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Some New Sequence Spaces of Generalized Fractional Order

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Abstract In this study, we define some new sequence spaces by means of a generalized fractional difference operator, compute their dual spaces, and characterize some matrix transformations on these sequence spaces. Finally, we study the statistical convergence of generalized fractional difference sequences.

1 Introduction

Let w be the space of all real valued sequences. Each subspace of w will be called a sequence space. ℓ_∞, c and c_0 denote the spaces of bounded, convergent, and null sequences, respectively. These sequence spaces are normed spaces with the norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$. These spaces are also $BK-$spaces with these norm.

The study of difference sequence spaces is a recent development in the summability theory. The notion of difference sequence spaces was introduced by Kizmaz [22] as follows:

$$U(\Delta) = \{x = (x_k) : \Delta x \in U\},$$

for $U \in \{\ell_\infty, c, c_0\}$ and $\Delta x = (x_k - x_{k+1})$. These sequence spaces are normed spaces with the norm

$$\|x\|_\Delta = |x_1| + \sup |\Delta x_k|.$$

Later, Et and Colak [13] generalized the sequence space $U(\Delta)$ to the followig sequence space as follows:

$$U(\Delta^m) = \{x = (x_k) : \Delta^m x \in U\},$$

where $m \in \mathbb{N}$ and $\Delta^m x = (\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1})$ and $\Delta^m x_k = \sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k+i}$.

Recently, a large body of work has been carried out by many mathematicians on various generalizations of sequence spaces. For a detailed account of difference sequence spaces, see ([3],[5],[6],[11],[15],[18],[21], [25]).

Baliarsingh [4] gave the fractional difference sequences with the help of the gamma function as follows:

For a proper fraction α, the fractional difference operator $\Delta^\alpha : w \to w$ is defined as follows:

$$(\Delta^\alpha x)_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}$$

for every $k \in \mathbb{N}$, where $\Gamma(p)$ is the Gamma function of a real number p and $p \notin \{0, -1, -2, -3, ...\}$. The Gamma function is given by

$$\Gamma(p) = \int_0^\infty e^{-t}t^{p-1}dt.$$

It is observed that $\Gamma(k+1) = k!$ $(k \in \mathbb{N})$ and for any $k \in \mathbb{R} - \{0, -1, -2, -3, ...\}, \Gamma(k+1) = k\Gamma(k)$.

In 2020, Meng and Mei [23] generalized the fractional difference operator as follows:
For a proper fraction α, the fractional difference operator B^{α} is defined by

$$B^{\alpha}x_k = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} r^{\alpha-i}s^i x_{k+i}$$

for each $k \in \mathbb{N}$ and $r, s \in \mathbb{R} - \{0\}$, for example

$$B^{\frac{1}{2}}x_k = r^{\frac{1}{2}}x_k + \frac{1}{2}r^{-\frac{1}{2}}sx_{k+1} - \frac{1}{8}r^{-\frac{3}{2}}s^2x_{k+2} + \frac{1}{16}r^{-\frac{5}{2}}s^3x_{k+3} - \dots$$

$$B^{-\frac{1}{2}}x_k = r^{-\frac{1}{2}}x_k - \frac{1}{2}r^{-\frac{3}{2}}sx_{k+1} + \frac{3}{8}r^{-\frac{5}{2}}s^2x_{k+2} - \frac{5}{16}r^{-\frac{7}{2}}s^3x_{k+3}\dots$$

For $X, Y \subset w$, the operator $B^{\tilde{\alpha}} : X \to Y$ defined by

$$B^{\tilde{\alpha}}x = \left(B^{\tilde{\alpha}}x_k\right)$$

is linear. Moreover, for two proper fractions $\alpha, \beta > 0$, the following properties hold:
(i) $B^{\tilde{\alpha}}\left(B^{\tilde{\beta}}(x_k)\right) = B^{\widetilde{\alpha+\beta}}(x_k)$
(ii) $B^{\tilde{\alpha}}\left(B^{\widetilde{-\alpha}}(x_k)\right) = B^{\widetilde{-\alpha}}\left(B^{\tilde{\alpha}}(x_k)\right) = x_k$.

