Palestine Journal of Mathematics

Vol.15(3), (2026) , 1100–1118 © Palestine Polytechnic University-PPU 2026

Labeled Class Graphs Determined by Regularly Normal Subpolygroups

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Communicated by Madeleine Al Tahan

Abstract. Let H be a polygroup and let N be a regularly normal subpolygroup of H. The N-class graph $G_N(H)$ has vertex set H, with distinct vertices adjacent when they represent different elements of H/N. We prove that the assignments

$$N \longmapsto \{[x]_N : x \in H\} \longmapsto G_N(H)$$

identify regularly normal subpolygroups with their quotient partitions and with the corresponding labeled complete multipartite graphs. Under these identifications, inclusion of subpolygroups is refinement of partitions and reverse containment of edge sets. We determine the behavior of the construction under strong homomorphisms, count the edges lost under a quotient refinement, and classify the resulting graphs up to isomorphism. The principal graph invariants, planarity, and adjacency and Laplacian spectra are then obtained from the quotient-class sizes. A multivalued example shows that the construction is not confined to ordinary groups.

2020 Mathematics Subject Classification. Primary: 05C25; Secondary: 20N20, 05C50.

Keywords and phrases: polygroup, regularly normal subpolygroup, quotient polygroup, complete multipartite graph, strong homomorphism, adjacency spectrum.

1 Introduction

Let H be a polygroup. A regularly normal subpolygroup N determines more than a quotient set: it determines a canonical partition of H into the fibers of

$$\pi_N : H \longrightarrow H/N, \qquad x \longmapsto [x]_N.$$

The purpose of this paper is to represent this partition by a graph without losing its order-theoretic relation to N. For distinct $x, y \in H$, put

$$\{x, y\} \in E(G_N(H)) \iff [x]_N \neq [y]_N \iff (x^{-1} \circ y) \cap N = \varnothing.$$

Thus nonadjacency is equality in the quotient, and the maximal independent sets of $G_N(H)$ are precisely the fibers of π_N.

The central point is not merely that $G_N(H)$ is complete multipartite. With the vertex set H retained, the three objects

$$N, \qquad \mathcal{C}_N := \{[x]_N : x \in H\}, \qquad G_N(H)$$

determine one another. Moreover, for regularly normal subpolygroups N_1, N_2,

$$N_1 \subseteq N_2 \iff \mathcal{C}_{N_1} \preceq \mathcal{C}_{N_2} \iff E(G_{N_2}(H)) \subseteq E(G_{N_1}(H)),$$

where $\preceq$ denotes refinement of partitions. This gives an order representation of the regularly normal subpolygroup poset by labeled graphs. We also identify the edges deleted when several N_1-classes merge into one N_2-class.

Strong homomorphisms supply the natural morphisms for this construction. If $f : H \to K$ is strong and M is regularly normal in K, then $f^{-1}(M)$ is regularly normal in H, and the induced map

$$H/f^{-1}(M) \longrightarrow K/M$$

preserves the quotient classes and the associated adjacency relation. In the surjective case, the partite sets are exactly the fibers of f. These results isolate the polygroup-theoretic content of the construction from the graph invariants that depend only on the class sizes.

Multivalued algebraic systems originate with Marty [1]; foundational developments and standard references include [2, 3, 5, 6, 7]. Polygroups and their quotient theory are treated in [4, 8, 9, 10, 11, 27, 28]. Related algebraic graph constructions appear in [12, 13, 14, 19, 20, 21, 18, 22, 23, 24]. We use the graph terminology of [15, 16].

Section 2 recalls the required polygroup and graph terminology. Section 3 establishes the quotient-partition representation and its functorial properties. Section 4 studies refinement, edge loss, and the numerical invariants determined by the class sizes. Section 5 treats planarity and the adjacency and Laplacian spectra.

2 Preliminaries

This section fixes the algebraic and graph-theoretic notation used in the paper. For general material on polygroups and hyperstructures, we refer to [4, 5, 6, 7, 27, 28]. The graph terminology is standard and follows [15, 16].

Let X be a nonempty set, and let $\mathcal{P}^*(X)$ denote the collection of its nonempty subsets. A hyperoperation on X is a map

$$\circ : X \times X \longrightarrow \mathcal{P}^*(X).$$

For nonempty subsets $A, B \subseteq X$, the operation is extended by

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b.$$

We use the abbreviations

$$x \circ A = \{x\} \circ A \qquad \text{and} \qquad A \circ x = A \circ \{x\}.$$

If $f : X \to Y$ and $A \subseteq X$, then

$$f(A) = \{f(a) : a \in A\}.$$

Definition 2.1. A *polygroup* is a system

$$(H, \circ, e, {}^{-1})$$

consisting of a nonempty set H, a hyperoperation $\circ$, an element $e \in H$, and a unary operation $x \mapsto x^{-1}$, such that the following conditions hold for all $x, y, z \in H$:

(i)
$$(x \circ y) \circ z = x \circ (y \circ z);$$

(ii)
$$e \circ x = x \circ e = \{x\};$$

(iii) if $z \in x \circ y$, then
$$x \in z \circ y^{-1} \qquad \text{and} \qquad y \in x^{-1} \circ z.$$

The last condition is the reversibility property. It gives, among other standard identities,

$$e \in (x \circ x^{-1}) \cap (x^{-1} \circ x),$$

$$e^{-1} = e, \qquad (x^{-1})^{-1} = x,$$

and

$$(x \circ y)^{-1} = y^{-1} \circ x^{-1}.$$

An ordinary group is a polygroup under the singleton-valued hyperoperation

$$x \circ y = \{xy\}.$$

Definition 2.2. A nonempty subset $N \subseteq H$ is a *subpolygroup* of H if

$$N \circ N \subseteq N \qquad \text{and} \qquad N^{-1} = N,$$

where

$$N^{-1} = \{x^{-1} : x \in N\}.$$

The identity e belongs to every nonempty subpolygroup. Indeed, if $x \in N$, then $x^{-1} \in N$ and

$$e \in x \circ x^{-1} \subseteq N.$$

Definition 2.3. A subpolygroup N of H is *normal* if

$$x^{-1} \circ N \circ x \subseteq N$$

for every $x \in H$.

The weaker condition needed for the quotient construction is regular normality.

Definition 2.4. A subpolygroup N of H is *regularly normal* if

$$x \circ N = N \circ x$$

for every $x \in H$.

Every normal subpolygroup is regularly normal, but the converse need not hold; see [28]. We use regular normality because it is preserved by kernels of strong homomorphisms and ensures that the quotient hyperoperation is well defined.

For an arbitrary subpolygroup N, define a relation on H by

$$x \sim_N y \quad \Longleftrightarrow \quad (x^{-1} \circ y) \cap N \neq \varnothing.$$

Proposition 2.5. *The relation $\sim_N$ is an equivalence relation on H.*

Proof. Reflexivity. Since $e \in x^{-1} \circ x$ and $e \in N$, we get $e \in (x^{-1} \circ x) \cap N$, so $x \sim_N x$.

Symmetry. Suppose $x \sim_N y$, and fix

$$a \in (x^{-1} \circ y) \cap N.$$

Since $N^{-1} = N$, taking inverses gives

$$a^{-1} \in (y^{-1} \circ x) \cap N,$$

so $y \sim_N x$.

