Palestine Journal of Mathematics

Vol.15(3), (2026) , 1071–1089 © Palestine Polytechnic University-PPU 2026

CONVERGENCE RESULTS FOR SAAM ITERATION AND SOLUTION OF AN INTEGRAL EQUATION

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Communicated by Hichem Ben-El-Mechaiekh

MSC 2010 Classifications: Primary 47H10; Secondary 54H25.

Keywords and phrases: Banach space; convergence; Reich–Suzuki-type nonexpansive mapping; stability; nonlinear integral equation.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract We introduce a new fixed point iterative scheme, called the Successive Acceleration Approximation Method (SAAM) iteration, designed to investigate the convergence behavior and approximation properties of nonlinear mappings in Banach spaces. The proposed method is studied for almost contraction mappings and for a class of generalized nonexpansive-type mappings. Unlike several existing iterative procedures that rely heavily on convex relaxation at boundary stages, the proposed iteration employs successive operator compositions, leading to improved error estimates and faster convergence. We establish convergence, strong convergence, and stability results for the new iteration and theoretically compare its convergence rate with several well-known fixed point iterative methods. The analysis shows that the proposed scheme achieves a stronger analytical error decay rate and faster convergence behavior. Numerical experiments further support the theoretical findings by demonstrating that the method reaches the fixed point in fewer iterations. Finally, an application to a nonlinear integral equation is presented to illustrate the practical effectiveness of the scheme.

1 Introduction

In 1922, Banach [4] introduced his famous contraction principle, establishing the existence of a unique fixed point for a contraction of a complete metric space into itself. The unique fixed point is obtained as the limit of a sequence of approximating Picard iterations. Various adaptations and extensions of the Banach contraction principle have continued to appear in the literature, including the works of Caristi [9], Edelstein [11], Kirk [20, 21], Matkowski [23, 24], and Ran and Reurings [30], among others([19],[29],[38],[39], [40]).

The theory of nonexpansive mappings constitutes one of the central areas in metric fixed point theory. Fundamental existence results were established independently by Browder [8], Göhde [13], and Kirk [20], whose celebrated theorem guarantees the existence of fixed points for nonexpansive mappings in Banach spaces. Subsequently, several iterative procedures for approximating fixed points were introduced, including the Mann iteration [22], Halpern iteration [14], and Ishikawa iteration [18]. These developments have led to a rich theory with numerous applications in nonlinear analysis, optimization, variational inequalities, and convex feasibility problems.

Although the Picard iteration converges for contractions, it may fail to converge for nonexpansive mappings. To overcome this limitation, Mann [22] introduced in 1953 the Mann iteration process for a non-expansive mapping $T : C \to C$, where C is a non-empty, closed and convex subset of Banach space U, defined by

$$w_{n+1} = (1 - \alpha_n)w_n + \alpha_n T w_n,$$

for all $n \geq 0$, where $w_0 \in C$ is arbitrary and $\alpha_n \in (0, 1)$.

For pseudo-contractive mappings, the Mann iteration also fails to converge to the fixed point. To resolve this problem, in 1974, Ishikawa [18] proposed a two-step iterative method for broader classes of nonlinear mappings, now known as the Ishikawa iteration, defined as

$$\left.\begin{aligned} w_{n+1} &= (1-\alpha_n)w_n + \alpha_n T x_n \\ x_n &= (1-\beta_n)w_n + \beta_n T w_n \end{aligned}\right\},$$

for all $n \geq 0$, where $w_0 \in C$ is arbitrary and $\alpha_n, \beta_n \in (0,1)$.

In 2000, Noor [25] introduced a three-step iterative process to improve approximation behavior for nonlinear operators, defined as

$$\left.\begin{aligned} w_{n+1} &= (1-\alpha_n)w_n + \alpha_n T x_n \\ x_n &= (1-\beta_n)w_n + \beta_n T y_n \\ y_n &= (1-\gamma_n)w_n + \gamma_n T w_n \end{aligned}\right\},$$

for all $n \geq 0$, where $w_0 \in C$ and $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

Motivated by these classical schemes, several researchers developed modified iterative procedures with improved convergence behavior and stability properties. In 2007, Agarwal et al. [2] introduced a two-step iteration process for nearly asymptotically nonexpansive mappings,

$$\left.\begin{aligned} w_{n+1} &= (1-\alpha_n)T w_n + \alpha_n T x_n \\ x_n &= (1-\beta_n)w_n + \beta_n T w_n \end{aligned}\right\},$$

for all $n \geq 0$, where $w_0 \in C$ and $\alpha_n, \beta_n \in (0,1)$. This process converges faster than Mann iteration process for contraction mappings.

In 2014, Abbas and Nazir [1] developed an iterative scheme which is faster than Agarwal et al.'s [2] scheme,

$$\left.\begin{aligned} w_{n+1} &= (1-\alpha_n)T x_n + \alpha_n T y_n \\ x_n &= (1-\beta_n)T w_n + \beta_n T y_n \\ y_n &= (1-\gamma_n)w_n + \gamma_n T w_n \end{aligned}\right\},$$

for all $n \geq 0$, where $w_0 \in C$ and $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

In 2016, Thakur et al. [37] developed an iterative procedure,

$$\left.\begin{aligned} w_{n+1} &= (1-\alpha_n)T y_n + \alpha_n T x_n \\ x_n &= (1-\beta_n)y_n + \beta_n T y_n \\ y_n &= (1-\gamma_n)w_n + \gamma_n T w_n \end{aligned}\right\}, \tag{1.1}$$

for all $n \geq 0$, where $w_0 \in C$ and $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

In 2018, Ullah and Arshad [42] developed an iteration process which converges faster than all the aforementioned processes,

$$\left.\begin{aligned} w_{n+1} &= T x_n \\ x_n &= T((1-\alpha_n)y_n + \alpha_n T y_n) \\ y_n &= (1-\beta_n)w_n + \beta_n T w_n \end{aligned}\right\},$$

for all $n \geq 0$, where $w_0 \in C$ and $\alpha_n, \beta_n \in (0,1)$.

In 2021, Ahmad et al. [3] presented the JK iteration,

$$\left.\begin{aligned} w_{n+1} &= T((1-\alpha_n)T y_n + \alpha_n T x_n) \\ x_n &= T y_n \\ y_n &= (1-\beta_n)w_n + \beta_n T w_n \end{aligned}\right\}, \tag{1.2}$$

for all $n \geq 0$, where $w_0 \in C$ and $\alpha_n, \beta_n \in (0,1)$.

In the same year, Rawat et al. [31] presented the S** iteration,

$$\left.\begin{aligned} w_{n+1} &= T\big((1-\alpha_n)T y_n + \alpha_n T x_n\big), \\ x_n &= T\big((1-\beta_n)y_n + \beta_n T y_n\big), \\ y_n &= T\big((1-\gamma_n)w_n + \gamma_n T w_n\big), \end{aligned}\right\}, \tag{1.3}$$

for all $n \geq 0$, where $w_0 \in C$ and $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

Then, Hammad et al. [16] presented the HR iteration,

$$\left.\begin{aligned} w_{n+1} &= Tx_n \\ x_n &= T((1-\gamma_n)y_n + \gamma_n T y_n) \\ y_n &= T((1-\beta_n)z_n + \beta_n T z_n) \\ z_n &= (1-\alpha_n)w_n + \alpha_n T w_n \end{aligned}\right\}, \tag{1.4}$$

for all $n \geq 0$, where $w_0 \in C$ and $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

Recently, Hammad et al. [15] introduced the HR* iteration for obtaining an approximate fixed point of almost contraction mappings and Reich–Suzuki-type nonexpansive mappings by generating the sequence $\{w_n\}$ iteratively, starting with an arbitrary element $w_0 \in C$, as

$$\left.\begin{aligned} z_n &= (1-\alpha_n)w_n + \alpha T w_n \\ y_n &= T((1-\beta_n)z_n + \beta_n T z_n) \\ x_n &= T(Ty_n) \\ w_{n+1} &= (1-\gamma_n)x_n + \gamma_n T x_n \end{aligned}\right\}. \tag{1.5}$$

The concept of stability of iterative procedures was introduced by Ostrowski [26] in 1967. An iterative process is said to be stable if every sequence generated with sufficiently small computational errors still converges to the same fixed point as the exact iteration. Ostrowski showed that the Picard iteration possesses this property for contractive mappings. Subsequently, Harder [17] generalized Ostrowski's framework by investigating the stability of more general iterative schemes under broader classes of contractive conditions. The study of stability is important in practical computations because exact evaluations of the operator involved in an iterative process are often impossible due to rounding errors, truncation errors, or numerical approximations. Consequently, instead of the exact sequence $\{x_n\}$, one frequently works with an approximate sequence $\{t_n\}$, and stability guarantees that such perturbations do not affect the convergence to the desired fixed point.

