\int_0^b (b-qt)_q^{\delta-1}\mathfrak{f}(t,x(t))\,d_qt\right).\end{aligned}$$

According to Lemma (3.1), the fixed points of the operator A represent the solutions to problem (1.1). The operator A is well define. i.e. for every $\varkappa \in C_{1-\zeta}(\mathbb{J},\mathbb{R})$ we have $A(\varkappa) \in C_{1-\zeta}(\mathbb{J},\mathbb{R})$. By using the assumption (H1), we have

$$\begin{aligned}|\mathfrak{f}(s,x(s))| &\leq |\mathfrak{f}(s,x(s))-\mathfrak{f}(s,0)|+|\mathfrak{f}(s,0)|\\ &\leq t_{\mathfrak{f}}\,|x(t)|+c_1,\ \forall s\in\mathbb{J},\end{aligned}$$

where $c_1=\sup_{s\in\mathbb{J}}|\mathfrak{f}(s,0)|$. By Lemma 2.2 and for any $\varkappa \in C_{1-\zeta}(\mathbb{J},\mathbb{R})$, we get

$$\begin{aligned}&\left|s^{1-\zeta}A\varkappa(s)\right|\\ &\leq \frac{s^{1-\zeta}}{\Gamma_q(\xi+\delta)}\int_0^s (s-qt)_q^{\xi+\delta-1}t^{\zeta-1}\left(t_{\mathfrak{f}}\left|t^{1-\zeta}x(t)\right|+c_1\right)d_qt\\ &+s^{1+\xi-\zeta}\left(\left(1+\frac{\Lambda_4}{|\Theta|}\right)\frac{1}{\Gamma_q(\delta)}\int_0^b (b-qt)_q^{\delta-1}t^{\zeta-1}\left(t_{\mathfrak{f}}\left|t^{1-\zeta}x(t)\right|+c_1\right)\,d_qt\right.\\ &\left.+\frac{\Lambda_3}{|\Theta|\,\Gamma_q(\xi+\delta)}\int_0^b (b-qt)_q^{\xi+\delta-1}t^{\zeta-1}\left(t_{\mathfrak{f}}\left|t^{1-\zeta}x(t)\right|+c_1\right)\,d_qt\right)\\ &+\left(\frac{\Lambda_1}{|\Theta|\,\Gamma_q(\xi+\delta)}\int_0^b (b-qt)_q^{\xi+\delta-1}t^{\zeta-1}\left(t_{\mathfrak{f}}\left|t^{1-\zeta}x(t)\right|+c_1\right)\,d_qt\right.\\ &\left.+\frac{\Lambda_2}{|\Theta|\,\Gamma_q(\delta)}\int_0^b (b-qt)_q^{\delta-1}t^{\zeta-1}\left(t_{\mathfrak{f}}\left|t^{1-\zeta}x(t)\right|+c_1\right)\,d_qt\right)\\ &\leq\left(\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\xi+\delta+\zeta)}+\left(\left(1+\frac{\Lambda_4}{|\Theta|}\right)\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\delta+\zeta)}+\frac{b^{2\xi+\delta}\Lambda_3\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)}\right)\right.\\ &\left.+\frac{\Lambda_1 b^{\zeta+\xi+\delta-1}\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)}+\frac{\Lambda_2\Gamma_q(\zeta)b^{\zeta+\delta-1}}{|\Theta|\,\Gamma_q(\delta+\zeta)}\right)\left(t_{\mathfrak{f}}\|x\|_{C_{1-\zeta}}+c_1\right).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\|Ax\|_{C_{1-\zeta}}\\ &\leq\left(\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\xi+\delta+\zeta)}+\left(\left(1+\frac{\Lambda_4}{|\Theta|}\right)\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\delta+\zeta)}+\frac{b^{2\xi+\delta}\Lambda_3\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)}\right)\right.\\ &\left.+\frac{\Lambda_1 b^{\zeta+\xi+\delta-1}\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)}+\frac{\Lambda_2\Gamma_q(\zeta)b^{\zeta+\delta-1}}{|\Theta|\,\Gamma_q(\delta+\zeta)}\right)\left(t_{\mathfrak{f}}\|x\|_{C_{1-\zeta}}+c_1\right).\end{aligned}$$

Which means that $A(\varkappa)\in C_{1-\zeta}(\mathbb{J},\mathbb{R})$.