Transitivity. Suppose $x \sim_N y$ and $y \sim_N z$, and fix

$$a \in (x^{-1} \circ y) \cap N, \qquad b \in (y^{-1} \circ z) \cap N.$$

By reversibility, $y \in x \circ a$ and $z \in y \circ b$, so

$$z \in (x \circ a) \circ b = x \circ (a \circ b).$$

Hence there exists $c \in a \circ b$ with $z \in x \circ c$, and reversibility gives $c \in x^{-1} \circ z$. Since $a, b \in N$ and $N \circ N \subseteq N$, also $c \in N$. Thus

$$c \in (x^{-1} \circ z) \cap N \neq \varnothing,$$

so $x \sim_N z$. □

The equivalence class containing x will be denoted by

$$[x]_N = \{y \in H : x \sim_N y\}.$$

If N is regularly normal, then

$$[x]_N = N \circ x = x \circ N.$$

In particular,

$$[e]_N = N.$$

The set of all classes is written

$$H/N = \{[x]_N : x \in H\}.$$

For regularly normal N, define

$$[x]_N \odot [y]_N = \{[z]_N : z \in x \circ y\}.$$

Proposition 2.6. *If N is regularly normal in H, then the preceding hyperoperation is well defined and*

$$(H/N, \odot, [e]_N, {}^{-1})$$

is a polygroup, where

$$[x]_N^{-1} = [x^{-1}]_N.$$

Proof. This is the quotient construction for regularly normal subpolygroups; see [28, Proposition 2.4]. □

We next recall the homomorphisms used later.

Definition 2.7. Let

$$(H_1, \circ_1, e_1, {}^{-1}) \qquad \text{and} \qquad (H_2, \circ_2, e_2, {}^{-1})$$

be polygroups. A map

$$\varphi : H_1 \longrightarrow H_2$$

is a *strong homomorphism* if

$$\varphi(e_1) = e_2$$

and

$$\varphi(x \circ_1 y) = \varphi(x) \circ_2 \varphi(y)$$

for every $x, y \in H_1$.

The kernel of φ is

$$\ker \varphi = \{x \in H_1 : \varphi(x) = e_2\},$$

and its image is

$$\operatorname{Im} \varphi = \{\varphi(x) : x \in H_1\}.$$

The following facts will be used in Section 3.

Proposition 2.8. *Let $\varphi : H_1 \to H_2$ be a strong homomorphism. Then* $\ker \varphi$ *is a regularly normal subpolygroup of H_1. Moreover,*

$$H_1/\ker \varphi \cong \operatorname{Im} \varphi.$$

Proof. The kernel assertion is [28, Proposition 2.6], and the isomorphism is the First Isomorphism Theorem [28, Theorem 2.7]. These results use the same notion of strong homomorphism as Definition 2.7. □

We now collect the graph-theoretic notation. Unless stated otherwise, graphs are simple and undirected; finiteness will be imposed when it is needed.

For a graph G, its vertex and edge sets are denoted by $V(G)$ and $E(G)$. The degree of a vertex x is denoted by $\deg_G(x)$, and

$$\delta(G) = \min\{\deg_G(x) : x \in V(G)\}$$

is the minimum degree when G is finite. A vertex is *universal* if it is adjacent to every other vertex. The graph is regular if all its vertices have the same degree.

For vertices x, y in the same connected component, $d_G(x, y)$ denotes their distance. If G is connected, its diameter is

$$\operatorname{diam}(G) = \max\{d_G(x, y) : x, y \in V(G)\}.$$

A set $S \subseteq V(G)$ is independent if no two of its vertices are adjacent. The maximum cardinality of an independent set is $\alpha(G)$. A clique is a set of pairwise adjacent vertices, and the maximum size of a clique is denoted by $\omega(G)$.

A proper coloring assigns different colors to adjacent vertices. The minimum number of colors in a proper coloring is $\chi(G)$.

A set $C \subseteq V(G)$ is a vertex cover if every edge has an endpoint in C. Its minimum possible cardinality is $\tau(G)$. For every finite graph,

$$\tau(G) = |V(G)| - \alpha(G).$$

A set $D \subseteq V(G)$ is dominating if every vertex outside D has a neighbor in D. The domination number is

$$\gamma(G) = \min\{|D| : D \text{ is a dominating set of } G\}.$$

For a connected graph G, the vertex connectivity $\kappa(G)$ is the minimum number of vertices whose deletion disconnects G or leaves a single vertex. In particular,

$$\kappa(K_n) = n - 1.$$

The complement of G, denoted by $\overline{G}$, has the same vertex set as G, with two distinct vertices adjacent in $\overline{G}$ exactly when they are nonadjacent in G.

For positive integers $n_1, \ldots, n_r$, the complete multipartite graph

$$K_{n_1,\ldots,n_r}$$

has a partition of its vertex set into independent sets of orders $n_1, \ldots, n_r$, with all possible edges between different parts.

A graph is split if its vertex set is the union of a clique and an independent set. It is chordal if it has no induced cycle of length at least four, and it is perfect if

$$\chi(F) = \omega(F)$$

for every induced subgraph F. A graph is planar if it admits a drawing in the plane without edge crossings.

For a finite connected graph G, its Wiener index is

$$W(G) = \sum_{\{x,y\} \subseteq V(G)} d_G(x, y),$$

where the sum is over all unordered pairs of distinct vertices.

Let $A(G)$ be the adjacency matrix of a finite graph G. Its eigenvalues form the adjacency spectrum, denoted by

$$\operatorname{Spec}(G).$$

If these eigenvalues are $\lambda_1, \ldots, \lambda_n$, the energy of G is

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The Laplacian matrix is

$$L(G) = D(G) - A(G),$$

where $D(G)$ is the diagonal matrix of vertex degrees. The eigenvalues of $L(G)$, counted with multiplicity, form the Laplacian spectrum of G.

A spanning tree is a spanning subgraph that is a tree. We denote the number of spanning trees of a connected graph G by $t(G)$.

For the remainder of the paper, H denotes a polygroup and N a regularly normal subpolygroup of H. Thus H/N is a well-defined quotient polygroup.

3 The graph determined by a regularly normal subpolygroup

Let H be a polygroup with identity e, and let N be a regularly normal subpolygroup of H. We write $[x]_N$ for the class of x in the quotient polygroup H/N, and

$$\pi_N : H \longrightarrow H/N, \qquad \pi_N(x) = [x]_N,$$

for the canonical map. The quotient relation gives

$$[x]_N = [y]_N \quad \Longleftrightarrow \quad (x^{-1} \circ y) \cap N \neq \varnothing.$$

In particular, $[e]_N = N$.

Definition 3.1. The *N-class graph of H*, denoted by $G_N(H)$, is the graph with vertex set H, where distinct vertices x and y are adjacent whenever

$$\pi_N(x) \neq \pi_N(y).$$

Equivalently,

$$\{x, y\} \in E(G_N(H)) \quad \Longleftrightarrow \quad (x^{-1} \circ y) \cap N = \varnothing.$$

Thus two elements are nonadjacent exactly when they have the same image in H/N. The graph therefore records the fibers of the quotient map, but not the multiplication between the quotient classes.

Theorem 3.2. *Let*

$$H/N = \{C_i : i \in I\}.$$

Then the sets C_i are precisely the maximal independent sets of $G_N(H)$, and every vertex of C_i is adjacent to every vertex of C_j whenever $i \neq j$. Consequently, $G_N(H)$ is a complete multipartite graph with partite sets C_i.

If H is finite and

$$H/N = \{C_1, \ldots, C_r\},$$

then

$$G_N(H) \cong K_{|C_1|,\ldots,|C_r|}$$

and

$$\overline{G_N(H)} \cong K_{|C_1|} \cup \cdots \cup K_{|C_r|}.$$

Proof. Fix $i \in I$. For $x, y \in C_i$, $\pi_N(x) = \pi_N(y)$, so $xy \notin E(G_N(H))$; hence C_i is independent.

Maximality. Let $z \notin C_i$. For every $x \in C_i$, $\pi_N(x) \neq \pi_N(z)$, so $\{x, z\} \in E(G_N(H))$; thus no vertex outside C_i can be adjoined to C_i while preserving independence.

Every independent set lies in a single class. Let $S \subseteq H$ be independent and fix $x \in S$. For every $y \in S$, $xy \notin E(G_N(H))$, so $\pi_N(x) = \pi_N(y)$; hence $S \subseteq [x]_N = C_i$ for the index i with $x \in C_i$. Combined with maximality, the sets C_i are exactly the maximal independent sets of $G_N(H)$.