In this paper, we introduce a new four-step iterative process, called the Successive Accelerated Approximation Method (SAAM), designed to improve convergence speed by strategically employing successive operator compositions while minimizing relaxation effects at critical stages of the iteration process. We study the convergence, strong convergence, and stability properties of the proposed scheme for almost contraction mappings and for a class of generalized nonexpansive-type mappings known as Reich–Suzuki-type nonexpansive mappings. The convergence rate of the proposed iteration is theoretically compared with several existing methods, including the Thakur iteration [37], JK iteration [3], HR iteration [16], and the recently introduced HR* iteration [15]. We further support the theoretical analysis through numerical experiments and demonstrate the applicability of the proposed method to the solution of a nonlinear integral equation.

2 Preliminaries

In this work, U is a Banach space and C is a non-empty, closed and convex subset of U.

Definition 2.1. A self-mapping T on C is
(i) a contraction if we have $\alpha \in (0,1)$, satisfying

$$\|Tu - Tv\| \leq \alpha \|u - v\|, \text{ for all } u, v \in C, \tag{2.1}$$

(ii) quasi-nonexpansive [10] if

$$\|Tu - v\| \leq \|u - v\|, u \in C \text{ and } v \in F(T), \tag{2.2}$$

A broader class of weak contractions that properly includes the family of Zamfirescu operators [43] was introduced by Berinde in [6]. This class has attracted considerable attention in fixed point theory and is often referred to in the literature as the class of almost contraction mappings (ACMs).

Definition 2.2. [6] A mapping $T : C \to C$ is called an ACM, if we have $\theta \in (0, 1)$ and some constant $l \geq 0$, so that for all $u, v \in C$,

$$\|Tu - Tv\| \leq \theta\|u - v\| + l\|u - Tu\|. \tag{2.3}$$

Suzuki [36] introduced an interesting generalization of nonexpansive mappings and established several existence and convergence results for this broader class of operators.

Definition 2.3. [36] A mapping $T : C \to C$ is said to satisfy condition (C) if

$$\frac{1}{2}\|u - Tu\| \leq \|u - v\| \implies \|Tu - Tv\| \leq \|u - v\|, u, v \in C.$$

In 2019, Pant and Pandey [27] introduced and studied the class of Reich–Suzuki-type nonexpansive (RSTN) mappings, establishing several fundamental properties and convergence results for this class of operators.

Definition 2.4. [27] A mapping $T : C \to C$ is said to be an RSTN mapping if there exists $\gamma \in [0, 1)$ so that

$$\begin{aligned} &\frac{1}{2}\|u - Tu\| \leq \|u - v\| \\ &\text{implies } \|Tu - Tv\| \leq \gamma\|u - Tu\| + \gamma\|v - Tv\| + (1 - 2\gamma)\|u - v\|, \end{aligned} \tag{2.4}$$

for all $u, v \in C$.

Remark 2.5. It is very clear that any mapping that meets condition (C) [36] is classified as an RSTN mapping [27] for $\gamma = 0$. The reverse implication of this is not true.

Definition 2.6. [7] Suppose that the real sequences $\{u_n\}$ and $\{v_n\}$ converge to u and v, respectively. Assume there exists

$$s = \lim_{n\to\infty} \frac{\|u_n - u\|}{\|v_n - v\|}.$$

(i) If $s = 0$, then the sequence $\{u_n\}$ converges faster to u than $\{v_n\}$ does to v.

(ii) If $s \in (0, \infty)$, then $\{u_n\}$ and $\{v_n\}$ exhibit identical convergence rates.

Definition 2.7. [35] A mapping $T : C \to C$ with nonempty fixed point set F in C is said to satisfy Condition (I) if there is a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(\delta) > 0$, for all $\delta > 0$ and $h(0) = 0$, such that $\|q - Tq\| \geq h(d(q, F(T)))$ for all $q \in C$, where $d(q, F(T)) = \inf_{q^* \in F(T)} \{\|q - q^*\|\}$.

The next definition of stability was given by Ostrowski [26].

Definition 2.8. [26] Let T be a self-mapping on a Banach space U. Consider $w_0 \in U$ and a sequence of points $w_n \in U$ defined by the iteration $w_{n+1} = f(T, w_n)$, approaching a fixed point p. Let $\{s_n\}$ and $\{\epsilon_n\}$ be sequences in U and $\mathbb{R}^+ = [0, \infty)$ respectively, such that by $\epsilon_n = \|s_{n+1} - f(T, s_n)\|$. Then the iterative process $w_{n+1} = f(T, w_n)$ is called stable in relation to T (T-stable) if $\lim_{n\to\infty} s_n = p$ if and only if $\lim_{n\to\infty} \epsilon_n = 0$.

Lemma 2.9. *[5] Consider a real number λ, $0 \leq \lambda < 1$, and a sequence of positive numbers $\{\epsilon_n\}$, so that*

$$\lim_{n\to\infty} \epsilon_n = 0.$$

Then, for any sequence of positive numbers $\{w_n\}$

$$w_{n+1} \leq \lambda w_n + \epsilon_n, \text{ for } n = 1, 2, ...,$$

we get $\lim_{n\to\infty} w_n = 0$.

Lemma 2.10. *[34] Let $\{w_n\}$ be any real sequence in a uniformly convex Banach space U satisfying $0 < a \leq w_n \leq b < 1$, $n \geq 1$. Assume that $\{u_n\}$ and $\{v_n\}$ are any two sequences in U for which $\limsup_{n\to\infty}\|u_n\| \leq r$, $\limsup_{n\to\infty}\|v_n\| \leq r$ and $\limsup_{n\to\infty}\|w_n u_n + (1-w_n)v_n\| = r$, $r \geq 0$. Then, $\limsup_{n\to\infty}\|u_n - v_n\| = 0$.*

Definition 2.11. [41] Assume that the sequence $\{w_n\}$ is bounded in U. For any $p \in U$, we fix $r(p, \{w_n\}) = \limsup_{n\to\infty}\|w_n - p\|$.
(i) $r(C, \{w_n\}) = \inf\{r(p, \{w_n\}) : p \in C\}$ is the set called the asymptotic radius of $\{w_n\}$ with respect to C.
(ii)The set

$$A(C, \{w_n\}) = \{p \in C : r(p, \{w_n\}) = r(C, \{w_n\})\}, \tag{2.5}$$

is the asymptotic center of $\{w_n\}$ with respect to C. It is easy to see that, $A(C, \{w_n\})$ contains exactly one point in a uniformly convex Banach space.