By (Hyp1), for each $\varkappa_1,\varkappa_2\in C_{1-\zeta}(\mathbb{J},\mathbb{R})$ and $\mathfrak{s}\in\mathbb{J}$, we get

$$\begin{aligned}&\left|s^{1-\zeta}\left(A\varkappa_1(s)-A\varkappa_2(s)\right)\right|\\ &\leq\frac{t_{\mathfrak{f}}s^{1-\zeta}}{\Gamma_q(\xi+\delta)}\int_0^s (s-qt)_q^{\xi+\delta-1}t^{\zeta-1}\left|t^{1-\zeta}\left(\varkappa_1-\varkappa_2\right)\right|d_qt\\ &+t_{\mathfrak{f}}s^{1+\xi-\zeta}\left(\left(1+\frac{\Lambda_4}{|\Theta|}\right)\frac{1}{\Gamma_q(\delta)}\int_0^b (b-qt)_q^{\delta-1}t^{\zeta-1}\left|t^{1-\zeta}\left(\varkappa_1-\varkappa_2\right)\right|\,d_qt\right.\\ &\left.+\frac{\Lambda_3}{|\Theta|\,\Gamma_q(\xi+\delta)}\int_0^b (b-qt)_q^{\xi+\delta-1}t^{\zeta-1}\left|t^{1-\zeta}\left(\varkappa_1-\varkappa_2\right)\right|d_qt\right)\\ &+\left(\frac{t_{\mathfrak{f}}\Lambda_1}{|\Theta|\,\Gamma_q(\xi+\delta)}\int_0^b (b-qt)_q^{\xi+\delta-1}t^{\zeta-1}\left|t^{1-\zeta}\left(\varkappa_1-\varkappa_2\right)\right|d_qt\right.\\ &\left.+\frac{t_{\mathfrak{f}}\Lambda_2}{|\Theta|\,\Gamma_q(\delta)}\int_0^b (b-qt)_q^{\delta-1}t^{\zeta-1}\left|t^{1-\zeta}\left(\varkappa_1-\varkappa_2\right)\right|d_qt\right)\end{aligned}$$

$$\leq \left(\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\xi+\delta+\zeta)} + \left(\left(1+\frac{\Lambda_4}{|\Theta|}\right) \frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\delta+\zeta)} + \frac{b^{2\xi+\delta}\Lambda_3\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)} \right) \right.$$
$$\left. + \frac{\Lambda_1 b^{\zeta+\xi+\delta-1}\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)} + \frac{\Lambda_2\Gamma_q(\zeta)b^{\zeta+\delta-1}}{|\Theta|\,\Gamma_q(\delta+\zeta)} \right) t_{\mathfrak{f}} \|\varkappa_1 - \varkappa_2\|_{C_{1-\zeta}}.$$

Thus, we obtain

$$\|A\varkappa_1 - A\varkappa_2\|_{C_{1-\zeta}}$$
$$\leq \left(\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\xi+\delta+\zeta)} + \left(\left(1+\frac{\Lambda_4}{|\Theta|}\right) \frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\delta+\zeta)} + \frac{b^{2\xi+\delta}\Lambda_3\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)} \right) \right.$$
$$\left. + \frac{\Lambda_1 b^{\zeta+\xi+\delta-1}\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)} + \frac{\Lambda_2\Gamma_q(\zeta)b^{\zeta+\delta-1}}{|\Theta|\,\Gamma_q(\delta+\zeta)} \right) t_{\mathfrak{f}} \|\varkappa_1 - \varkappa_2\|_{C_{1-\zeta}}$$
$$= t_{\mathfrak{f}}\Delta \|\varkappa_1 - \varkappa_2\|_{C_{1-\zeta}}.$$