Between classes. If $x \in C_i$, $y \in C_j$, and $i \neq j$, then $\pi_N(x) \neq \pi_N(y)$, so $\{x, y\} \in E(G_N(H))$. Together with independence within each C_i, this proves the complete multipartite description. Two vertices are adjacent in $\overline{G_N(H)}$ exactly when they lie in the same C_i, which gives the stated form of the complement. □

Results below depending only on the numbers $|C_i|$ are standard consequences for complete multipartite graphs; those involving kernels, induced quotient maps, or comparisons of different regularly normal subpolygroups are genuinely polygroup-theoretic.

Theorem 3.2 immediately gives the two extreme cases:

$$G_N(H) \text{ is empty} \quad \Longleftrightarrow \quad N = H,$$

and

$$G_N(H) \text{ is complete} \quad \Longleftrightarrow \quad \text{every } N\text{-class is a singleton.}$$

Indeed, the first condition says that the quotient has only one class, while the second says that every partite set has order one.

The construction may also be viewed through homomorphisms.

Theorem 3.3. *Let*

$$\varphi : H \longrightarrow K$$

be a surjective strong homomorphism of polygroups, and let $N = \ker \varphi$. *By the first isomorphism theorem for polygroups,* N *is regularly normal and the map*

$$\overline{\varphi} : H/N \longrightarrow K, \qquad \overline{\varphi}([x]_N) = \varphi(x),$$

is an isomorphism. For $x, y \in H$,

$$[x]_N = [y]_N \quad \Longleftrightarrow \quad \varphi(x) = \varphi(y).$$

Consequently,

$$\{x, y\} \in E(G_N(H)) \quad \Longleftrightarrow \quad \varphi(x) \neq \varphi(y).$$

If $K = \{a_1, \ldots, a_r\}$ *and* H *is finite, then*

$$G_N(H) \cong K_{m_1,\ldots,m_r}, \qquad m_i = |\varphi^{-1}(a_i)|.$$

Proof. Since $\overline{\varphi}$ is injective,

$$[x]_N = [y]_N \iff \overline{\varphi}([x]_N) = \overline{\varphi}([y]_N) \iff \varphi(x) = \varphi(y),$$

using the definition of $\overline{\varphi}$ at the second step. Definition 3.1 now yields the stated characterization of adjacency, and the fibers of φ are exactly the N-classes. Applying Theorem 3.2 gives $G_N(H) \cong K_{m_1,\ldots,m_r}$. □

The preceding result has a useful functorial extension that does not require surjectivity.

Theorem 3.4. *Let* $f : H \to K$ *be a strong homomorphism, let* M *be a regularly normal subpolygroup of* K, *and put* $N = f^{-1}(M)$. *Then* N *is regularly normal in* H, *and*

$$\widetilde{f} : H/N \longrightarrow K/M, \qquad \widetilde{f}([x]_N) = [f(x)]_M,$$

is a well-defined injective strong homomorphism onto $\mathrm{Im}(\pi_M \circ f)$. *Moreover, for distinct* $x, y \in H$,

$$\{x, y\} \in E(G_N(H)) \quad \Longleftrightarrow \quad [f(x)]_M \neq [f(y)]_M.$$

Thus f, *regarded as a map on vertices from* $G_N(H)$ *to* $G_M(K)$, *preserves and reflects adjacency. If* f *is surjective, then* $\widetilde{f} : H/N \to K/M$ *is an isomorphism.*

Proof. The canonical projection $\pi_M : K \to K/M$ is a strong homomorphism by the quotient construction [28, Proposition 2.4], so $\pi_M \circ f$ is strong and

$$\ker(\pi_M \circ f) = \{x \in H : f(x) \in M\} = N.$$

By [28, Proposition 2.6], N is regularly normal, and the First Isomorphism Theorem [28, Theorem 2.7] gives the asserted injective strong homomorphism $\widetilde{f}$ and identifies its image. In particular,

$$[x]_N = [y]_N \quad \Longleftrightarrow \quad [f(x)]_M = [f(y)]_M,$$

which is the contrapositive of the adjacency equivalence stated above. Surjectivity of f implies surjectivity of $\widetilde{f}$. □

Corollary 3.5. *Let*

$$\mathrm{Aut}(H)_N = \{f \in \mathrm{Aut}(H) : f(N) = N\}.$$

Every member of $\mathrm{Aut}(H)_N$ *is an automorphism of the labeled graph* $G_N(H)$*. Consequently,*

$$\mathrm{Aut}(H)_N \leq \mathrm{Aut}(G_N(H)).$$

The induced action on H/N *has kernel consisting precisely of those polygroup automorphisms* f *for which* $[f(x)]_N = [x]_N$ *for every* $x \in H$*.*

Proof. Apply Theorem 3.4 with $K = H$, $M = N$, and $f \in \mathrm{Aut}(H)$ satisfying $f^{-1}(N) = N$, i.e. $f \in \mathrm{Aut}(H)_N$; the adjacency equivalence there shows f is a graph automorphism of $G_N(H)$. The final statement is the definition of the kernel of the induced permutation of H/N. □

The next result describes exactly how much quotient information is retained by the graph.

Theorem 3.6. *Let* H *and* K *be finite polygroups, and let* N *and* M *be regularly normal subpolygroups of* H *and* K*, respectively. Write*

$$H/N = \{C_1, \ldots, C_r\}, \qquad K/M = \{D_1, \ldots, D_s\}.$$

Then

$$G_N(H) \cong G_M(K)$$

if and only if $r = s$ *and there is a permutation* $\sigma \in S_r$ *such that*

$$|C_i| = |D_{\sigma(i)}| \qquad (1 \leq i \leq r).$$

Proof. ($\Leftarrow$) If the class sizes agree up to a permutation, then by Theorem 3.2 both graphs are complete multipartite with the same multiset of part sizes, hence isomorphic.

($\Rightarrow$) Let $f : G_N(H) \to G_M(K)$ be a graph isomorphism; then f is also an isomorphism of the complements. By Theorem 3.2,

$$\overline{G_N(H)} = \bigsqcup_{i=1}^{r} K_{|C_i|}, \qquad \overline{G_M(K)} = \bigsqcup_{j=1}^{s} K_{|D_j|}.$$

An isomorphism of graphs carries connected components onto connected components of the same order, so f induces a bijection between $\{1, \ldots, r\}$ and $\{1, \ldots, s\}$, forcing $r = s$ and a permutation $\sigma \in S_r$ with $|C_i| = |D_{\sigma(i)}|$ for every i. □

Corollary 3.7. *Suppose that* H *is finite and every* N*-class has cardinality* $|N|$*. Then*

$$|N| = \alpha(G_N(H)),$$

$$|H/N| = \omega(G_N(H)) = \chi(G_N(H)),$$

and

$$|H| = \alpha(G_N(H))\chi(G_N(H)).$$

Proof. By Theorem 3.2 the maximal independent sets are the N-classes, all of size $|N|$, so $\alpha(G_N(H)) = |N|$. A clique meets each class in at most one vertex, and choosing one vertex per class produces a clique of order $|H/N|$; hence $\omega(G_N(H)) = |H/N|$. Coloring each C_i by a distinct color uses $|H/N|$ colors, and $\chi(G_N(H)) \geq \omega(G_N(H))$, so $\chi(G_N(H)) = |H/N|$. The final identity is $|H| = |N| \cdot |H/N|$. □

Remark 3.8. The graph determines the number and sizes of the N-classes, but it does not generally determine the hyperoperation on H or on H/N. Thus two nonisomorphic polygroups may give rise to isomorphic class graphs. Theorem 3.6 describes precisely the information preserved by this construction.

Example 3.9. Let

$$H = \mathbb{Z}_{12}$$

be the additive group of integers modulo 12, regarded as a polygroup under the singleton-valued hyperoperation

$$x \circ y = \{x + y\}.$$

Define

$$\varphi : \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_4, \qquad \varphi(x) = x \pmod 4.$$

This map is well defined and surjective. Moreover, for all $x, y \in \mathbb{Z}_{12}$,

$$\varphi(x \circ y) = \varphi(\{x + y\}) = \{\varphi(x + y)\} = \{\varphi(x) + \varphi(y)\} = \varphi(x) \circ \varphi(y).$$

Hence φ is a surjective strong homomorphism.