3 Main Results

Following the direction of Hammad et al. [15], we introduce a novel iteration process

$$\left.\begin{aligned} z_n &= Tw_n \\ y_n &= T((1-\alpha_n)z_n + \alpha_n Tz_n) \\ x_n &= T((1-\beta_n)y_n + \beta_n Ty_n) \\ w_{n+1} &= Tx_n \end{aligned}\right\}, \tag{3.1}$$

for $n \geq 0$, where $w_0 \in C$ and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0, 1)$. We call it SAAM iteration.

We will first establish that, for ACM, the SAAM iteration converges faster than the HR* iteration and is stable. We then prove some weak as well as strong convergence conclusions for RSTN mappings via SAAM iteration. In the end, by providing a numerical experiment, we demonstrate that this newly introduced iteration converges more rapidly as compared to all aforementioned iteration processes.

The SAAM iteration is structurally related to the SADMR iteration introduced in [32], but the first two intermediate sequences are constructed differently.

Theorem 3.1. *Consider a self-mapping T on C that is an ACM, where C is a non-empty, convex and closed subset of a Banach space U. Consider $\{w_n\}$ to be a sequence generated by the SAAM iteration (3.1). Then $\{w_n\} \to w^*$, where w^* is a unique fixed point of T.*

Proof. Consider $w^* \in Fix(T)$. Based on ACM (2.3),

$$\begin{aligned} \|z_n - w^*\| &= \|Tw_n - Tw^*\| \\ &\leq \theta\|w_n - w^*\|. \end{aligned} \tag{3.2}$$

From the first step of SAAM iteration (3.1) and inequality (3.2), we obtain

$$\begin{aligned} \|y_n - w^*\| &= \|T((1-\alpha_n)z_n + \alpha_n Tz_n) - Tw^*\| \\ &\leq \theta\|(1-\alpha_n)z_n + \alpha_n Tz_n - w^*\| \\ &\leq \theta[(1-\alpha_n)\|z_n - w^*\| + \alpha_n\|Tz_n - Tw^*\|] \\ &\leq \theta[(1-\alpha_n)\|z_n - w^*\| + \alpha_n\theta\|z_n - w^*\|] \\ &= \theta(1-\alpha_n(1-\theta))\|z_n - w^*\| \\ &\leq \theta^2(1-\alpha_n(1-\theta))\|w_n - w^*\|. \end{aligned} \tag{3.3}$$

From the second step of SAAM iteration (3.1) and inequality (3.3), we obtain

$$\begin{aligned}\|x_n - w^*\| &= \|T((1-\beta_n)y_n + \beta_n Ty_n) - Tw^*\| \\ &\leq \theta\|(1-\beta_n)y_n + \beta_n Ty_n - w^*\| \\ &\leq \theta[(1-\beta_n)\|y_n - w^*\| + \beta_n\|Ty_n - Tw^*\|] \\ &\leq \theta[(1-\beta_n)\|y_n - w^*\| + \beta_n\theta\|y_n - w^*\|] \\ &\leq \theta[(1-\beta_n(1-\theta))\|y_n - w^*\|] \\ &\leq \theta[(1-\beta_n(1-\theta))\theta^2(1-\alpha_n(1-\theta))\|w_n - w^*\|] \\ &= \theta^3(1-\beta_n(1-\theta))(1-\alpha_n(1-\theta))\|w_n - w^*\|. \end{aligned} \tag{3.4}$$

From the third step of SAAM iteration (3.1) and inequality (3.4), we obtain

$$\begin{aligned}\|w_{n+1} - w^*\| &= \|Tx_n - Tw^*\| \\ &\leq \theta\|x_n - w^*\| \\ &\leq \theta^4(1-\beta_n(1-\theta))(1-\alpha_n(1-\theta))\|w_n - w^*\|. \end{aligned} \tag{3.5}$$

As $\theta \in (0,1)$ and $0 < \alpha_n, \beta_n < 1$, it follows that $(1-\alpha_n(1-\theta)) < 1$ and $(1-\beta_n(1-\theta)) < 1$, hence

$$\begin{aligned}\|w_{n+1} - w^*\| &\leq \theta^4\|w_n - w^*\| \\ &\leq \theta^8\|w_{n-1} - w^*\| \\ &\cdot \\ &\cdot \\ &\cdot \\ &\leq \theta^{4(n+1)}\|w_0 - w^*\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.6}$$

Hence, $\{w_n\} \to w^*$. Now, let w_1 and w_2 be two fixed points of T. Consider

$$\begin{aligned}\|Tw_1 - Tw_2\| &\leq \theta\|w_1 - w_2\| + l\|w_2 - Tw_2\| \\ &= \theta\|w_1 - w_2\|, \end{aligned}$$

that is, $\|w_1 - w_2\| < \|w_1 - w_2\|$, which is a contradiction. Hence, the fixed point is unique. □

Theorem 3.2. *Let C be a non-empty, convex and closed subset of a Banach space U and $T : C \to C$ be an ACM. If $\{w_n\}$ is a sequence generated by the SAAM iteration (3.1). Then $\{w_n\}$ converges more rapidly than the sequence $\{s_n\}$, which is generated by HR* iteration (1.5).*

Proof. From inequality (3.6) of Theorem 3.1, we obtain

$$\|w_{n+1} - w^*\| \leq \theta^{4(n+1)}\|w_0 - w^*\|.$$

Additionally, from the first step of HR* iteration (1.5), we obtain

$$\begin{aligned}\|z_n - w^*\| &= \|(1-\alpha_n)s_n + \alpha_n Ts_n - Tw^*\| \\ &\leq (1-\alpha_n)\|s_n - w^*\| + \alpha_n\|Ts_n - Tw^*\| \\ &\leq (1-\alpha_n)\|s_n - w^*\| + \alpha_n\theta\|s_n - w^*\| \\ &= (1-\alpha_n(1-\theta))\|s_n - w^*\|. \end{aligned} \tag{3.7}$$

From the second step of HR* iteration (1.5) and inequality (3.7), we obtain

$$\begin{aligned}\|y_n - w^*\| &= \|T((1-\beta_n)z_n + \beta_n T z_n) - Tw^*\| \\ &\le \theta\|(1-\beta_n)z_n + \beta_n T z_n - w^*\| \\ &= \theta[(1-\beta_n)\|z_n - w^*\| + \beta_n\|Tz_n - Tw^*\|] \\ &\le \theta[(1-\beta_n)\|z_n - w^*\| + \beta_n\theta\|z_n - w^*\|] \\ &\le \theta(1-\beta_n(1-\theta))\|z_n - w^*\| \\ &\le \theta(1-\alpha_n(1-\theta))(1-\beta_n(1-\theta))\|s_n - w^*\|. \end{aligned} \tag{3.8}$$

From the third step of HR* iteration (1.5) and inequality (3.8), we obtain

$$\begin{aligned}\|x_n - w^*\| &= \|T(Ty_n) - Tw^*\| \\ &\le \theta\|Ty_n - w^*\| \\ &= \theta\|Ty_n - Tw^*\| \\ &\le \theta^2\|y_n - w^*\| \\ &\le \theta^3(1-\alpha_n(1-\theta))(1-\beta_n(1-\theta))\|s_n - w^*\|. \end{aligned} \tag{3.9}$$

From the fourth step of HR* iteration (1.5) and inequality (3.9), we obtain

$$\begin{aligned}\|s_{n+1} - w^*\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n) - Tw^*\| \\ &\le (1-\gamma_n)\|x_n - w^*\| + \gamma_n\|Tx_n - Tw^*\| \\ &\le (1-\gamma_n)\|x_n - w^*\| + \gamma_n\theta\|x_n - w^*\| \\ &= (1-\gamma_n(1-\theta))\|x_n - w^*\| \\ &\le \theta^3(1-\alpha_n(1-\theta))(1-\beta_n(1-\theta))(1-\gamma_n(1-\theta))\|s_n - w^*\|. \end{aligned} \tag{3.10}$$

Using PMI, we have

$$\|s_{n+1} - w^*\| \le \theta^{3(n+1)}\|s_0 - w^*\|. \tag{3.11}$$

On dividing inequality (3.6) by inequality (3.11), we have

$$\frac{\|w_{n+1} - w^*\|}{\|s_{n+1} - w^*\|} \le \frac{\theta^{4(n+1)}\|w_0 - w^*\|}{\theta^{3(n+1)}\|s_0 - w^*\|} = \theta^{n+1}\frac{\|w_0 - w^*\|}{\|s_0 - w^*\|} \to 0 \text{ as } n \to \infty,$$

which implies that $\{w_n\}$ converges more rapidly than $\{s_n\}$. □

The first two steps of the HR iteration (1.4) and the HR* iteration (1.5) are identical. The difference arises in the third and fourth steps, where the placement of the convex combination is modified. This modification leads to a different error estimate under the ACM condition.