The main purpose of this work is to define the sequence space $U(B_v^{\tilde{\alpha}})$ for $U \in \{\ell_\infty, c, c_0\}$, investigate its topological properties, and determine the α-,β-,and γ-duals of the spaces $\ell_\infty(B_v^{\tilde{\alpha}})$, $c(B_v^{\tilde{\alpha}})$ and $c_0(B_v^{\tilde{\alpha}})$.

2 Main Results

Summability theory's main goal is to use transformations represented by special infinite matrices to give divergent sequences and series a limit. It is more natural and practical to examine matrix transformations rather than abstract linear operators as the most general linear mapping from one sequence space to another may be described by an infinite matrix. Therefore, we will examine the matrix $B^{\tilde{\alpha}}$.

We now start this section with a new definition.

Definition 2.1. Let $v = (v_n)$ be any sequence such that $v_n \neq 0$ for all $k \in \mathbb{N}$ and $B^{\tilde{\alpha}}x_n$ defined by

$$B^{\tilde{\alpha}}x_n = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} r^{\alpha-i}s^i x_{k+i}.$$

For a proper fraction α, we define the sequence spaces $\ell_\infty(B_v^{\tilde{\alpha}})$, $c(B_v^{\tilde{\alpha}})$ and $c_0(B_v^{\tilde{\alpha}})$ as follows:

$$\ell_\infty(B_v^{\tilde{\alpha}}) = \left\{ x = (x_k) \in w : \left(\sum_{j=0}^{k} v_j B^{\alpha}x_j \right) \in \ell_\infty \right\}$$

$$c(B_v^{\tilde{\alpha}}) = \left\{ x = (x_k) \in w : \left(\sum_{j=0}^{k} v_j B^{\alpha}x_j \right) \in c \right\}$$

$$c_0(B_v^{\tilde{\alpha}}) = \left\{ x = (x_k) \in w : \left(\sum_{j=0}^{k} v_j B^{\alpha}x_j \right) \in c_0 \right\}$$

Let V denote the set of all sequences $v = (v_n)$ and write $\frac{1}{v} = (\frac{1}{v_n})$ for $v \in V$, where $v_k = 0$ for $k < 0$ and

$$y_k = \sum_{j=0}^{k} v_j B^{\tilde{\alpha}}x_j = v_0 B^{\tilde{\alpha}}x_0 + v_1 B^{\tilde{\alpha}}x_1 + \ldots + v_k B^{\tilde{\alpha}}x_k$$

$$= r^{\alpha}v_0x_0 + (r^{\alpha}v_1 + \alpha r^{\alpha-1}sv_0)x_1 + \left(r^{\alpha}v_2 + \alpha r^{\alpha-1}sv_1 + \frac{\alpha(\alpha-1)}{2!}r^{\alpha-2}s^2v_0 \right)x_2 + \ldots$$

Thus, we can define the matrix $\tilde{B} = (B^{\tilde{\alpha}}_{nk})$ as follows:

$$B^{\tilde{\alpha}}_{nk} = \begin{cases} \sum_{i=0}^{\infty} \dfrac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} r^{\alpha-i} s^i v_{k-i} & , k \geq n \\ 0 & , k < n \end{cases}.$$

Clearly

$$B^{\tilde{\alpha}}_{nk} = \begin{bmatrix} r^{\alpha}v_k & 0 & 0 & 0 & \dots \\ \alpha r^{\alpha-1}sv_{k-1} & r^{\alpha}v_k & 0 & 0 & \dots \\ \dfrac{\alpha(\alpha-1)}{2!}r^{\alpha-2}s^2v_{k-2} & \alpha r^{\alpha-1}sv_{k-1} & r^{\alpha}v_k & 0 & \dots \\ \dfrac{\alpha(\alpha-1)(\alpha-2)}{3!}r^{\alpha-3}s^3v_{k-3} & \dfrac{\alpha(\alpha-1)}{2!}r^{\alpha-2}s^2v_{k-2} & \alpha r^{\alpha-1}sv_{k-1} & r^{\alpha}v_k & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem 2.2. *The space $U(B^{\tilde{\alpha}}_v)$ is a $BK-$space with the norm*

$$\|x\|_{U(B^{\tilde{\alpha}}_v)} = \sup_k \left| \sum_{j=0}^{k} v_j B^{\tilde{\alpha}} x_j \right|,$$

where $U \in \{\ell_\infty, c, c_0\}$.