Its kernel is

$$N = \ker \varphi = \{0, 4, 8\}.$$

Therefore N is a regularly normal subpolygroup of $\mathbb{Z}_{12}$. By Theorem 3.3, the N-classes coincide with the fibers of φ. They are

$$C_0 = \varphi^{-1}(0) = \{0, 4, 8\}, \qquad C_1 = \varphi^{-1}(1) = \{1, 5, 9\},$$

$$C_2 = \varphi^{-1}(2) = \{2, 6, 10\}, \qquad C_3 = \varphi^{-1}(3) = \{3, 7, 11\}.$$

Thus two distinct elements of $\mathbb{Z}_{12}$ are adjacent in $G_N(\mathbb{Z}_{12})$ exactly when they have different images under φ. Equivalently, vertices in the same set C_i are pairwise nonadjacent, while every vertex of C_i is adjacent to every vertex of C_j whenever $i \neq j$. Consequently,

$$G_N(\mathbb{Z}_{12}) \cong K_{3,3,3,3}.$$

It follows that

$$\alpha(G_N(\mathbb{Z}_{12})) = 3$$

and

$$\omega(G_N(\mathbb{Z}_{12})) = \chi(G_N(\mathbb{Z}_{12})) = 4.$$

Every vertex $v \in \mathbb{Z}_{12}$ has degree

$$\deg_{G_N(\mathbb{Z}_{12})}(v) = 12 - 3 = 9,$$

and hence the graph is 9-regular. Its number of edges is

$$|E(G_N(\mathbb{Z}_{12}))| = \sum_{0 \leq i < j \leq 3} |C_i||C_j| = \binom{4}{2} 3^2 = 54.$$

In this example, the graph also recovers

$$|N| = \alpha(G_N(\mathbb{Z}_{12})) = 3$$

and

$$|H/N| = \omega(G_N(\mathbb{Z}_{12})) = \chi(G_N(\mathbb{Z}_{12})) = 4.$$

Therefore

$$|H| = \alpha(G_N(\mathbb{Z}_{12}))\chi(G_N(\mathbb{Z}_{12})) = 12.$$

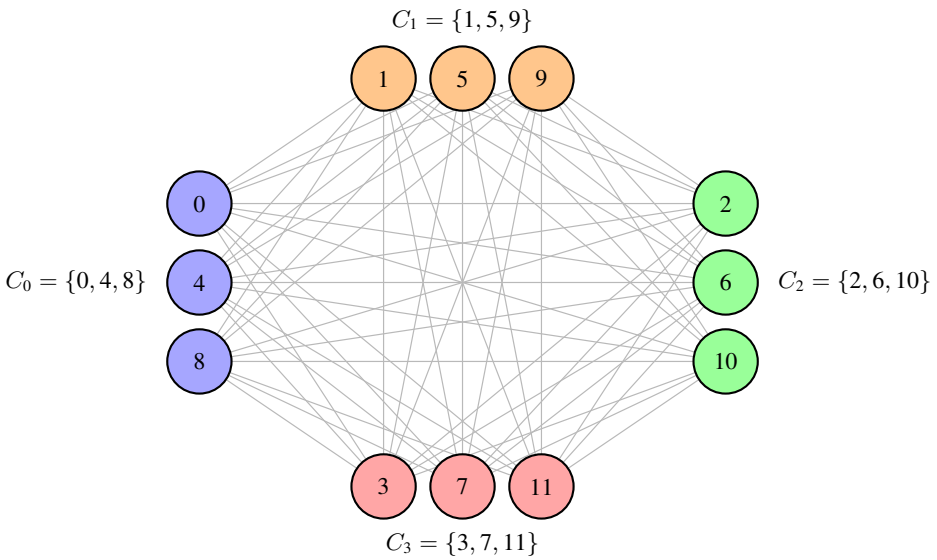


Figure 1. The class graph $G_N(\mathbb{Z}_{12}) \cong K_{3,3,3,3}$. Vertices of the same color form an N-class and are pairwise nonadjacent; every two vertices of different colors are adjacent.

The preceding example is the group special case. The following example shows that the construction also applies to a genuinely multivalued polygroup.

Example 3.10. Let $P = \{e, a\}$, with e an identity, $e^{-1} = e$, $a^{-1} = a$, and

$$a \circ a = \{e, a\}.$$

Together with $e \circ x = x \circ e = \{x\}$, this is a polygroup. Indeed, the only associativity check not involving e is

$$(a \circ a) \circ a = \{e, a\} \circ a = \{e, a\} = a \circ \{e, a\} = a \circ (a \circ a),$$

and reversibility follows immediately from $a^{-1} = a$ and $a \circ a = \{e, a\}$. In particular, the operation is not singleton-valued. Let

$$H = P \times \mathbb{Z}_3$$

be the direct hyperproduct, so that

$$(p, i) * (q, j) = \{(r, i + j) : r \in p \circ q\},$$

where the second coordinate is taken modulo 3. Associativity, the identity law, inversion, and reversibility hold coordinatewise, so H is a polygroup. The projection

$$\varphi : H \longrightarrow \mathbb{Z}_3, \qquad \varphi(p, i) = i,$$

is a surjective strong homomorphism. Its kernel is

$$N = P \times \{0\} = \{(e, 0), (a, 0)\}.$$

By [28, Proposition 2.6], N is regularly normal, and by [28, Theorem 2.7] its quotient classes are the three fibers

$$C_i = P \times \{i\} \qquad (i \in \mathbb{Z}_3).$$

Therefore

$$G_N(H) \cong K_{2,2,2}.$$

This example is genuinely multivalued because, for instance,

$$(a, 0) * (a, 0) = \{(e, 0), (a, 0)\}.$$

It also illustrates the algebraic content of Theorem 3.3: the partite sets are exactly the fibers of a strong homomorphism, not merely an arbitrarily chosen partition of the vertex set.

4 Structural properties of $G_N(H)$

Throughout this section, H is a polygroup and N is a regularly normal subpolygroup of H. We use the notation

$$H/N = \{C_i : i \in I\}.$$

By Theorem 3.2, the sets C_i are precisely the partite sets of $G_N(H)$. Thus vertices in the same N-class are nonadjacent, while vertices in distinct classes are adjacent.

We begin with the degrees and distances in $G_N(H)$.

Proposition 4.1. *Suppose that H is finite. If $x \in H$, then*

$$\deg_{G_N(H)}(x) = |H| - |[x]_N|.$$

Consequently, $G_N(H)$ is regular if and only if all N-classes have the same cardinality.

Proof. For $y \neq x$, $xy \notin E(G_N(H)) \iff [y]_N = [x]_N$, so the neighbors of x are exactly $H \setminus [x]_N$; hence $\deg_{G_N(H)}(x) = |H| - |[x]_N|$. Vertices in classes of equal size therefore have equal degree, and $G_N(H)$ is regular precisely when all partite sets have equal cardinality. □

Theorem 4.2. *Suppose that $N \neq H$. Then $G_N(H)$ is connected and*

$$\operatorname{diam}(G_N(H)) = \begin{cases} 1, & \text{if every } N\text{-class is a singleton}, \\ 2, & \text{otherwise}. \end{cases}$$

Moreover, a vertex $x \in H$ is universal if and only if

$$[x]_N = \{x\}.$$

Proof. Since $N \neq H$, $|H/N| \geq 2$. Let $x, y \in H$ be distinct. If $[x]_N \neq [y]_N$, then $\{x, y\} \in E(G_N(H))$. If $[x]_N = [y]_N$, choose z in a different N-class; then $\{x, z\}, \{z, y\} \in E(G_N(H))$, so x-z-y is a path of length two. Hence $G_N(H)$ is connected with $\operatorname{diam}(G_N(H)) \leq 2$.

If every class is a singleton, every two distinct vertices lie in different classes and are therefore adjacent, so the diameter is 1. Otherwise some class has two distinct elements, at distance exactly 2, so the diameter equals 2.