For the HR iteration, the third step contains the convex combination

$$x_n = (1-\gamma_n)y_n + \gamma_n T y_n.$$

Consequently,

$$\|x_n - w^*\| \le (1-\gamma_n(1-\theta))\|y_n - w^*\|.$$

Thus, the resulting estimate depends on both the contraction coefficient θ and the relaxation parameter γ_n. In the HR* iteration, the third step is replaced by

$$x_n = T(Ty_n).$$

Applying the contractive condition twice yields

$$\|x_n - w^*\| = \|T(Ty_n) - Tw^*\| \le \theta^2\|y_n - w^*\|.$$

Hence, the error estimate involves the factor θ^2. Consequently, an additional factor of θ appears in the estimate when compared with a single operator evaluation.

The proposed SAAM iteration further modifies the structure by beginning with the direct mapping

$$z_n = Tw_n.$$

Therefore,

$$\|z_n - w^*\| \leq \theta\|w_n - w^*\|.$$

Combining this estimate with the subsequent steps of the iteration gives

$$\|w_{n+1} - w^*\| \leq \theta^4(1 - \alpha_n(1-\theta))(1 - \beta_n(1-\theta))\|w_n - w^*\|.$$

Since

$$(1 - \alpha_n(1-\theta)) < 1, \qquad (1 - \beta_n(1-\theta)) < 1,$$

it follows that

$$\|w_{n+1} - w^*\| \leq \theta^{4(n+1)}\|w_0 - w^*\|.$$

Similarly, for the HR* iteration,

$$\|s_{n+1} - w^*\| \leq \theta^{3(n+1)}\|s_0 - w^*\|.$$

Consequently,

$$\lim_{n\to\infty} \frac{\|w_{n+1} - w^*\|}{\|s_{n+1} - w^*\|} \leq \lim_{n\to\infty} \theta^{n+1} \frac{\|w_0 - w^*\|}{\|s_0 - w^*\|} = 0,$$

because $0 < \theta < 1$. By Definition 2.6, the SAAM iteration converges faster than the HR* iteration.

Iteration Scheme	Analytical Error Bound
HR	$O(\theta^{3n})$
HR*	$O(\theta^{3n})$
SAAM	$O(\theta^{4n})$

Table 1. Comparison of dominant analytical error estimates.

These analytical findings are consistent with the MATLAB numerical experiments reported in Table 2, where the SAAM iteration reaches the fixed point $(0,0)$ in fewer iterations than the existing schemes. The additional direct operator evaluation in the SAAM iteration contributes an extra factor of θ in the error estimate, resulting in the bound $O(\theta^{4n})$.

Remark 3.3. Similarly, one can show that the SAAM iteration converges more rapidly than the HR [16], S** [31], Thakur [37] and JK [3] iterations.

Example 3.4. Let $U = \mathbb{R}^2$ and $C = \{u = (u_1, u_2) : (u_1, u_2) \in [0,5] \times [0,5]\}$ be a subset of U with norm $\|p\| = \|(u_1, u_2)\| = \max\{|u_1|, |u_2|\}$. Let $T : C \to C$ be defined by

$$T(u_1, u_2) = \left(\frac{u_1}{5}, \frac{u_2}{5}\right).$$

$(0,0)$ is the only fixed point of T. Now, we show that T is an ACM. If $u, v \in [0,5]$, then

$$\begin{aligned}
\|u - Tu\| &= \|(u_1, u_2) - \left(\frac{u_1}{5}, \frac{u_2}{5}\right)\| \\
&= \left\|\left(\frac{4u_1}{5}, \frac{4u_2}{5}\right)\right\| \\
&= \max\left\{\left|\frac{4u_1}{5}\right|, \left|\frac{4u_2}{5}\right|\right\} \\
&= \frac{4}{5}\max\{|u_1|, |u_2|\}.
\end{aligned}$$

Now, consider

$$\begin{aligned}\|Tu - Tv\| &= \left\|\left(\frac{u_1}{5}, \frac{u_2}{5}\right) - \left(\frac{v_1}{5}, \frac{v_2}{5}\right)\right\| \\ &= \left\|\left(\frac{u_1 - v_1}{5}, \frac{u_2 - v_2}{5}\right)\right\| \\ &= \max\left\{\left|\frac{u_1 - v_1}{5}\right|, \left|\frac{u_2 - v_2}{5}\right|\right\} \\ &= \frac{1}{5}\max\{|u_1 - v_1|, |u_2 - v_2|\} \\ &= \frac{1}{5}\|(u_1, u_2) - (v_1, v_2)\| \\ &= \frac{1}{5}\|u - v\| \\ &\leq \frac{1}{5}\|u - v\| + \frac{4}{5}\max\{|u_1|, |u_2|\}.\end{aligned}$$

Therefore, $\|Tu - Tv\| \leq \theta\|u - v\| + l\|u - Tu\|$, for $\theta = \frac{1}{5}$ and $l = 1$, that is, T is an ACM. Using MATLAB, we compare the numbers of iterations required by the SAAM and HR* iterations to converge to the fixed point. The comparison can be seen in Table 2. Clearly, the SAAM iteration converges faster.

Number of iterations	SAAM	HR*
1	(3.0, 3.5)	(3.0,3.5)
2	(0.000768,0.000896)	(0.009464888889,0.01104237037)
3	(0.000000196608,0.000000229376)	(0.000016355328,0.000019081216)
4	(0.000000000050331,0.000000000058720)	(0.000000018397519,0.000000021463773)
5	(0,0)	(0.000000000014957,0.000000000017451)
6	(0,0)	(0,0)

Table 2. The rate of convergence with $w_0 = (3.0, 3.5)$ and $\alpha_n = \beta_n = \gamma_n = 0.75$.