Proof. Since the spaces ℓ_∞, c and c_0 are $BK-$ spaces and the matrix $\tilde{B}$ is triangle the space $U(B^{\tilde{\alpha}}_v)$ is also $BK-$space for $U \in \{\ell_\infty, c, c_0\}$ (see [29]). □

Theorem 2.3. *The space $U(B^{\tilde{\alpha}}_v)$ is linearly isomorphic to the space U, where $U \in \{\ell_\infty, c, c_0\}$.*

Proof. We will prove the theorem only for the space $c_0(B^{\tilde{\alpha}}_v)$. To prove the theorem, we must show that there is a linear bijection between $c_0(B^{\tilde{\alpha}}_v)$ and c_0. Consider the transformation defined $T : c_0(B^{\tilde{\alpha}}_v) \to c_0$ by $Tx \to y = Tx$ The linearity of T is obvious and $x = \theta = (0, 0, \dots)$, where $Tx = \theta$ hence T is injective. The following equation can be considered to show that T is surjective.

$$x_k = \frac{1}{v_k} \sum_{i=0} \frac{\Gamma(-\alpha+1) r^{-\alpha-i} s^i}{i!\Gamma(-\alpha-i+1)} (y_i - y_{i-1}).$$

Since $B^{\tilde{\alpha}}$ is linear, we may write

$$\sum_{j=0}^{k} v_j B^{\tilde{\alpha}} x_j = \sum_{j=0}^{k} v_j B^{\tilde{\alpha}} \left(\frac{1}{v_j} \sum_{i=0} (-1)^i \frac{\Gamma(-\alpha+1) r^{-\alpha-i} s^i}{i!\Gamma(-\alpha-i+1)} (y_i - y_{i-1}) \right)$$

$$= \sum_{j=0}^{k} v_j B^{\tilde{\alpha}} (\widetilde{B^{-\alpha}} (\frac{(y_i - y_{i-1})}{v_j}) = \sum_{j=0}^{k} (y_i - y_{i-1}).$$

Thus

$$\lim_{k\to\infty} \left(\sum_{j=0}^{k} v_j B^{\tilde{\alpha}} x_j \right) = \lim_{k\to\infty} y_k = 0.$$

Therefore $x \in c_0(B^{\tilde{\alpha}}_v)$ and so T is surjective and norm preserving. This completes the proof. □

3 Dual Spaces

In this section we compute the α-,β-, and γ-duals of the spaces $\ell_\infty(B^{\tilde{\alpha}}_v), c_0(B^{\tilde{\alpha}}_v)$ and $c(B^{\tilde{\alpha}}_v)$. Before we continue, we need review the definitions of the α-,β-, and γ-duals.

Definition 3.1. The α-,β-, and γ-duals of a sequence space X are defined by

$$X^{\alpha} = \{a = (a_k) \in w : ax = (a_k x_k) \in \ell_1, \quad \forall x \in X\},$$
$$X^{\beta} = \{a = (a_k) \in w : ax = (a_k x_k) \in cs, \quad \forall x \in X\},$$
$$X^{\gamma} = \{a = (a_k) \in w : ax = (a_k x_k) \in bs, \quad \forall x \in X\},$$

respectively.

The generalized duals of some sequence spaces were studied by Et [15], Chandra and Tripathy [9], Basar [5].

We now present the following results of Stieglitz and Tietz [28], which will be useful in the sequel. In what follows, we denote the collection of all finite subsets of $\mathbb{N}$ by F.

Lemma 3.2. *i)* $A = (a_{nk}) \in (\ell_\infty, \ell_1)$ *if and only if*

$$\sup_{K \in F} \sum_{n=0}^{\infty} \left| \sum_{k \in K} a_{nk} \right| < \infty. \tag{3.1}$$

ii) $A = (a_{nk}) \in (\ell_\infty, c)$ *if and only if*

$$\lim_{n \to \infty} \sum_{k} |a_{nk}| = \sum_{k} |\alpha_k| < \infty, \tag{3.2}$$