Finally, x is adjacent to every vertex outside $[x]_N$ and to no vertex of $[x]_N \setminus \{x\}$; hence x is universal $\iff [x]_N \setminus \{x\} = \varnothing \iff [x]_N = \{x\}$. □

The domination number also depends only on whether a singleton class is present.

Theorem 4.3. *Suppose that $N \neq H$. Then*

$$\gamma(G_N(H)) = \begin{cases} 1, & \text{if some } N\text{-class is a singleton}, \\ 2, & \text{otherwise}. \end{cases}$$

Proof. If $\{x\} \in H/N$, then x is universal by Theorem 4.2, so $\{x\}$ is dominating and $\gamma(G_N(H)) = 1$.

If instead every N-class has at least two elements, no vertex is universal, so $\gamma(G_N(H)) \geq 2$. Fix $x \in C_i$, $y \in C_j$ with $i \neq j$. Every vertex of $C_i \setminus \{x\}$ is adjacent to y, every vertex of $C_j \setminus \{y\}$ is adjacent to x, and every vertex outside $C_i \cup C_j$ is adjacent to both. Hence $\{x, y\}$ dominates H, and $\gamma(G_N(H)) = 2$. □

We next compare the graphs obtained from nested regularly normal subpolygroups.

Theorem 4.4. *Let H be a finite polygroup, and let N_1 and N_2 be regularly normal subpolygroups of H such that*

$$N_1 \subseteq N_2.$$

Then every N_1-class is contained in a unique N_2-class and

$$E(G_{N_2}(H)) \subseteq E(G_{N_1}(H)).$$

For each $D \in H/N_2$, let

$$C_{D,1}, \dots, C_{D,k_D}$$

be the N_1-classes contained in D. Then

$$E(G_{N_1}(H)) \setminus E(G_{N_2}(H))$$

consists precisely of the pairs $\{x, y\}$ for which x and y belong to distinct N_1-classes contained in the same N_2-class. Consequently,

$$|E(G_{N_1}(H))| - |E(G_{N_2}(H))| = \sum_{D \in H/N_2} \sum_{1 \le i < j \le k_D} |C_{D,i}|\,|C_{D,j}|.$$

Equivalently,

$$|E(G_{N_1}(H))| - |E(G_{N_2}(H))| = \frac{1}{2} \sum_{D \in H/N_2} \left(|D|^2 - \sum_{i=1}^{k_D} |C_{D,i}|^2 \right).$$

Proof. Refinement. Suppose $[x]_{N_1} = [y]_{N_1}$, so $(x^{-1} \circ y) \cap N_1 \neq \varnothing$. Since $N_1 \subseteq N_2$, also $(x^{-1} \circ y) \cap N_2 \neq \varnothing$, i.e. $[x]_{N_2} = [y]_{N_2}$. Thus every N_1-class lies in a single N_2-class, and this N_2-class is unique because the N_2-classes partition H.

Edge inclusion. Let $\{x, y\} \in E(G_{N_2}(H))$, so $[x]_{N_2} \neq [y]_{N_2}$. If $[x]_{N_1} = [y]_{N_1}$, refinement would force $[x]_{N_2} = [y]_{N_2}$, a contradiction; hence $[x]_{N_1} \neq [y]_{N_1}$, i.e. $\{x, y\} \in E(G_{N_1}(H))$. Thus $E(G_{N_2}(H)) \subseteq E(G_{N_1}(H))$.

Deleted edges. An edge $\{x, y\}$ lies in $E(G_{N_1}(H)) \setminus E(G_{N_2}(H))$ exactly when

$$[x]_{N_1} \neq [y]_{N_1} \qquad \text{and} \qquad [x]_{N_2} = [y]_{N_2},$$

i.e. exactly when x and y lie in distinct N_1-classes contained in a common N_2-class D. Fixing $D \in H/N_2$, each pair $1 \le i < j \le k_D$ contributes the $|C_{D,i}|\,|C_{D,j}|$ edges between $C_{D,i}$ and $C_{D,j}$ (all present in $G_{N_1}(H)$, none in $G_{N_2}(H)$, since both endpoints lie in D). Summing over $1 \le i < j \le k_D$ and then over $D \in H/N_2$ gives

$$|E(G_{N_1}(H))| - |E(G_{N_2}(H))| = \sum_{D \in H/N_2} \sum_{1 \le i < j \le k_D} |C_{D,i}|\,|C_{D,j}|.$$

Since $D = \bigsqcup_{i=1}^{k_D} C_{D,i}$,

$$|D|^2 = \sum_{i=1}^{k_D} |C_{D,i}|^2 + 2 \sum_{1 \le i < j \le k_D} |C_{D,i}|\,|C_{D,j}|,$$

which gives the equivalent expression. □

Corollary 4.5. *Let H be a finite polygroup, and let N_1 and N_2 be regularly normal subpolygroups of H. Then*

$$N_1 \subseteq N_2 \iff E(G_{N_2}(H)) \subseteq E(G_{N_1}(H)).$$

Consequently,

$$N_1 = N_2 \iff G_{N_1}(H) = G_{N_2}(H)$$

as graphs on the common vertex set H. Moreover,

$$N_1 \subsetneq N_2 \iff E(G_{N_2}(H)) \subsetneq E(G_{N_1}(H)).$$

Proof. ($\Rightarrow$) This is Theorem 4.4.

($\Leftarrow$) Suppose $E(G_{N_2}(H)) \subseteq E(G_{N_1}(H))$, and let $x \in N_1$; we show $x \in N_2$. Suppose not. Since $e \in N_2$, $x \neq e$. As $[e]_{N_1} = N_1 \ni x$, e and x lie in the same N_1-class, so $\{e, x\} \notin E(G_{N_1}(H))$. But $[e]_{N_2} = N_2 \not\ni x$, so $[e]_{N_2} \neq [x]_{N_2}$ and hence

$$\{e, x\} \in E(G_{N_2}(H)) \subseteq E(G_{N_1}(H)),$$

a contradiction. Thus $x \in N_2$, and $N_1 \subseteq N_2$.

Equality. If $N_1 = N_2$ the graphs coincide by definition. Conversely, $G_{N_1}(H) = G_{N_2}(H)$ gives edge inclusion in both directions, hence $N_1 \subseteq N_2$ and $N_2 \subseteq N_1$ by the equivalence just proved, so $N_1 = N_2$.

Strict inclusion. Suppose $N_1 \subsetneq N_2$ and fix $x \in N_2 \setminus N_1$. Since $x \notin [e]_{N_1} = N_1$, $\{e, x\} \in E(G_{N_1}(H))$; since $e, x \in N_2 = [e]_{N_2}$, $\{e, x\} \notin E(G_{N_2}(H))$. Together with $E(G_{N_2}(H)) \subseteq E(G_{N_1}(H))$, this gives $E(G_{N_2}(H)) \subsetneq E(G_{N_1}(H))$. Conversely, strict edge inclusion gives $N_1 \subseteq N_2$ by the equivalence above, and $N_1 = N_2$ would force equal edge sets, contrary to strictness; so $N_1 \subsetneq N_2$. □

For a fixed finite polygroup H, let $\mathcal{R}(H)$ denote the set of regularly normal subpolygroups of H. For $N \in \mathcal{R}(H)$, put

$$\mathcal{C}_N = \{[x]_N : x \in H\}.$$

Let

$$\mathfrak{C}(H) = \{\mathcal{C}_N : N \in \mathcal{R}(H)\} \quad \text{and} \quad \mathfrak{G}(H) = \{G_N(H) : N \in \mathcal{R}(H)\},$$

where every graph in $\mathfrak{G}(H)$ retains the labeled vertex set H. For partitions $\mathcal{C}, \mathcal{D}$ of H, write $\mathcal{C} \preceq \mathcal{D}$ if every block of $\mathcal{C}$ is contained in a block of $\mathcal{D}$.