Theorem 3.5. *Let $T : C \to C$ be an ACM on a non-empty, convex and closed subset of a Banach space U. Suppose w^* is a fixed point of T and $\{w_n\}$ be a sequence generated by the SAAM iteration (3.1). Then SAAM iteration (3.1) is T-stable.*

Proof. Let $\{q_n\} \subset C$ be any sequence in C. Set $\epsilon_n = \|q_{n+1} - Tf_n\|$, where $f_n = T((1-t_n)g_n + t_nTg_n), g_n = T((1-s_n)h_n + s_nTh_n), h_n = Tq_n$. Assume that $\lim_{n\to\infty} \epsilon_n = 0$. From ACM (2.3), we have

$$\begin{aligned}\|h_n - w^*\| &= \|Tq_n - Tw^*\| \\ &\leq \theta\|q_n - w^*\|. \end{aligned} \tag{3.12}$$

From the second step of SAAM iteration (3.1) and inequality (3.12), we obtain

$$\begin{aligned}\|g_n - w^*\| &= \|T((1-s_n)h_n + s_nTh_n) - Tw^*\| \\ &\leq \theta\|(1-s_n)h_n + s_nTh_n - w^*\| \\ &\leq \theta[(1-s_n)\|h_n - w^*\| + s_n\|Th_n - Tw^*\|] \\ &\leq \theta(1 - s_n(1-\theta))\|h_n - w^*\| \\ &\leq \theta^2(1 - s_n(1-\theta))\|q_n - w^*\|. \end{aligned} \tag{3.13}$$

From the third step of SAAM iteration (3.1) and inequality (3.13), we obtain

$$\begin{aligned}\|f_n - w^*\| &= \|T((1-t_n)g_n + t_nTg_n) - Tw^*\| \\ &\leq \theta\|(1-t_n)g_n + t_nTg_n - w^*\| \\ &\leq \theta[(1-t_n)\|g_n - w^*\| + t_n\|Tg_n - Tw^*\|] \\ &\leq \theta(1-t_n(1-\theta))\|g_n - w^*\| \\ &\leq \theta^3(1-s_n(1-\theta))(1-t_n(1-\theta))\|q_n - w^*\|. \end{aligned} \tag{3.14}$$

Then, by triangle inequality, ACM (2.3) and the fourth step of SAAM iteration (3.1), we get

$$\begin{aligned}\|q_{n+1} - w^*\| &= \|q_{n+1} - Tf_n\| + \|Tf_n - w^*\| \\ &= \epsilon_n + \|Tf_n - Tw^*\| \\ &\leq \epsilon_n + \theta\|f_n - w^*\| \\ &\leq \epsilon_n + \theta^4(1-s_n(1-\theta))(1-t_n(1-\theta))\|q_n - w^*\|.\end{aligned}$$

Since $0 < \theta < 1$ and $s_n, t_n \in (0,1)$, we obtain

$$\theta^4(1-t_n(1-\delta))(1-s_n(1-\delta)) < 1.$$

From Lemma 2.9, we obtain

$$\lim_{n\to\infty}\|q_n - w^*\| = 0,$$

that is, $\{q_n\} \to w^*$. Conversely, let $\lim_{n\to\infty} q_n = w^*$. Then

$$\begin{aligned}\epsilon_n &= \|q_{n+1} - Tf_n\| \\ &= \|q_{n+1} - w^* + w^* - Tf_n\| \\ &\leq \|q_{n+1} - w^*\| + \|Tf_n - Tw^*\| \\ &\leq \|q_{n+1} - w^*\| + \theta\|f_n - w^*\| \\ &\leq \|q_{n+1} - w^*\| + \theta\|T((1-t_n)g_n + t_nTg_n) - w^*\| \\ &\leq \|q_{n+1} - w^*\| + \theta^2\|(1-t_n)g_n + t_nTg_n - w^*\| \\ &\leq \|q_{n+1} - w^*\| + \theta^2(1-t_n)\|g_n - w^*\| + \theta^2t_n\|Tg_n - Tw^*\| \\ &\leq \|q_{n+1} - w^*\| + \theta^2(1-t_n(1-\theta))\|g_n - w^*\| \\ &\leq \|q_{n+1} - w^*\| + \theta^2(1-t_n(1-\theta))\|T((1-s_n)h_n + s_nTh_n) - Tw^*\| \\ &\leq \|q_{n+1} - w^*\| + \theta^3(1-t_n(1-\theta))\|(1-s_n)h_n + s_nTh_n - w^*\| \\ &\leq \|q_{n+1} - w^*\| + \theta^3(1-t_n(1-\theta))[(1-s_n)\|h_n - w^*\| + s_n\|Th_n - Tw^*\|] \\ &\leq \|q_{n+1} - w^*\| + \theta^3(1-t_n(1-\theta))(1-s_n(1-\theta))\|h_n - w^*\| \\ &\leq \|q_{n+1} - w^*\| + \theta^3(1-t_n(1-\theta))(1-s_n(1-\theta))\|Tq_n - Tw^*\| \\ &\leq \|q_{n+1} - w^*\| + \theta^4(1-t_n(1-\theta))(1-s_n(1-\theta))\|q_n - w^*\|. \end{aligned} \tag{3.15}$$

Inequality (3.15) gives $\lim_{n\to\infty}\epsilon_n = 0$. □

Lemma 3.6. *[3] Assume that $T : C \to C$ is an RSTN mapping and $F(T) \neq \emptyset$, then for arbitrary point $v \in C$ and $w^* \in F(T)$, $\|Tv - Tw^*\| \leq \|v - w^*\|$.*

Theorem 3.7. *Let $T : C \to C$ be an RSTN mapping on a non-empty, convex and closed subset of a Banach space U and $F(T) \neq \emptyset$. For $w_0 \in C$, the sequence $\{w_n\}$ is initiated by the SAAM iteration (3.1). Then $\lim_{n\to\infty}\|w_n - w^*\|$ exists for $w^* \in F(T)$.*

Proof. Assume $w^* \in F(T)$. Using Lemma 3.6, we get

$$\begin{aligned} \|z_n - w^*\| &= \|Tw_n - w^*\| \\ &\leq \|w_n - w^*\|, \end{aligned} \tag{3.16}$$

$$\begin{aligned} \|y_n - w^*\| &= \|T((1-\alpha_n)z_n + \alpha_n Tz_n) - Tw^*\| \\ &\leq \|(1-\alpha_n)z_n + \alpha_n Tz_n - w^*\| \\ &\leq (1-\alpha_n)\|z_n - w^*\| + \alpha_n\|Tz_n - Tw^*\| \\ &\leq \|z_n - w^*\| \\ &\leq \|w_n - w^*\|, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \|x_n - w^*\| &= \|T((1-\beta_n)y_n + \beta_n Ty_n) - w^*\| \\ &\leq \|(1-\beta_n)y_n + \beta_n Ty_n - w^*\| \\ &\leq (1-\beta_n)\|y_n - w^*\| + \beta_n\|Ty_n - Tw^*\| \\ &\leq (1-\beta_n)\|y_n - w^*\| + \beta_n\|y_n - w^*\| \\ &\leq \|y_n - w^*\| \\ &\leq \|w_n - w^*\|, \end{aligned}$$

$$\begin{aligned} \|w_{n+1} - w^*\| &= \|Tx_n - w^*\| \\ &\leq \|x_n - w^*\| \\ &\leq \|w_n - w^*\|. \end{aligned}$$

This proves that $\{\|w_n - w^*\|\}$ is a bounded and decreasing sequence, that is, $\lim_{n\to\infty}\|w_n - w^*\|$ exists for each $w^* \in F(T)$. □

Lemma 3.8. *Let C be a subset of a uniformly convex Banach space U. Assume that T is an RSTN mapping on C. Then*

$$\|u - Tv\| \leq (1+\gamma)\|Tu - u\| + \|u - v\|,$$

holds for $u, v \in C$ such that $v \in Fix(T)$.