From (3.5), A is a contraction. As an outcome of Banach's F.P.T , A has a unique fixed point which is a unique solution of (1.1) on $\mathbb{J}$. □

3.2 Existence Results via Schauder's fixed piont theorem

Theorem 3.3. *Assume that conditions (Hyp1)-((Hyp2) are fulfilled. If*

$$q_2^* \triangle < 1, \tag{3.6}$$

then, the problem (1.1) has at least one solution on $\mathbb{J}$, where $q_2^* = \sup_{s\in J} q_2(s)$ and Δ is given by (3.1).

Proof. Let us fix

$$R \geq \frac{q_1^* \triangle}{1 - q_2^* \triangle}.$$

Consider the non-empty closed bounded and convex subset $\mathcal{B}_R = \{\varkappa \in C_{1-\zeta}(\mathbb{J},\mathbb{R}) : \|\varkappa\|_{C_{1-\zeta}} \leq R\}$. We shall demonstrate that A holds the requirements of Schauder's fixed piont theorem, by presenting the argument in three distinct steps.

Step1 The operator A transforms $\mathcal{B}_R$ into itself.

For every $\varkappa \in \mathcal{B}_R$, we have

$$\left|s^{1-\zeta}A\varkappa(s)\right|$$
$$\leq \frac{s^{1-\zeta}}{\Gamma_q(\xi+\delta)} \int_0^s (s-qt)_q^{\xi+\delta-1} t^{\zeta-1} \left(t^{1-\zeta}q_1(t) + q_2(t)\left|t^{1-\zeta}\varkappa(t)\right|\right) d_q t$$
$$+ s^{1+\xi-\zeta} \left(\left(1+\frac{\Lambda_4}{|\Theta|}\right) \frac{1}{\Gamma_q(\delta)} \int_0^b (b-qt)_q^{\delta-1} t^{\zeta-1} \left(t^{1-\zeta}q_1(t) + q_2(t)\left|t^{1-\zeta}\varkappa(t)\right|\right) d_q t \right.$$
$$\left. + \frac{\Lambda_3}{|\Theta|\,\Gamma_q(\xi+\delta)} \int_0^b (b-qt)_q^{\xi+\delta-1} t^{\zeta-1} \left(t^{1-\zeta}q_1(t) + q_2(t)\left|t^{1-\zeta}\varkappa(t)\right|\right) d_q t \right)$$
$$+ \left(\frac{\Lambda_1}{|\Theta|\,\Gamma_q(\xi+\delta)} \int_0^b (b-qt)_q^{\xi+\delta-1} t^{\zeta-1} \left(t^{1-\zeta}q_1(t) + q_2(t)\left|t^{1-\zeta}\varkappa(t)\right|\right) d_q t \right.$$
$$\left. + \frac{\Lambda_2}{|\Theta|\,\Gamma_q(\delta)} \int_0^b (b-qt)_q^{\delta-1} t^{\zeta-1} \left(t^{1-\zeta}q_1(t) + q_2(t)\left|t^{1-\zeta}\varkappa(t)\right|\right) d_q t \right)$$
$$\leq \left(\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\xi+\delta+\zeta)} + \left(\left(1+\frac{\Lambda_4}{|\Theta|}\right) \frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\delta+\zeta)} + \frac{b^{2\xi+\delta}\Lambda_3\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)} \right) \right.$$
$$\left. + \frac{\Lambda_1 b^{\zeta+\xi+\delta-1}\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)} + \frac{\Lambda_2\Gamma_q(\zeta)b^{\zeta+\delta-1}}{|\Theta|\,\Gamma_q(\delta+\zeta)} \right) (q_1^* + q_2^* R),$$

thus, we get

$$\|A\varkappa\|_{C_{1-\zeta}} \leq \triangle\left(q_1^* + q_2^* R\right) \leq R,$$

which means that $A\left(\mathcal{B}_R\right) \subset \mathcal{B}_R$ and and the set $A\left(\mathcal{B}_R\right)$ is uniformly bounded.