Theorem 4.6 (Representation theorem). *The maps*

$$\Phi : \mathcal{R}(H) \longrightarrow \mathfrak{C}(H), \qquad \Phi(N) = \mathcal{C}_N,$$

and

$$\Psi : \mathfrak{C}(H) \longrightarrow \mathfrak{G}(H), \qquad \Psi(\mathcal{C}_N) = G_N(H),$$

are bijections. Moreover, for $N_1, N_2 \in \mathcal{R}(H)$,

$$\begin{aligned} N_1 \subseteq N_2 &\iff \mathcal{C}_{N_1} \preceq \mathcal{C}_{N_2} \\ &\iff E(G_{N_2}(H)) \subseteq E(G_{N_1}(H)). \end{aligned}$$

Consequently,

$$(\mathcal{R}(H), \subseteq) \cong (\mathfrak{C}(H), \preceq) \cong (\mathfrak{G}(H), \leq_{\text{rev}}),$$

where

$$G_1 \leq_{\text{rev}} G_2 \quad \iff \quad E(G_2) \subseteq E(G_1).$$

Proof. Since $[e]_N = N$,

$$\mathcal{C}_{N_1} = \mathcal{C}_{N_2} \Longrightarrow [e]_{N_1} = [e]_{N_2} \Longrightarrow N_1 = N_2.$$

Thus Φ is injective, and it is surjective by the definition of $\mathfrak{C}(H)$.

By Theorem 3.2, the blocks of $\mathcal{C}_N$ are exactly the maximal independent sets of $G_N(H)$. Hence

$$G_{N_1}(H) = G_{N_2}(H) \Longrightarrow \mathcal{C}_{N_1} = \mathcal{C}_{N_2},$$

and Ψ is bijective.

If $N_1 \subseteq N_2$, Theorem 4.4 gives

$$\mathcal{C}_{N_1} \preceq \mathcal{C}_{N_2} \quad \text{and} \quad E(G_{N_2}(H)) \subseteq E(G_{N_1}(H)).$$

Conversely,

$$\mathcal{C}_{N_1} \preceq \mathcal{C}_{N_2} \Longrightarrow [e]_{N_1} \subseteq [e]_{N_2} \Longrightarrow N_1 \subseteq N_2.$$

The converse from edge containment is Corollary 4.5. □

Remark 4.7. Under the hypotheses of Theorem 4.4, define

$$\rho_{12} : H/N_1 \longrightarrow H/N_2, \qquad \rho_{12}([x]_{N_1}) = [x]_{N_2}.$$

The refinement established in the proof shows that ρ_{12} is well defined. It is plainly surjective. Moreover,

$$\begin{aligned}\rho_{12}\big([x]_{N_1} \odot [y]_{N_1}\big) &= \rho_{12}\big(\{[z]_{N_1} : z \in x \circ y\}\big)\\ &= \{[z]_{N_2} : z \in x \circ y\}\\ &= [x]_{N_2} \odot [y]_{N_2}.\end{aligned}$$

Thus ρ_{12} is a surjective strong homomorphism. Its kernel is

$$\ker \rho_{12} = \{[x]_{N_1} : x \in N_2\} = N_2/N_1.$$

Consequently, the Third Isomorphism Theorem [28, Theorem 2.9] gives

$$(H/N_1)/(N_2/N_1) \cong H/N_2.$$

An edge of $G_{N_1}(H)$ disappears in $G_{N_2}(H)$ precisely when its endpoints represent distinct N_1-classes that are identified by ρ_{12}. Hence the deleted edges record exactly the collapse of quotient classes under the canonical quotient homomorphism.

The principal graph invariants of $G_N(H)$ are now read off from the sizes of the quotient classes; compare [15, 16].

Theorem 4.8. *Suppose that H is finite and*

$$H/N = \{C_1, \dots, C_r\}.$$

Put

$$n = |H|, \qquad n_i = |C_i|, \qquad q = \max_{1 \le i \le r} n_i.$$

Then

$$\omega(G_N(H)) = \chi(G_N(H)) = r,$$
$$\alpha(G_N(H)) = q, \qquad \tau(G_N(H)) = n - q,$$

and

$$|E(G_N(H))| = \sum_{1 \le i < j \le r} n_i n_j = \frac{1}{2}\left(n^2 - \sum_{i=1}^{r} n_i^2\right).$$

Proof. A clique meets each N-class in at most one vertex, and choosing one vertex from each class produces a clique of order r; hence $\omega(G_N(H)) = r$. Coloring each C_i by a distinct color gives a proper r-coloring, and $\chi(G_N(H)) \ge \omega(G_N(H)) = r$ forces $\chi(G_N(H)) = r$.

An independent set is contained in a single N-class, and every N-class is independent, so $\alpha(G_N(H)) = q$; the identity $\tau(G) = |V(G)| - \alpha(G)$ then gives $\tau(G_N(H)) = n - q$.

Every vertex of C_i is adjacent to every vertex of C_j for $i \ne j$, contributing $n_i n_j$ edges per pair, so

$$|E(G_N(H))| = \sum_{i<j} n_i n_j = \tfrac{1}{2}\Big(\big(\textstyle\sum_i n_i\big)^2 - \sum_i n_i^2\Big) = \tfrac{1}{2}\Big(n^2 - \textstyle\sum_i n_i^2\Big).$$

□

The connectivity of $G_N(H)$ admits a similar expression.

Theorem 4.9. *Suppose that H is finite, $N \ne H$, and*

$$q = \max\{|C| : C \in H/N\}.$$

Then

$$\kappa(G_N(H)) = |H| - q.$$

Consequently,

$$\kappa(G_N(H)) = \delta(G_N(H)) = |H| - q,$$

where κ and δ denote vertex connectivity and minimum degree, respectively.

Proof. Upper bound. Let C be an N-class with $|C| = q$. If $q \geq 2$, removing $H \setminus C$ (of size $|H| - q$) leaves an edgeless graph on C, which is disconnected; hence $\kappa(G_N(H)) \leq |H| - q$. If $q = 1$, then $G_N(H)$ is complete and $\kappa(G_N(H)) = |H| - 1 = |H| - q$.

Lower bound. Let $S \subseteq H$ with $|S| < |H| - q$, and let $x, y \in H \setminus S$ be distinct. If $[x]_N \neq [y]_N$, then $\{x, y\} \in E(G_N(H))$. If $x, y \in C_i$, then, since

$$|H \setminus C_i| = |H| - n_i \geq |H| - q > |S|,$$

there exists $z \in (H \setminus C_i) \setminus S$, and x-z-y is a path in $G_N(H) - S$. Hence $G_N(H) - S$ is connected for every such S, so $\kappa(G_N(H)) \geq |H| - q$. Combined with the upper bound, $\kappa(G_N(H)) = |H| - q$.

By Proposition 4.1, $\delta(G_N(H)) = |H| - \max_i n_i = |H| - q$. □

Proposition 4.10. *Suppose that H is finite and $N \neq H$. If*

$$H/N = \{C_1, \ldots, C_r\}, \qquad n_i = |C_i|, \qquad n = |H|,$$

then the Wiener index of $G_N(H)$ is

$$W(G_N(H)) = \binom{n}{2} + \sum_{i=1}^{r} \binom{n_i}{2}.$$

Proof. By Theorem 4.2, $d_G(x, y) = 1$ if x, y lie in different classes and $d_G(x, y) = 2$ if they lie in the same class. Assigning distance 1 to every one of the $\binom{n}{2}$ unordered pairs, then adding one extra unit for each of the $\sum_i \binom{n_i}{2}$ same-class pairs, gives

$$W(G_N(H)) = \binom{n}{2} + \sum_{i=1}^{r} \binom{n_i}{2}.$$

□

Corollary 4.11. *Suppose that H is finite, $|H/N| = r$, and every N-class has cardinality m. Then*

$$G_N(H) \cong K_{\underbrace{m, \ldots, m}_{r \text{ parts}}},$$

and $G_N(H)$ is $(r-1)m$-regular. Moreover,

$$|V(G_N(H))| = rm,$$

$$|E(G_N(H))| = \binom{r}{2} m^2,$$

$$\alpha(G_N(H)) = m, \qquad \omega(G_N(H)) = \chi(G_N(H)) = r,$$

and, when $r \geq 2$,

$$\kappa(G_N(H)) = (r-1)m.$$

Proof. Apply Proposition 4.1, Theorems 4.8 and 4.9 with

$$n = rm \qquad \text{and} \qquad q = m.$$

□

5 Planarity and spectral properties

Throughout this section, H is finite, N is a regularly normal subpolygroup of H, and

$$H/N = \{C_1, \ldots, C_r\}.$$

Set

$$n_i = |C_i| \qquad (1 \leq i \leq r)$$

and

$$n = |H| = \sum_{i=1}^{r} n_i.$$

By Theorem 3.2,

$$G_N(H) \cong K_{n_1,\ldots,n_r}.$$

We first identify two familiar classes of graphs that arise from this decomposition.