Proof. Consider

$$\begin{aligned} \|u - Tv\| &\leq \|u - Tu\| + \|Tu - Tv\| \\ &\leq \|Tu - u\| + \gamma\|Tu - u\| + \gamma\|Tv - v\| + (1-2\gamma)\|u - v\| \\ &= (1+\gamma)\|Tu - u\| + (1-2\gamma)\|u - v\| \\ &\leq (1+\gamma)\|Tu - u\| + \|u - v\|. \end{aligned}$$

□

Theorem 3.9. *Let $T : C \to C$ be an RSTN mapping on a non-empty, convex and closed subset of a uniformly convex Banach space U. Consider $w_0 \in C$ to be arbitrary and the sequence $\{w_n\}$ to be initiated by the SAAM iteration (3.1). Then $\lim_{n\to\infty}\|Tw_n - w_n\| = 0$ if and only if $F(T) \neq \emptyset$.*

Proof. Consider $F(T) \neq \emptyset$ and assume $w^* \in F(T)$. Utilizing Theorem 3.7, $\lim_{n\to\infty}\|w_n - w^*\|$ exists and $\{w_n\}$ is bounded. Let

$$\lim_{n\to\infty}\|w_n - w^*\| = c. \tag{3.18}$$

From inequalities (3.16) and (3.18), we obtain

$$\limsup_{n\to\infty}\|z_n - w^*\| \leq \limsup_{n\to\infty}\|w_n - w^*\| = c. \tag{3.19}$$

By Lemma 3.6, we obtain

$$\limsup_{n\to\infty}\|Tw_n - w^*\| \le \limsup_{n\to\infty}\|w_n - w^*\| = c. \tag{3.20}$$

On the other hand

$$\begin{aligned}\|w_{n+1} - w^*\| &= \|Tx_n - q\| \\ &\le \|x_n - q\| \\ &\le \|T((1-\beta_n)y_n + \beta_n Ty_n) - w^*\| \\ &\le (1-\beta_n)\|y_n - w^*\| + \beta_n\|Ty_n - w^*\| \\ &\le \|y_n - w^*\|.\end{aligned}$$

Using inequality (3.17), we obtain

$$\|w_{n+1} - w^*\| \le \|z_n - w^*\|.$$

Therefore,

$$c \le \liminf_{n\to\infty}\|z_n - w^*\|. \tag{3.21}$$

From inequalities (3.19) and (3.21), we obtain

$$\begin{aligned}c &= \lim_{n\to\infty}\|z_n - w^*\| \\ &= \lim_{n\to\infty}\|Tw_n - w^*\| \\ &\le \lim_{n\to\infty}\|(1-\alpha_n)(Tw_n - w^*) + \alpha_n(w_n - w^*)\|.\end{aligned} \tag{3.22}$$

From inequalities (3.18), (3.20), (3.22) and Lemma 2.10, $\lim_{n\to\infty}\|Tw_n - w_n\| = 0$.

Conversely, consider $\{w_n\}$ to be bounded and $\lim_{n\to\infty}\|Tw_n - w_n\| = 0$ and $w^* \in A(C,\{w_n\})$. From Lemma 3.8, we get

$$\begin{aligned}r(Tw^*,\{w_n\}) &= \limsup_{n\to\infty}\|w_n - Tw^*\| \\ &\le \limsup_{n\to\infty}[(1+\gamma)\|Tw_n - w_n\| + \|w_n - w^*\|] \\ &\le \limsup_{n\to\infty}\|w_n - w^*\| \\ &= r(w^*,\{w_n\}),\end{aligned}$$

that is, $Tw^* \in A(C,\{w_n\})$. Since $A(C,\{w_n\})$ is a singleton, U is uniformly convex. So, $Tw^* = w^*$, that is, $F(T) \ne \emptyset$. □

Theorem 3.10. *Let U be a uniformly convex Banach space and C be its non-empty, convex and closed subset. Suppose a self-mapping T on C is an RSTN mapping, $F(T) \ne \emptyset$ and $\{w_n\}$ is defined by the SAAM iteration (3.1). Then the sequence $\{w_n\}$ converges to some point of $F(T)$ if and only if $\lim_{n\to\infty}\inf d(w_n, F(T)) = 0$, where $d(w_n, F(T)) = \inf\{\|w_n - q\| : q \in F(T)\}$.*

Proof. It is very obvious that if $\{w_n\}$ is a convergent sequence that converges to a fixed point, then $\lim_{n\to\infty}\inf d(w_n, F(T)) = 0$.

Conversely, consider $\lim_{n\to\infty}\inf d(w_n, F(T)) = 0$. Using Theorem 3.7, $\lim_{n\to\infty}\|w_n - q\|$ exists for $q \in F(T)$.

For any given $\epsilon > 0$, we have $N \in \mathbb{N}$ so that for every $n > N$, $d(w_n, F(T)) < \frac{\epsilon}{2}$. To be specific $\inf\{\|w_N - q\| : q \in F(T)\} < \frac{\epsilon}{2}$, that is, we have some q^* satisfying $\|w_N - q^*\| < \frac{\epsilon}{2}$. So we get

$$\begin{aligned}\|w_{m+n} - w_n\| &\le \|w_{m+n} - q^*\| + \|w_n - q^*\| \\ &\le 2\|w_N - q^*\| \\ &< \epsilon,\ n > N \text{ and } m \ge 0,\end{aligned}$$

that is, $\{w_n\}$ is a Cauchy sequence in C. Using the closedness of C, there exists an element w^* in C, so that $\lim_{n\to\infty} w_n = w^*$. Also $\lim_{n\to\infty} \inf d(w_n, F(T)) = 0$ leads to $\lim_{n\to\infty} \inf d(w^*, F(T)) = 0$, i.e., $w^* \in F(T)$. □

Theorem 3.11. *Let $T : C \to C$ be an RSTN mapping on a non-empty, convex and compact subset of a uniformly convex Banach space U. Consider $\{w_n\}$ to be the SAAM iteration (3.1). Then $\{w_n\}$ converges strongly to a fixed point of T.*

Proof. Utilizing Theorem 3.9, we get $\lim_{n\to\infty} \|Tw_n - w_n\| = 0$. Using the compactness of C, we have a subsequence (w_{n_i}) of $\{w_n\}$ converging strongly to w^* in C. From Lemma 3.8, we obtain

$$\|w_{n_i} - Tw^*\| \leq (1+\gamma)\|Tw_{n_i} - w_{n_i}\| + \|w_{n_i} - w^*\|.$$

As $i \to \infty$, $Tw^* = w^*$, that is, $w^* \in F(T)$. Utilizing Theorem 3.7, $\lim_{n\to\infty} \|w_n - w^*\|$ exists for $w^* \in F(T)$. Hence, $\{w_n\}$ converges strongly to $w^* \in F(T)$. □

Theorem 3.12. *Let $T : C \to C$ be an RSTN mapping, where C is a non-empty, convex and closed subset of a uniformly convex Banach space U. Consider $\{w_n\}$ to be the SAAM iteration (3.1) and $F(T) \neq \emptyset$. If T satisfies condition (I), then the sequence $\{w_n\}$ converges to a point of $F(T)$.*

Proof. From Theorem 3.9, $\lim_{n\to\infty} \|Tw_n - w_n\| = 0$. Using condition (I), we have

$$0 \leq \lim_{n\to\infty} f(d(w_n, F(T))) \leq \lim_{n\to\infty} \|Tw_n - w_n\| = 0.$$

Thus, $\lim_{n\to\infty} f(d(w_n, F(T))) = 0$. As $f : (0,\infty) \to (0,\infty)$ is a nondecreasing function so that $f(0) = 0$ and for all $w > 0$, $f(w) > 0$, we get $\lim_{n\to\infty} d(w_n, F(T)) = 0$. As all the hypotheses of Theorem 3.10 have been satisfied, we conclude that $\{w_n\}$ converges to $w^* \in F(T)$. □

4 Numerical Experiment

First, we describe a mapping T which is an RSTN mapping but does not satisfy condition (C).