Step 2 The mapping A is continuous.

Let $(\varkappa_n)_{n\in N} \subset \mathcal{B}_R$ be a sequence such that $\varkappa_n \to \varkappa$ in $C_{1-\zeta}(J,\mathbb{R})$. By Lemma 2.2 and (Hyp1), we have

$$\begin{aligned}
&\left|s^{1-\zeta}\left(A\varkappa_n(s) - A\varkappa(s)\right)\right| \\
&\leq \frac{t_{\mathfrak{f}} s^{1-\zeta}}{\Gamma_q(\xi+\delta)} \int_0^s (s-qt)_q^{\xi+\delta-1} t^{\zeta-1} \left|t^{1-\zeta}\left(\varkappa_n - \varkappa\right)\right| d_q t \\
&+ t_{\mathfrak{f}} s^{1+\xi-\zeta} \left(\left(1 + \frac{\Lambda_4}{|\Theta|}\right) \frac{1}{\Gamma_q(\delta)} \int_0^b (b-qt)_q^{\delta-1} t^{\zeta-1} \left|t^{1-\zeta}\left(\varkappa_n - \varkappa\right)\right| d_q t \right. \\
&\left. + \frac{\Lambda_3}{|\Theta|\,\Gamma_q(\xi+\delta)} \int_0^b (b-qt)_q^{\xi+\delta-1} t^{\zeta-1} \left|t^{1-\zeta}\left(\varkappa_n - \varkappa\right)\right| d_q t \right) \\
&+ \left(\frac{t_{\mathfrak{f}}\Lambda_1}{|\Theta|\,\Gamma_q(\xi+\delta)} \int_0^b (b-qt)_q^{\xi+\delta-1} t^{\zeta-1} \left|t^{1-\zeta}\left(\varkappa_n - \varkappa\right)\right| d_q t \right. \\
&\left. + \frac{t_{\mathfrak{f}}\Lambda_2}{|\Theta|\,\Gamma_q(\delta)} \int_0^b (b-qt)_q^{\delta-1} t^{\zeta-1} \left|t^{1-\zeta}\left(\varkappa_n - \varkappa\right)\right| d_q t \right) \\
&\leq \left(\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\xi+\delta+\zeta)} + \left(\left(1 + \frac{\Lambda_4}{|\Theta|}\right) \frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\delta+\zeta)} + \frac{b^{2\xi+\delta}\Lambda_3\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)} \right) \right. \\
&\left. + \frac{\Lambda_1 b^{\zeta+\xi+\delta-1}\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)} + \frac{\Lambda_2\Gamma_q(\zeta) b^{\zeta+\delta-1}}{|\Theta|\,\Gamma_q(\delta+\zeta)} \right) t_{\mathfrak{f}} \|\varkappa_n - \varkappa\|_{C_{1-\zeta}},
\end{aligned}$$

thus, we obtain

$$\begin{aligned}
&\|A\varkappa_n - A\varkappa\|_{C_{1-\zeta}} \\
&\leq \left(\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\xi+\delta+\zeta)} + \left(\left(1 + \frac{\Lambda_4}{|\Theta|}\right) \frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\delta+\zeta)} + \frac{b^{2\xi+\delta}\Lambda_3\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)} \right) \right. \\
&\left. + \frac{\Lambda_1 b^{\zeta+\xi+\delta-1}\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)} + \frac{\Lambda_2\Gamma_q(\zeta) b^{\zeta+\delta-1}}{|\Theta|\,\Gamma_q(\delta+\zeta)} \right) t_{\mathfrak{f}} \|\varkappa_n - \varkappa\|_{C_{1-\zeta}} \\
&= t_{\mathfrak{f}}\Delta \|\varkappa_n - \varkappa\|_{C_{1-\zeta}},
\end{aligned}$$

which gives

$$\|A\varkappa_n - A\varkappa\|_{C_{1-\zeta}} \to 0 \text{ as } n \to \infty.$$

Therefore, A is continuous.