$$\lim_{n \to \infty} |a_{nk}| = \alpha_k \text{ for all } k. \tag{3.3}$$

iii) $A = (a_{nk}) \in (\ell_\infty, \ell_\infty)$ *if and only if*

$$\sup_{n \in \mathbb{N}} \sum_{k} |a_{nk}| < \infty. \tag{3.4}$$

Theorem 3.3. *Define the matrices* $B = (b_{nk})$ *and* $C = (c_{nk})$ *as follows:*

$$b_{nk} = \begin{cases} \sum_{i=0}^{\infty} \dfrac{(-1)^i}{v_i} \dfrac{\Gamma(1-\alpha) r^{-\alpha-i} s^i}{i! \Gamma(1-\alpha-i)} a_k & , k \geq n \\ 0 & , k < n \end{cases}$$

and

$$c_{nk} = \begin{cases} \sum_{i=0}^{\infty} \dfrac{(-1)^i}{v_i} \dfrac{\Gamma(1-\alpha)}{i! \Gamma(1-\alpha-i)} \sum_{j=n}^{k} \dfrac{a_j}{v_j} & , k \geq n \\ 0 & , k < n \end{cases}$$

and set

$$\check{D}_1 = \left\{ a = (a_k) \in w : \sup_{K \in F} \sum_{n} \left| \sum_{k \in K} \frac{1}{v_k} \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(1-\alpha) r^{-\alpha-i} s^i}{i! \Gamma(1-\alpha-i)} a_n \right| < \infty \right\}.$$

$$\check{D}_2 = \left\{ a = (a_k) \in w : \sum_{n} \sum_{k} \left| \sum_{i=0}^{\infty} \left[\frac{1}{v_k} (-1)^i \frac{\Gamma(1-\alpha) r^{-\alpha-i} s^i}{i! \Gamma(1-\alpha-i)} a_n \right] \right| < \infty \right\},$$

$$\check{D}_3 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k} |c_{nk}| = \sum_{k} \left| \lim_{n \to \infty} c_{nk} \right| \right\},$$

$$\check{D}_4 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} (c_{nk} - \alpha_k) = 0 \text{ for all } k \right\},$$

$$\check{D}_5 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \left(\sum_{k} c_{nk} - \alpha_k' \right) = 0 \right\},$$

$$\check{D}_6 = \left\{ a = (a_k) \in w : \sup_{n} \sum_{k} |c_{nk}| < \infty \right\},$$

then

a) $[\ell_\infty(B_v^{\tilde{\alpha}})]^\alpha = \check{D}_1$

b) $[\ell_\infty(B_v^{\tilde{\alpha}})]^\beta = \check{D}_3 \cap \check{D}_4$

c) $[\ell_\infty(B_v^{\tilde{\alpha}})]^\gamma = \check{D}_6$

d) $[c_0(B_v^{\tilde{\alpha}})]^\alpha = \check{D}_1$

e) $[c_0(B_v^{\tilde{\alpha}})]^\beta = \check{D}_4 \cap \check{D}_6$

f) $[c_0(B_v^{\tilde{\alpha}})]^\gamma = \check{D}_6 \cap \check{D}_4$

g) $[c(B_v^{\tilde{\alpha}})]^\alpha = \check{D}_1 \cap \check{D}_2$

h) $[c(B_v^{\tilde{\alpha}})]^\beta = \check{D}_4 \cap \check{D}_5 \cap \check{D}_6$

i) $[c(B_v^{\tilde{\alpha}})]^\gamma = \check{D}_6.$

4 Matrix Transformations

In this section, we derive necessary and sufficient criteria for matrix mappings from any one of the spaces $\ell_\infty(B_v^{\tilde{\alpha}})$, $c_0(B_v^{\tilde{\alpha}})$, and $c(B_v^{\tilde{\alpha}})$ to any one of the spaces $\ell_\infty, c, c_0, \ell_1$. The following theorem, which is derived directly from Kirişci and Başar [21], is crucial to our research.