Example 4.1. Let $(\mathbb{R}, \|.\|)$ be a Banach space, endowed with the usual norm. Let $C = [0, 3]$. Let $T : C \to C$ be described as

$$Tw = \begin{cases} \frac{2w+14}{9} & \text{if } w \in [0, 3) \\ \frac{14}{9} & \text{if } w = 3. \end{cases}$$

Clearly, condition (C) does not hold for T because if $w = \frac{5}{2}$ and $v = 3$.

$$\frac{1}{2}\left|\frac{5}{2} - T\left(\frac{5}{2}\right)\right| = \frac{1}{2}\left|\frac{5}{2} - \frac{19}{9}\right| = \frac{1}{2}.\frac{7}{18} = \frac{7}{36} < 0.5 = |w - v|.$$

But

$$|Tw - Tv| = \left|\frac{19}{9} - \frac{14}{9}\right| = \frac{5}{9} > 0.5 = |w - v|.$$

Now we demonstrate that T is an RSTN mapping.

(i) If $w, v < 3$ and $\gamma = \frac{1}{2}$

$$|Tw - Tv| = \left|\frac{2w+14}{9} - \frac{2v+14}{9}\right| = \frac{2}{9}|w - v|,$$

and

$$\begin{aligned} \frac{1}{2}|w - Tw| + \frac{1}{2}|v - Tv| &= \frac{1}{2}\left|w - \frac{2w+14}{9}\right| + \frac{1}{2}\left|v - \frac{2v+14}{9}\right| \\ &= \frac{7}{18}|w+2| + \frac{7}{18}|v+2| \\ &\geq \frac{7}{18}|(w+2) - (v+2)| \\ &= \frac{7}{18}|w - v|. \end{aligned}$$

Therefore, T is an RSTN mapping.

(ii) If $w < 3$, $v = 3$ and $\gamma = \frac{1}{2}$

$$|Tw - Tv| = \left|\frac{2w+14}{9} - \frac{14}{9}\right| = \frac{2w}{9},$$

and

$$\begin{aligned}\frac{1}{2}|w - Tw| + \frac{1}{2}|v - Tv| &= \frac{1}{2}\left|w - \frac{2w+14}{9}\right| + \frac{1}{2}\left|3 - \frac{14}{9}\right| \\ &= \frac{7}{18}|w+2| + \frac{13}{18} \\ &= \frac{7w}{18} + \frac{14}{18} + \frac{13}{18} \\ &= \frac{7w}{18} + \frac{27}{18}.\end{aligned}$$

Therefore, T is an RSTN mapping.

(iii) If $w = 3$, $v < 3$ and $\gamma = \frac{1}{2}$, then T is an RSTN mapping just like in Case (ii).

(iv) If $w = v = 3$ and $\gamma = \frac{1}{2}$, we have

$$|Tw - Tv| = \left|\frac{14}{9} - \frac{14}{9}\right| = 0 < \frac{13}{9} = \frac{1}{2}|w - Tw| + \frac{1}{2}|v - Tv|.$$

for any $0 < \delta < 1$ and $L \geq 0$.

Hence, T is an RSTN mapping with a unique fixed point 2.

We now show numerically that our new SAAM iteration process (3.1) converges more quickly than each one of HR* [15], HR [16], S** [31], JK [3] and Thakur [37] iteration processes. We have compared the results for two different initial guesses and different values of parameters α_n, β_n and γ_n. Figures 1 and 3 are obtained corresponding to Tables 3 and 4, respectively.

New	HR*	S**	HR	JK	Thakur
3.0	3.0	3.0	3.0	3.0	3.0
2.001847388097	2.007235603381	2.008267389286	2.007235603381	2.037409608969	2.189236904943
2.00000272533	2.000037359736	2.000053887894	2.000037359736	2.001117555763	2.030630939346
2.000000003335	2.000000145707	2.000000286527	2.000000145707	2.000027689782	2.004294181167
2.000000000003	2.000000000448	2.000000001276	2.000000000448	2.000000585432	2.000528465129
2.0	2.000000000001	2.000000000005	2.000000000001	2.000000010797	2.0000578122
2.0	2.0	2.0	2.0	2.000000000177	2.00000568518
2.0	2.0	2.0	2.0	2.000000000003	2.000000507514
2.0	2.0	2.0	2.0	2.0	2.000000041481
2.0	2.0	2.0	2.0	2.0	2.000000003128
2.0	2.0	2.0	2.0	2.0	2.000000000219

Table 3. The rate of convergence with $w_0 = 3$ and $\alpha_n = \beta_n = \gamma_n = \frac{n}{n+5}$.

5 An Application to an Integral Equation

Let $F = C([0,a], \mathbb{R})$ be the set of real-valued continuous functions with domain $[0,a]$. Consider

$$u(t) = \int_0^a f(t,s)\phi(s,u(s))ds, \text{ where } t \in [0,a], \tag{5.1}$$

satisfying the following

(i) $\phi : [0,a] \times \mathbb{R} \to \mathbb{R}$ is continuous.

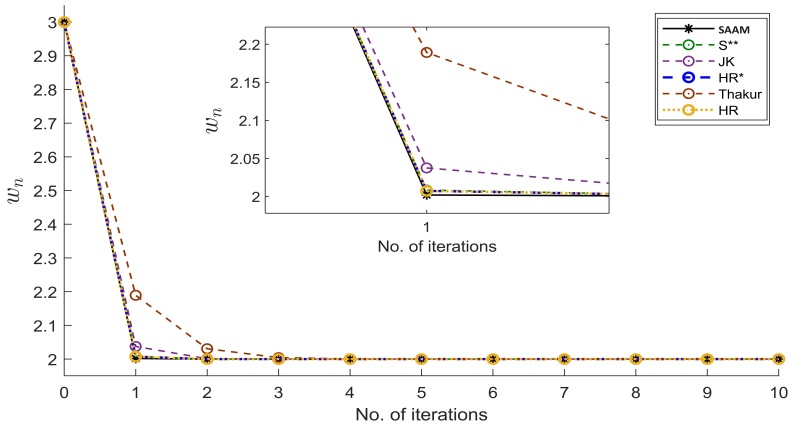


Figure 1. Convergence of SAAM, HR*, S**, HR and Thakur iterations with $w_0 = 3$ and $\alpha_n = \beta_n = \gamma_n = \frac{n}{n+5}$.

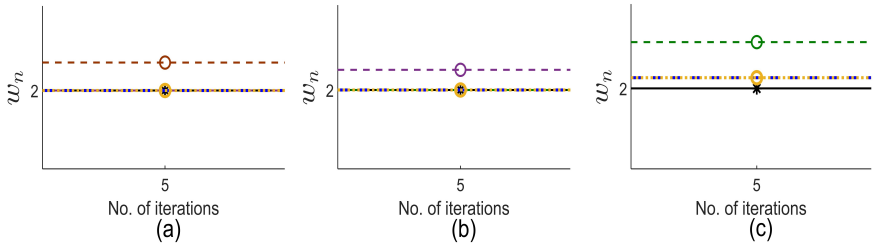


Figure 2. Zoomed Figure 1 at the fixed point.

New	HR*	S**	HR	JK	Thakur
0	0	0	0	0	0
1.996194347068	1.984872529594	1.982949956816	1.984872529594	1.922935528121	1.614277777778
1.999992758503	1.99988557982	1.999854648014	1.99988557982	1.997030533587	1.925609183642
1.999999986221	1.999999134556	1.999998760871	1.999999134556	1.99988557982	1.985652904501
1.999999999974	1.999999993454	1.999999989436	1.999999993454	1.999995591135	1.997233003221
2.0	1.99999999995	1.99999999991	1.99999999995	1.999999830117	1.999466353927
2.0	2.0	1.999999999999	2.0	1.999999993454	1.999897080425
2.0	2.0	2.0	2.0	1.999999999748	1.999980150816
2.0	2.0	2.0	2.0	1.99999999999	1.999996171864
2.0	2.0	2.0	2.0	2.0	1.999999261702
2.0	2.0	2.0	2.0	2.0	1.999999857611

Table 4. The rate of convergence with $w_0 = 0$ and $\alpha_n = \beta_n = \gamma_n = 0.15$.