Step 3. $A(\mathcal{B}_R)$ is a equicontinuous set.

For $s_1, s_2 \in \mathbb{J},\ s_1 < s_2$ and $\varkappa \in \mathcal{B}$, we have

$$\begin{aligned}
&|s_2^{1-\zeta} A\varkappa(s_2) - s_1^{1-\zeta} A\varkappa(s_1)| \\
&= \left| \frac{1}{\Gamma_q(\xi+\delta)} \int_0^{s_2} s_2^{1-\zeta} (s_2 - qt)_q^{\xi+\delta-1} t^{\zeta-1} \left(t^{1-\zeta} q_1(t) + q_2(t) \left|t^{1-\zeta}\varkappa(t)\right|\right) d_q t \right. \\
&- \frac{1}{\Gamma_q(\xi+\delta)} \int_0^{s_1} s_1^{1-\zeta} (s_1 - qt)_q^{\xi+\delta-1} t^{\zeta-1} \left(t^{1-\zeta} q_1(t) + q_2(t) \left|t^{1-\zeta}\varkappa(t)\right|\right) d_q t \\
&+ \left(s_2^{1+\xi-\zeta} - s_1^{1+\xi-\zeta}\right) \left(\left(1 + \frac{\Lambda_4}{|\Theta|}\right) \frac{1}{\Gamma_q(\delta)} \int_0^b (b-qt)_q^{\delta-1} t^{\zeta-1} \left(t^{1-\zeta} q_1(t) + q_2(t) \left|t^{1-\zeta}\varkappa(t)\right|\right) d_q t \right. \\
&\left. \left. + \frac{\Lambda_3}{|\Theta|\Gamma_q(\xi+\delta)} \int_0^b (b-qt)_q^{\xi+\delta-1} t^{\zeta-1} \left(t^{1-\zeta} q_1(t) + q_2(t) \left|t^{1-\zeta}\varkappa(t)\right|\right) d_q t \right) \right|
\end{aligned}$$

$$\leq \left(\frac{(q_1^*+q_2^*R)}{\Gamma_q(\xi+\delta)}\int_0^{s_1}\left(s_2^{1-\zeta}(s_2-qt)_q^{\xi+\delta-1}-s_1^{1-\zeta}(s_1-qt)_q^{\xi+\delta-1}\right)t^{\zeta-1}d_qt\right.$$
$$+\frac{(q_1^*+q_2^*R)}{\Gamma_q(\xi+\delta)}\int_{s_1}^{s_2}s_2^{1-\zeta}(s_2-qt)_q^{\xi+\delta-1}t^{\zeta-1}d_qt$$
$$\left.+\left(s_2^{1+\xi-\zeta}-s_1^{1+\xi-\zeta}\right)(q_1^*+q_2^*R)\left(\left(1+\frac{\Lambda_4}{|\Theta|}\right)\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\delta+\zeta)}+\frac{b^{2\xi+\delta}\Lambda_3\Gamma_q(\zeta)}{|\Theta|\Gamma_q(\xi+\delta+\zeta)}\right)\right),$$