Theorem 4.1. *Let U be any one of the spaces ℓ_∞, c , c_0 and $Y \subset w$. Define $W^{(n)} = \left(w_{pk}^{(n)}\right)$ and $W = (w_{nk})$ by*

$$w_{pk}^{(n)} = \begin{cases} \sum_{t=k}^{p} \dfrac{(-1)^t}{v_t} \dfrac{\Gamma(1-\alpha) r^{-\alpha-t} s^t}{t!\Gamma(1-\alpha-t)} a_{nt} & , 0 \le k \le p \\ 0 & , p > n \end{cases}$$

and

$$w_{nk} = \sum_{t=k}^{p} \frac{(-1)^t}{v_t} \frac{\Gamma(1-\alpha) r^{-\alpha-t} s^t}{t!\Gamma(1-\alpha-t)} a_{nt}$$

$\forall n, k \in \mathbb{N}^0$, respectively. Then $A = (a_{nk}) \in (U(B_v^{\tilde{\alpha}}, Y)$ if and only if $W^{(n)} = \left(w_{pk}^{(n)}\right) \in (U, c)$ for all $n \in \mathbb{N}^0$ and $W = w_{nk} \in (U, Y)$.

Using the matrix mapping characterization findings from Stieglitz and Tietz [28] in conjunction with Theorem 4.1, we derive the following results:

Corollary 4.2. *The following propositions are true:*

(i) $A \in (c_0(B_v^{\tilde{\alpha}}, \ell_\infty)$ *if and only if*

$$\sup_{p \in \mathbb{N}^0} \sum_{k=0}^{\infty} |w_{pk}^{(n)}| < \infty, \tag{4.1}$$

$$\lim_{p \to \infty} w_{pk}^{(n)} \text{ exists, for all } k \in \mathbb{N}^0 \tag{4.2}$$

hold and

$$\sup_{n \in \mathbb{N}^0} \sum_{k=0}^{\infty} |w_{nk}| < \infty \tag{4.3}$$

also holds.

(ii) $A \in (c_0(B_v^{\tilde{\alpha}}, c)$ *if and only if (4.1), (4.2) and (4.3) holds, and*

$$\lim_{n \to \infty} w_{nk} \text{ exists, for all } k \in \mathbb{N}^0 \tag{4.4}$$

also holds.

(iii) $A \in (c_0(B_v^{\tilde{\alpha}}, c_0)$ *if and only if (4.1), (4.2) and (4.3) holds, and*

$$\lim_{n\to\infty} w_{nk} = 0, \text{ for all } k \in \mathbb{N}^0 \tag{4.5}$$

also holds.

(iv) $A \in (c_0(B_v^{\tilde{\alpha}}, \ell_1)$ *if and only if (4.1) and (4.2) holds, and*

$$\sup_{N\in F} \sum_{k=0}^{\infty} \left| \sum_{n\in N} w_{nk} \right| < \infty, \tag{4.6}$$

also holds.

Corollary 4.3. *The following propositions are true:*

(i) $A \in (c(B_v^{\tilde{\alpha}}, \ell_\infty)$ *if and only if (4.1), (4.2) and (4.3) holds, and*

$$\lim_{p\to\infty} \sum_{k=0}^{\infty} |w_{pk}^{(n)}| \quad \text{exists for each } n \tag{4.7}$$

also holds.

(ii) $A \in (c(B_v^{\tilde{\alpha}}, c)$ *if and only if (4.1), (4.2), (4.3) ,(4.4) and (11) holds, and*

$$\lim_{p\to\infty} \sum_{k=0}^{n} w_{nk} \text{ exists}$$

also holds.

(iii) $A \in (c(B_v^{\tilde{\alpha}}, c_0)$ *if and only if (4.1), (4.2), (4.3), (4.5) and (4.7) holds, and*

$$\lim_{p\to\infty} \sum_{k=0}^{n} w_{nk} = 0$$

also holds.

(iv) $A \in (c(B_v^{\tilde{\alpha}}, \ell_1)$ *if and only if (4.1), (4.2), (4.6) and (4.7) hold.*