Theorem 5.1. *The graph $G_N(H)$ is perfect. Furthermore, the following conditions are equivalent:*

(i) $G_N(H)$ *is split;*

(ii) $G_N(H)$ *is chordal;*

(iii) at most one of $n_1, \ldots, n_r$ *is greater than one.*

Proof. Every induced subgraph F of $K_{n_1,\ldots,n_r}$ is complete multipartite, with $\omega(F) = \chi(F)$ equal to its number of nonempty parts; hence $G_N(H)$ is perfect.

((3)⇒(1),(2)) If at most one class has cardinality greater than one, take that class as the independent part and the (singleton) remaining classes as a clique; $G_N(H)$ is split, and every split graph is chordal.

(¬(3)⇒ ¬(1),¬(2)) Suppose C_i and C_j $(i \neq j)$ each have at least two elements; choose $x_1 \neq x_2 \in C_i$ and $y_1 \neq y_2 \in C_j$. The subgraph induced on $\{x_1, x_2, y_1, y_2\}$ is $K_{2,2}$, an induced 4-cycle, so $G_N(H)$ is not chordal and, since every induced subgraph of a split graph is split while C_4 is not split, $G_N(H)$ is not split. □

We next determine exactly when $G_N(H)$ is planar.

Theorem 5.2. *Assume, after reindexing, that*

$$1 \leq n_1 \leq n_2 \leq \cdots \leq n_r.$$

Then $G_N(H)$ is planar if and only if one of the following holds:

(i) $r = 1$*;*

(ii) $r = 2$ *and* $n_1 \leq 2$*;*

(iii) $r = 3$*, and either*

$$n_1 = n_2 = 1$$

or

$$n_3 \leq 2;$$

(iv) $r = 4$*, and*

$$(n_1, n_2, n_3, n_4) \in \{(1,1,1,1), (1,1,1,2)\}.$$

Proof. Case $r \geq 5$. One vertex from each of five distinct classes induces $K_5 \subseteq G_N(H)$, so $G_N(H)$ is nonplanar.

Case $r = 4$. If $n_4 \geq 3$, take three vertices from C_4 and one from each of C_1, C_2, C_3; the bipartite edges between these two sets (ignoring the extra edges among C_1, C_2, C_3) form $K_{3,3} \subseteq G_N(H)$, so $G_N(H)$ is nonplanar. If instead $n_3 \geq 2$ (so $n_4 = 2$, since $n_4 \geq 3$ was excluded and $n_4 \geq n_3 \geq 2$), then $G_N(H)$ contains $K_{1,1,2,2}$ as a subgraph, a graph on 6 vertices with 13 edges, exceeding the planar bound $3 \cdot 6 - 6 = 12$; hence $G_N(H)$ is nonplanar. What remains is $n_1 = n_2 = n_3 = 1$ and $n_4 \in \{1, 2\}$, giving $K_{1,1,1,1} = K_4$ or $K_{1,1,1,2} = K_5 - e$, both planar.

Case $r = 3$. If $n_1 = n_2 = 1$, then $G_N(H) \cong K_{1,1,n_3}$, which is planar for every n_3 (draw the two singleton vertices on either side of the n_3-vertex row, with one arc between them routed outside the figure). If $n_3 \leq 2$, then $G_N(H)$ is a subgraph of the planar octahedral graph $K_{2,2,2}$. Otherwise $n_3 \geq 3$ and $n_1 + n_2 \geq 3$; choosing three vertices from C_3 and three from $C_1 \cup C_2$, and keeping only the edges between the two sets, gives $K_{3,3} \subseteq G_N(H)$, so $G_N(H)$ is nonplanar.

Case $r = 2$. $G_N(H) \cong K_{n_1,n_2}$, planar $\iff$ $n_1 \leq 2$ (else $K_{3,3} \subseteq G_N(H)$).

Case $r = 1$. $G_N(H)$ is edgeless, hence planar. □

We now turn to the adjacency spectrum; the calculations are classical for complete multipartite graphs (compare [17]) and are included to display the dependence on the quotient-class sizes.

Theorem 5.3. *The adjacency characteristic polynomial of $G_N(H)$ is*

$$\phi_A(\lambda) = \lambda^{n-r}\left[\prod_{i=1}^{r}(\lambda+n_i) - \sum_{i=1}^{r} n_i \prod_{\substack{1\le j\le r\\ j\neq i}} (\lambda+n_j)\right].$$

Proof. Order the vertices so that each class occupies consecutive indices; then $A = A(G_N(H))$ has zero diagonal blocks and all-one off-diagonal blocks. For $1 \le i \le r$, let

$$U_i = \Big\{u \in \mathbb{R}^{C_i} : \textstyle\sum_{x\in C_i} u_x = 0\Big\}, \qquad \dim U_i = n_i - 1,$$

extended by zero off C_i; then $Au = 0$ for every $u \in U_i$. Let W be the space of vectors constant on each class, so

$$\mathbb{R}^n = \Big(\bigoplus_{i=1}^{r} U_i\Big) \oplus W,$$

with both summands A-invariant. The restriction of A to W, in the basis of class-indicator vectors, is $B = (b_{ij})_{r\times r}$ with $b_{ii} = 0$ and $b_{ij} = n_j$ for $i \neq j$; hence

$$\phi_A(\lambda) = \lambda^{n-r}\det(\lambda I_r - B).$$

Let $D = \operatorname{diag}(n_1,\dots,n_r)$, $\mathbf{1} = (1,\dots,1)^{\mathsf{T}}$, $\mathbf{n} = (n_1,\dots,n_r)^{\mathsf{T}}$. Since $B = \mathbf{1}\mathbf{n}^{\mathsf{T}} - D$,

$$\lambda I_r - B = \lambda I_r + D - \mathbf{1}\mathbf{n}^{\mathsf{T}}.$$

For $\lambda \neq -n_i$ $(1 \le i \le r)$, the rank-one determinant identity gives

$$\det(\lambda I_r - B) = \prod_{i=1}^{r}(\lambda+n_i)\Big(1 - \sum_{i=1}^{r}\frac{n_i}{\lambda+n_i}\Big) = \prod_{i=1}^{r}(\lambda+n_i) - \sum_{i=1}^{r} n_i \prod_{\substack{1\le j\le r\\ j\neq i}} (\lambda+n_j).$$

Both sides are polynomials in λ, so the identity extends to $\lambda = -n_i$ by continuity; multiplying by λ^{n-r} completes the proof. □

Corollary 5.4. *Suppose that $r \ge 2$ and every N-class has cardinality m. Then*

$$\operatorname{Spec}(G_N(H)) = \Big\{((r-1)m)^{(1)}, (-m)^{(r-1)}, 0^{(r(m-1))}\Big\}.$$

Consequently,

$$\mathcal{E}(G_N(H)) = 2(r-1)m.$$

Proof. With $n = rm$, setting $n_i = m$ for all i in Theorem 5.3 gives

$$\phi_A(\lambda) = \lambda^{r(m-1)}\big(\lambda - (r-1)m\big)(\lambda+m)^{r-1},$$

which yields the stated spectrum. Hence

$$\mathcal{E}(G_N(H)) = (r-1)m + (r-1)\cdot m = 2(r-1)m.$$

□

Remark 5.5. If $r = 1$, then $G_N(H)$ is empty, with adjacency spectrum $0^{(n)}$ and energy 0.