(ii) $f : [0, a] \times [0, a] \to \mathbb{R}^2$ is continuous and measurable at $s \in [0, a]$, for all $t \in [0, a]$.

(iii) $f(t, s) \geq 0, t, s \in [0, a]$ and $\int_0^a f(t, s)ds \leq 1$.

Clearly, $F = C([0, a], \mathbb{R})$ is a Banach space and the maximum norm $\|u - v\|_\infty = \max_{t\in[0,a]} |u(t) - v(t)|$, for all $u, v \in C([0, a], \mathbb{R})$.

Theorem 5.1. *Let U be a non-empty, convex and closed subset of F and $T : U \to U$ be defined as*

$$Tu(t) = \int_0^a f(t, s)\phi(s, u(s))ds, \text{ where } t \in [0, a].$$

Suppose that conditions 1-3 hold and there exists $0 < \theta < 1$ so that

$$|\phi(t, u(t)) - \phi(t, v(t))| \leq \theta|u(t) - v(t)|,$$

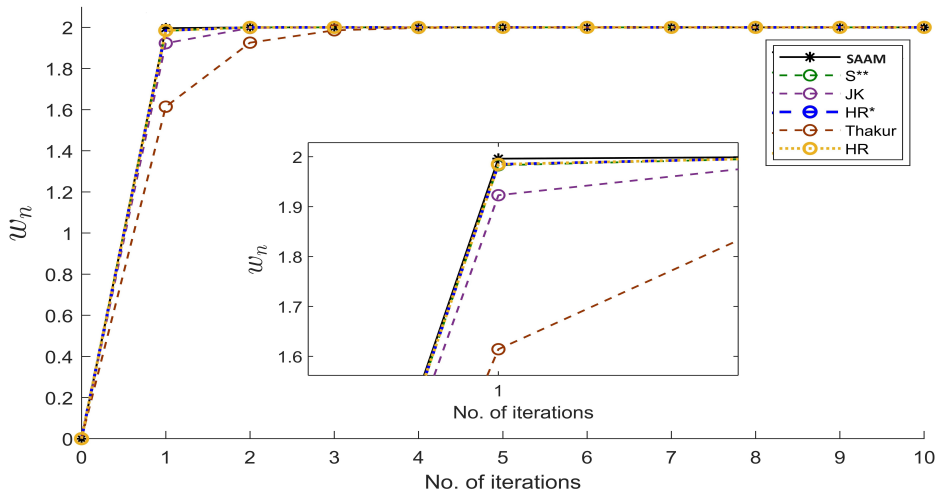


Figure 3. Convergence of SAAM, HR*, S**, HR and Thakur iterations with $w_0 = 0$ and $\alpha_n = \beta_n = \gamma_n = 0.15$.

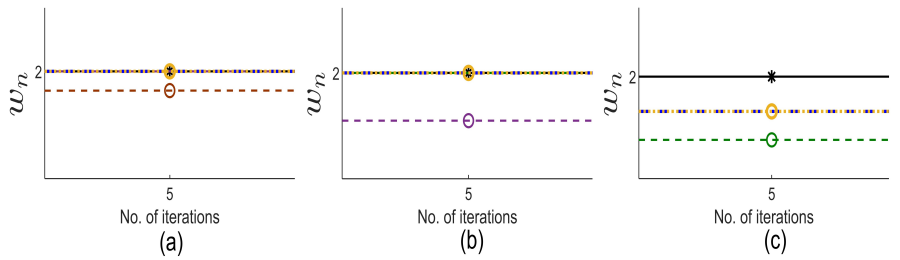


Figure 4. Zoomed Figure 3 at the fixed point.

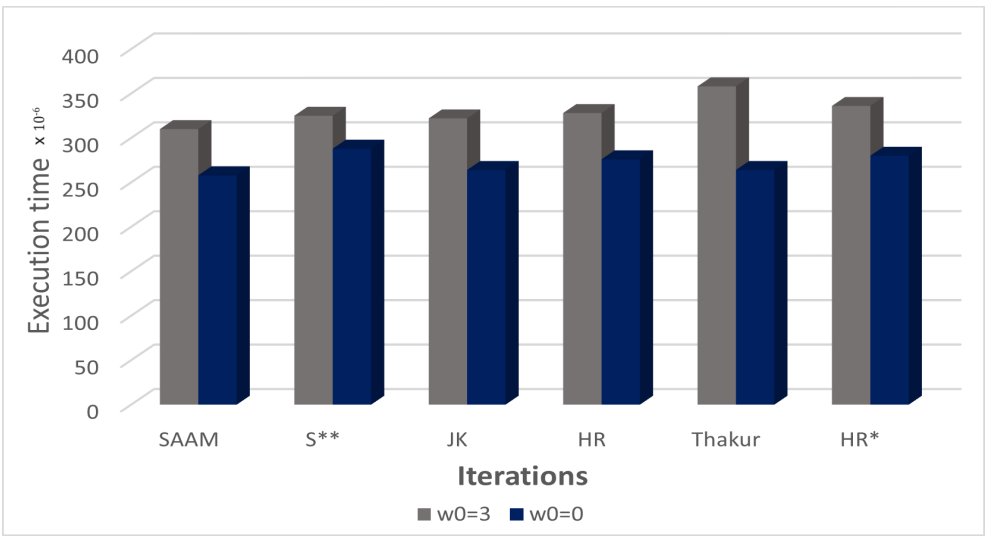


Figure 5. Time taken by the iterations to converge to the fixed point.

for every $t \in [0, a]$ *and* $u, v \in C([0, a], \mathbb{R})$. *Then, integral equation* (5.1) *has a unique solution in* $U \subseteq C([0, a], \mathbb{R})$ *provided that* T *has a unique fixed point.*

Proof. Consider $u, v \in F \times F$, then

$$\begin{aligned}
\|Tu - Tv\| &= \max_{t\in[0,a]} |Tu(t) - Tv(t)| \\
&\leq \max_{t\in[0,a]} \left| \int_0^a f(t,s)\phi(s,u(s))ds - \int_0^a f(t,s)\phi(s,v(s))ds \right| \\
&\leq \max_{t\in[0,a]} \int_0^a f(t,s)|\phi(s,u(s))ds - \phi(s,v(s))|ds \\
&\leq \max_{t\in[0,a]} \int_0^a f(t,s)\left[\theta|u(t) - v(t)|\right] ds \\
&\leq \max_{t\in[0,a]} \theta|u(t) - v(t)| \int_0^a f(t,s)ds \\
&\leq \theta\|u(t) - v(t)\| \\
&\leq \theta\|u - v\| + L\|u - Tu\|, \text{ for some } L \geq 0.
\end{aligned}$$

Therefore, T is an ACM, so by Theorem 3.1, the sequence described by the SAAM iteration (3.1) converges to a unique fixed point, which is also the solution of problem (5.1). □

6 Conclusion remarks

We have initiated a new SAAM iteration scheme designed to enhance accuracy and convergence speed in the approximation of fixed points and established weak and strong convergence results for ACM and RSTN via SAAM iteration. We demonstrated the fastest convergence speed and stability of the newly introduced iteration compared to existing methods, including the Thakur iteration [37], JK iteration [3], and HR* iteration [15], using theoretical calculations. An example is given to support our results. Additionally, we demonstrated the significance of SAAM iteration through a numerical example under changes in the initial guesses and an application to the solution of integral equations. A comparative analysis is presented with some earlier introduced iterative procedures along with the time estimate to approximate the fixed point.

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Received: 2025-05-03
Accepted: 2026-09-03