Corollary 4.4. *The following propositions are true:*

(i) $A \in (\ell_\infty(B_v^{\tilde{\alpha}}, \ell_\infty)$ *if and only if*

$$\lim_{p\to\infty} \sum_{t=k}^{p} \frac{(-1)^t}{v_t} \frac{\Gamma(1-\alpha) r^{-\alpha-t} s^t}{t!\Gamma(1-\alpha-t)} a_{nt} = w_{nk}, \quad \text{for all } n, k \in \mathbb{N}^+, \tag{4.8}$$

$$\lim_{p\to\infty} \sum_{k=1}^{p} \left| \sum_{t=k}^{p} \frac{(-1)^t}{v_t} \frac{\Gamma(1-\alpha) r^{-\alpha-t} s^t}{t!\Gamma(1-\alpha-t)} a_{nt} \right| = \sum_k |w_{nk}|, \quad \text{for all } n \in \mathbb{N}^+, \tag{4.9}$$

$$\sup_{m\in\mathbb{N}} \sum_k |w_{nk}| < \infty, \tag{4.10}$$

also holds

(ii) $A \in (\ell_\infty(B_v^{\tilde{\alpha}}, c)$ *if and only if (4.8), (4.9) and*

$$\lim_{n\to\infty} w_{nk} = \tilde{\alpha}_k; \quad k \in \mathbb{N}, \tag{4.11}$$

$$\lim_{n\to\infty} \sum_k |w_{nk}| = \sum_k |\tilde{\alpha}_k|, \tag{4.12}$$

also holds.

(iii) $A \in (\ell_\infty(B_v^{\tilde{\alpha}}, c_0)$ *if and only if (4.8), (4.9) and*

$$\lim_{n\to\infty} \sum_{k=0}^{n} w_{nk} = 0$$

also holds.

(iv) $A \in (\ell_\infty(B_v^{\tilde{\alpha}}, \ell_1)$ *if and only if (4.8), (4.9) and*

$$\sup_{N,K\in F} \left| \sum_{n\in N} \sum_{k\in K} w_{nk} \right| < \infty$$

also holds.

5 Results on Statistical Convergence

The concept of statistical convergence was introduced by Steinhaus [27] and Fast [16], then reintroduced independently by Schoenberg [26], and the notion was associated with summability theory by Braha et al. [8], Connor [12], Fridy [17] , Cinar et al. ([10],[11]), Et et al. [14], Belen et al. [7], Mohiuddine et al. [24], Atabey et al. [2], Güngör et al. ([18],[19]), Işık and Akbas ([1],[20]) and many others.

In this section, we give the definition of $B^{\tilde{\alpha}}$-statistical convergence with the fractional order α.

Definition 5.1. Let $x = (x_k)$ be a sequence of real or complex numbers. A sequence $x = (x_k)$ is said $B^{\tilde{\alpha}}-$statistical convergence to L, if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |B^{\tilde{\alpha}} x_k - L| \ge \varepsilon\}| = 0.$$

The set of all $B^{\tilde{\alpha}}-$statistically convergent sequences will be denoted by $B^{\tilde{\alpha}}(S)$. In this case we write $x_k \to L(B^{\tilde{\alpha}}(S))$.

The concept of $B^{\tilde{\alpha}}$-statistical convergence, which we defined in this section, is different from other types of convergence. For this let us take a look at the following examples

Example 5.2. Let us consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1 & \text{if } k = n^2, \quad n = 0, 1, 2, ... \\ 0 & \text{if } k \neq n^2, \quad n = 0, 1, 2, ... \end{cases}$$

This sequence is not convergent but statistically convergent to zero. If the generalized fractional difference operator is applied to this sequence, it can be seen that it is not $B^{\tilde{\alpha}}-$ statistically convergent.

Example 5.3. Let $x = (x_k)$ be a sequence and set $y_k = (B^{\tilde{\alpha}} x_k)$. Let's define the sequence $y = (y_k)$ as follow:

$$y_k = \begin{cases} 1 & \text{if } k = n^3, \quad n = 0, 1, 2, ... \\ 0 & \text{if } k \neq n^3, \quad n = 0, 1, 2, ... \end{cases}$$

Then $y = (y_k)$ is statistically convergent but it is not convergence.

Definition 5.4. Let α a proper fraction and $x = (x_k)$ be a sequence of real or complex numbers. If

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |B^{\tilde{\alpha}} x_k - L| = 0$$

for some L, then $x = (x_k)$ is said to be strongly $B^{\tilde{\alpha}}$-Cesaro summable to L. We denote the set of all strongly $B^{\tilde{\alpha}}$-Cesaro summable sequences by $B^{\tilde{\alpha}}(w_p)$. In this case we write $x_k \to L(B^{\tilde{\alpha}}(w_p))$.