thus , we get

$$|s_2^{1-\zeta}A\varkappa(s_2)-s_1^{1-\zeta}A\varkappa(s_1)|$$
$$\leq \left(\frac{(q_1^*+q_2^*R)}{\Gamma_q(\xi+\delta)}\int_0^{s_1}\left(s_2^{1-\zeta}(s_2-qt)_q^{\xi+\delta-1}-s_1^{1-\zeta}(s_1-qt)_q^{\xi+\delta-1}\right)t^{\zeta-1}d_qt\right.$$
$$+\frac{(q_1^*+q_2^*R)}{\Gamma_q(\xi+\delta)}\int_{s_1}^{s_2}s_2^{1-\zeta}(s_2-qt)_q^{\xi+\delta-1}t^{\zeta-1}d_qt$$
$$\left.+\left(s_2^{1+\xi-\zeta}-s_1^{1+\xi-\zeta}\right)(q_1^*+q_2^*R)\left(\left(1+\frac{\Lambda_4}{|\Theta|}\right)\frac{b^{\xi+\delta}\Gamma_q(\zeta)}{\Gamma_q(\delta+\zeta)}+\frac{b^{2\xi+\delta}\Lambda_3\Gamma_q(\zeta)}{|\Theta|\,\Gamma_q(\xi+\delta+\zeta)}\right)\right).$$

As $s_1\to s_2$, the right-hand side of the above inequality tends to zero. That is to say that $A(\mathcal{B}_{\mathcal{R}})$ is equicontinuous, then by Ascoli-Arzela theorem, we conclude that the operator A is compact.

Consequently, all assumptions of Schauder's fixed point theorem hold, Thus there a fixed point $\varkappa\in\mathcal{B}_{\mathcal{R}}$, which is indeed a solution of (1.1) on $\mathbb{J}$. □

4 Example

As an application of our result, we consider the following BVP of fractional q-difference equation:

$$^{H}\mathcal{D}_{\frac{1}{2}}^{\frac{1}{3},\eta}\left(^{C}\mathcal{D}_{\frac{1}{2}}^{\frac{1}{4}}\varkappa(s)\right)=\frac{\sin s}{5+e^s}\varkappa(s),\ s\in[0,1],$$
$$\varkappa(1)=0,\ ^{C}\mathcal{D}_{\frac{1}{2}}^{\frac{1}{4}}\varkappa(0)=0, \tag{4.1}$$

here $\xi=\frac{1}{4}$, $\zeta=\frac{7}{16}$, $\delta=\frac{3}{4}$, $q=\frac{1}{2}$, $b=1$, and $\mathfrak{f}(s,\varkappa)=\frac{\sin s}{5+e^s}\varkappa(s)$. Using these data we find $\Theta=0.159\,83\neq 0$ and

$$C_{1-\zeta}([0,1],\mathbb{R})=C_{\frac{9}{16}}([0,1],\mathbb{R})=\left\{h:(0,1]\to\mathbb{R}:s^{\frac{9}{16}}h\in C([0,1],\mathbb{R})\right\}.$$

Set

$$\mathfrak{f}(s,\varkappa)=\frac{\sin s}{5+e^s}\varkappa(s),\ \ (s,\varkappa)\in[0,1]\times\mathbb{R}.$$

For each $\varkappa_1,\varkappa_2\in\mathbb{R}$, we have

$$|\mathfrak{f}(s,\varkappa_1)-\mathfrak{f}(s,\varkappa_2)|\leq\frac{1}{6}|\varkappa_1-\varkappa_2|,\ s\in[0,1].$$

Hence, the assumption (Hyp1) is satisfied with $t_{\mathfrak{f}}=\frac{1}{6}$. We will check that condition (3.5) is holds. Indeed

$$t_{\mathfrak{f}}\Delta\approx 0.44<1.$$

Therefore, it follows from Theorem 3.2 that the problem (4.1) has a unique solution in the space $C_{\frac{9}{16}}([0,1],\mathbb{R})$.

5 Conclusion

This paper has investigated a novel class of boundary value problems for fractional q-difference equations that incorporate q-Caputo and q-Hilfer derivatives. The existence of solutions is established using Schauder's fixed point theorem and the Banach contraction mapping principle. A concrete example is provided to demonstrate the applicability of the theoretical findings. In future work, numerically approximating such problems remains a promising yet challenging direction due to the combined influence of fractional operators and q-calculus. Furthermore, applying these models to real-world phenomena with memory effects and long-range interactions would reinforce the motivation and shed new light on our theoretical results.

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