The Laplacian spectrum admits a closed formula for arbitrary class sizes.

Theorem 5.6. *If $r \geq 2$, then the Laplacian spectrum of $G_N(H)$ is*

$$\left\{0^{(1)}, n^{(r-1)}, (n-n_1)^{(n_1-1)}, \ldots, (n-n_r)^{(n_r-1)}\right\},$$

where terms with multiplicity zero are omitted.

Proof. Let $L = D(G_N(H)) - A(G_N(H))$, and reuse the subspaces U_i, W from the proof of Theorem 5.3. Every vertex of C_i has degree $n - n_i$, and $Au = 0$ for $u \in U_i$, so $Lu = (n-n_i)u$ for every $u \in U_i$; thus $n - n_i$ is a Laplacian eigenvalue of multiplicity $n_i - 1$.

For $v \in W$, write $v|_{C_i} = a_i \mathbf{1}_{n_i}$. For $x \in C_i$,

$$(Lv)_x = na_i - \sum_{j=1}^{r} n_j a_j,$$

since x has degree $n - n_i$ and its neighbors contribute $\sum_{j\neq i} n_j a_j = \sum_j n_j a_j - n_i a_i$. Thus on

$$W_0 = \left\{v \in W : \textstyle\sum_j n_j a_j = 0\right\}, \qquad \dim W_0 = r - 1,$$

we have $Lv = nv$; and the all-ones vector spans the complementary line in W, with $L\mathbf{1} = 0$. Since

$$1 + (r-1) + \sum_{i=1}^{r} (n_i - 1) = n,$$

this accounts for the full spectrum. □

Remark 5.7. When $r = 1$, the graph is empty and its Laplacian spectrum is $0^{(n)}$.

The number of spanning trees follows directly from the Laplacian spectrum.

Corollary 5.8. *Suppose that $r \geq 2$. Then*

$$t(G_N(H)) = n^{r-2} \prod_{i=1}^{r} (n-n_i)^{n_i-1}.$$

Proof. Since $r \geq 2$, $G_N(H)$ is connected, so by the Matrix–Tree Theorem $t(G_N(H))$ equals $1/n$ times the product of the nonzero Laplacian eigenvalues. By Theorem 5.6, this product is $n^{r-1} \prod_i (n-n_i)^{n_i-1}$, so

$$t(G_N(H)) = \frac{1}{n} \cdot n^{r-1} \prod_{i=1}^{r} (n-n_i)^{n_i-1} = n^{r-2} \prod_{i=1}^{r} (n-n_i)^{n_i-1}.$$

□

Remark 5.9. The characteristic polynomial, Laplacian spectrum, energy in the uniform case, and number of spanning trees depend only on the multiset

$$\{|C| : C \in H/N\}.$$

Thus these quantities recover information about the sizes of the quotient classes, but not the hyperoperation on H/N.

References

[1] F. Marty, Sur une généralisation de la notion de groupe, in *Proceedings of the Eighth Scandinavian Mathematical Congress*, Stockholm, 1934, pp. 45–49.

[2] M. Dresher and O. Ore, Theory of multigroups, *American Journal of Mathematics* **60** (1938), no. 3, 705–733.

[3] M. Koskas, Groupoïdes, demi-hypergroupes et hypergroupes, *Journal de Mathématiques Pures et Appliquées* **49** (1970), 155–192.

[4] S. D. Comer, Polygroups derived from cogroups, *Journal of Algebra* **89** (1984), no. 2, 397–405. doi:10.1016/0021-8693(84)90225-4.

[5] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani Editore, Tricesimo, Italy, 1993.

[6] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003. doi:10.1007/978-1-4757-3714-1.

[7] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press, Palm Harbor, FL, 1994.

[8] B. Davvaz, Applications of the γ^*-relation to polygroups, *Communications in Algebra* **35** (2007), no. 9, 2698–2706. doi:10.1080/00927870701302123.

[9] B. Davvaz and H. Karimian, On γ-complete hypergroups, *European Journal of Combinatorics* **28** (2007), no. 1, 86–93.

[10] V. Leoreanu-Fotea and B. Davvaz, n-hypergroups and binary relations, *European Journal of Combinatorics* **29** (2008), no. 5, 1207–1218.

[11] D. Freni, A new characterization of derived hypergroups via strongly regular equivalences, *Communications in Algebra* **30** (2002), no. 8, 3977–3989.

[12] B. Satyanarayana, K. S. Prasad, and K. B. Srinivas, Graph of a nearring with respect to an ideal, *Communications in Algebra* **38** (2010), no. 5, 1957–1967.

[13] K. B. Srinivas, J. B. Jagadeesha, and K. S. Prasad, Different prime graphs of a nearring with respect to an ideal, in *Nearrings, Nearfields and Related Topics*, World Scientific, Singapore, 2017, pp. 185–203.

[14] T. Tamizh Chelvam and S. Nithya, Zero-divisor graph of an ideal of a near-ring, *Discrete Mathematics, Algorithms and Applications* **5** (2013), no. 1, Article 1350007. doi:10.1142/S1793830913500073.

[15] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, Vol. 244, Springer, New York, 2008. doi:10.1007/978-1-84628-970-5.

[16] D. B. West, *Introduction to Graph Theory*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2001.

[17] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Universitext, Springer, New York, 2012. doi:10.1007/978-1-4614-1939-6.

[18] S. Al-Kaseasbeh and J. Coykendall, Adjacency-like conditions and induced ideal graphs, *Communications in Algebra* **51** (2023), no. 9, 3708–3728. doi:10.1080/00927872.2023.2176342.

[19] S. Al-Kaseasbeh and A. Erfanian, A bipartite graph associated to elements and cosets of subgroups of a finite group, *AIMS Mathematics* **6** (2021), no. 10, 10395–10404. doi:10.3934/math.2021603.

[20] S. Al-Kaseasbeh and A. Erfanian, The structure of Cayley graphs of dihedral groups of valencies 1, 2, and 3, *Proyecciones* **40** (2021), no. 6, 1683–1691.

[21] D. D. Anderson and S. Al-Kaseasbeh, The intersection subgroup graph of a group, *Communications in Algebra* **51** (2023), no. 9, 3556–3573. doi:10.1080/00927872.2023.2169957.

[22] A. Das, M. Saha, and S. Al-Kaseasbeh, On co-maximal subgroup graph of a group, *Ricerche di Matematica* **73** (2024), 2075–2089. doi:10.1007/s11587-023-00823-w.

[23] A. Erfanian, M. Al Tahan, and S. Al-Kaseasbeh, Hamiltonian degree of finite groups, *International Journal of Group Theory* **15** (2026), no. 4, 193–206. doi:10.22108/ijgt.2025.144951.1956.

[24] A. Erfanian, S. Al-Kaseasbeh, and M. Al Tahan, A bipartite graph associated to Hamiltonian degrees of finite groups, *Journal of Discrete Mathematical Sciences and Cryptography* **29** (2026), no. 3, 1421–1434. doi:10.47974/JDMSC-2594.

[25] M. Al Tahan, S. Hošková-Mayerová, B. Davvaz, and A. C. Sonea, On subpolygroup commutativity degree of finite polygroups, *AIMS Mathematics* **8** (2023), no. 10, 23786–23799. doi:10.3934/math.20231211.

[26] M. Al Tahan, B. Davvaz, and A. C. Sonea, Relative commutativity degree of a subpolygroup of a finite polygroup, *Journal of Mathematical Sciences* **289** (2024), no. 4, 586–600. doi:10.1007/s10958-024-07302-x.

[27] B. Davvaz, Isomorphism theorems of polygroups, *Bulletin of the Malaysian Mathematical Sciences Society* **33** (2010), no. 3, 385–392.

[28] L. K. Shehu and H. A. Khashan, New extensions of polygroups by polygroups, *Filomat* **33** (2019), no. 12, 3655–3668. doi:10.2298/FIL1912655S.

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Received: 2025-03-30
Accepted: 2026-08-08