Theorem 5.5. *Let α a proper fraction and $x = (x_k)$ be a sequence of real and complex numbers. If $x_k \to L(B^{\tilde{\alpha}}(w_p))$ then $x_k \to L(B^{\tilde{\alpha}}(S))$.*

Proof. Let $x = (x_k) \to L(B^{\tilde{\alpha}}(w_p))$ and $\varepsilon > 0$. We can write

$$\sum_{k=1}^{n} |B^{\tilde{\alpha}} x_k - L| \geq |\{k \leq n : |B^{\tilde{\alpha}} x_k - L| \geq \varepsilon\}|.\varepsilon$$

If the limit is taken for $n \to \infty$, then we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |B^{\tilde{\alpha}} x_k - L| \geq \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |B^{\tilde{\alpha}} x_k - L| \geq \varepsilon\}| \varepsilon = 0.$$

This shows that $x \to L(B^{\tilde{\alpha}}(S))$. □

Theorem 5.6. *Let α a proper fraction and $x = (x_k)$ be a sequence of real or complex numbers. If $(x_k) \in B^{\tilde{\alpha}}(\ell_\infty)$ and $x_k \to L(B^{\tilde{\alpha}}(S))$, then $x_k \to L(B^{\tilde{\alpha}}(w_p))$.*

Proof. Since $x \in B^{\tilde{\alpha}}(\ell_\infty)$ and $x_k \to L(B^{\tilde{\alpha}}(S))$ we write $B^{\tilde{\alpha}}(x_k) \in \ell_\infty$ and

$$\frac{1}{n} |\{k \leq n : |B^{\tilde{\alpha}}(x_k) - L| \geq \varepsilon\}| \to 0, \quad n \to \infty$$

Let $\varepsilon > 0$

$$\frac{1}{n} |\{k \leq n : |B^{\tilde{\alpha}}(x_k) - L| \geq \varepsilon\}| \leq \frac{\varepsilon}{\|B^{\tilde{\alpha}} x\|_\infty + L}$$

where $\sum_{i=1}^{\infty} |x_i| = L$ for all $n > n_0$. Since $x = (x_k) \in B^{\tilde{\alpha}}(\ell_\infty)$ we can write

$$|B^{\tilde{\alpha}}(x_k) - L| \leq sup|B^{\tilde{\alpha}}(x_k) - L| \leq \left\|B^{\tilde{\alpha}}(x_k) - L\right\|_{B^{\tilde{\alpha}}} \leq L + \left\|B^{\tilde{\alpha}}(x_k)\right\|_\infty = \|x\|_{B^{\tilde{\alpha}}}.$$

Therefore

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} |B^{\tilde{\alpha}}(x_k) - L| &= \frac{1}{n} \left(\sum_{\substack{k=1 \\ |B^{\tilde{\alpha}}(x_k) - L| \geq \varepsilon}}^{n} |B^{\tilde{\alpha}}(x_k) - L| + \sum_{\substack{k=1 \\ |B^{\tilde{\alpha}}(x_k) - L| < \varepsilon}}^{n} |B^{\tilde{\alpha}}(x_k) - L| \right) \\ &< \frac{1}{n}(n\varepsilon + n\varepsilon) = 2\varepsilon. \end{aligned}$$

Hence $x_k \to L(B^{\tilde{\alpha}}(w_p))$. □

6 Conclusion

In this paper, we introduced several new sequence spaces generated by a generalized fractional difference operator. These spaces extend certain classical sequence spaces and provide a broader framework for studying the behavior of sequences defined by fractional order operators. After defining these spaces, we investigated their fundamental properties and obtained representations for their dual spaces. In addition, we characterized the matrix transformations between these newly defined spaces and some classical sequence spaces.

Furthermore, we examined the statistical convergence of sequences associated with the generalized fractional difference operator. The obtained results reveal that the proposed sequence spaces offer a suitable setting for studying various convergence behaviors and operator-theoretic properties. Therefore, the approach developed in this paper contributes to the ongoing research on fractional difference operators and sequence space theory.

It is expected that the results obtained in this work may stimulate further research in this direction. In particular, the newly defined sequence spaces may be used to investigate different summability methods, compact operators, and various types of convergence in future studies.

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Declarations

The authors state that their interests do not conflict with each other.

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