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The Classical Kronecker Function Rings in Pullback Diagrams

Elhousseine EDDAMANE [1,2]

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Corresponding Author: Elhousseine EDDAMANE

Abstract. In this paper, we study the classical Kronecker function ring of a $D+M$ construction. We apply this to $R = D + M$, which results from $T = K + M$, a valuation domain, where $R \subseteq T$ forms a Prüfer extension.

1 Introduction

In this article, all rings are commutative with identity. Let $R \subseteq S$ be a ring extension. In [7], Knebusch and Kaiser define a star operation on a ring extension $R \subseteq S$ as a map $* : \Im(R, S) \to \Im(R, S)$, where $\Im(R, S)$ is the set of all R-submodules of S, satisfying the following conditions: for all $A, B \in \Im(R, S)$,

(i) $A \subseteq A^*$ and $(A^*)^* = A^*$.

(ii) $AB^* \subseteq (AB)^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$.

A star operation $*$ on a ring extension $R \subseteq S$ is said to be e.a.b. if the following holds: if A, B, and C are finitely generated R-submodules of S such that $AS = S$ and $(AB)^* \subseteq (AC)^*$, then $B^* \subseteq C^*$ [7, Definition 1, p. 152].

They also introduced the notion of the Kronecker function ring of a ring extension with respect to a star operation. This leads to a generalization of the classical notion of the Kronecker function ring.

The Kronecker function ring of a ring extension arising from a pullback diagram of type $\square$, corresponding to a Prüfer extension $R \subseteq S$, was introduced by Paudel and Tchamna in [9]. While the preliminary notions above hold for general commutative rings, this paper focuses on the classical Kronecker function ring of a Prüfer extension in a pullback setting. To this end, unless explicitly stated otherwise, all rings in our new results are assumed to be integral domains (and in particular, T is assumed to be a valuation domain).

2 Preliminaries

Let R be an integral domain with quotient field L. Let $F(R)$ denote the set of nonzero fractional ideals of R. A star operation on R is defined as an application $* : F(R) \to F(R)$, where $I \mapsto I^*$, such that for all $I, J \in F(R)$ and $x \in L \setminus \{0\}$, the following conditions hold:

(c_1) $R^* = R$ and $(xI)^* = xI^*$.

(c_2) $I \subseteq I^*$ and if $I \subseteq J$, then $I^* \subseteq J^*$.

(c_3) $(I^*)^* = I^*$.

A star operation $*$ is called *endlich arithmetisch brauchbar* (in short, e.a.b.) if for any finitely generated $I, J, H \in F(R)$, $(IJ)^* \subseteq (IH)^*$ implies $J^* \subseteq H^*$.

Given an e.a.b. star operation, the Kronecker function ring of R with respect to $*$ is defined by:

$$Kr(*) \text{ or } R^* = \left\{ \frac{f}{g} \mid f, g \in R[X],\ c_R(f)^* \subseteq c_R(g)^* \right\},$$

where $c_R(f)$ denotes the content of the polynomial $f(X)$ (i.e., $c_R(f)$ is the ideal of R generated by the coefficients of f). It is known that $Kr(*)$ is a Bézout domain with quotient field $K(X)$ such that $Kr(*) \cap K = R$ (see [4, Section 32]).

In particular, if b_R denotes the star operation of R defined on the fractional ideals I of R by $I \to I^{b_R} := \cap\{IV \mid V \text{ is a valuation overring of } R\}$, then Arnold proved that R is a Prüfer domain if and only if $R(X) = R^{b_R}$.

We refer to [4, Sections 32 and 34] for the basic properties of star operations and the b-operation.

For a ring S and an additive totally ordered abelian group Γ, let $\Gamma \cup \infty = \Gamma \cup \{\infty\}$, where $\infty + g = g + \infty = \infty$ for all $g \in \Gamma \cup \infty$, and $g < \infty$ for all $g \in \Gamma$. A valuation on S with values in Γ is a map $v : S \to \Gamma \cup \infty$ such that:

(1) $v(xy) = v(x) + v(y)$ for all $x, y \in S$.

(2) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in S$.

(3) $v(1) = 0$ and $v(0) = \infty$.

The set $V = \{x \in S : v(x) \geq 0\}$ is called the valuation subring of S. If $v(S) = \{0, \infty\}$, then v is said to be trivial; otherwise, v is called non-trivial. If $v(S) = \Gamma \cup \infty$, then v is called a Manis valuation on S and $V = \{x \in S : v(x) \geq 0\}$ is called a Manis subring of S.

For a subring R of S, if there exists a Manis valuation $v : S \to \Gamma \cup \infty$ such that $R = \{x \in S : v(x) \geq 0\}$, then the extension $R \subseteq S$ is called a Manis extension. In this case, (R, p) is called a Manis pair in S, where $p = \{x \in S : v(x) > 0\}$.

For a ring extension $R \subseteq S$ and an R-submodule M of S, if τ is a multiplicative subset of R, we denote by $M_{[\tau]}$ the set of all $x \in S$ such that $tx \in M$ for some $t \in \tau$.

Let $R \subseteq S$ be a ring extension. The ring S is called a Prüfer extension of R if $(R_{[p]}, p_{[p]})$ is a Manis pair in S for every maximal ideal p of R. In this case, we say that R is Prüfer in S. More on Manis valuations and Prüfer extensions can be found in [6].

For any R-submodule I of S, denote by $[R : I]$ the set of all $x \in S$ such that $xI \subseteq R$, and $(R : I) := R \cap [R : I]$. The R-submodule I of S is called S-invertible if there exists an R-submodule J of S such that $IJ = R$ [6, Definition 3, p. 90]. The inverse of I is uniquely determined and it is $I^{-1} := [R : I]$ [6, Remark 1.10, p. 90]. The R-module I is said to be S-regular if $IS = S$ [6, Definition 1, p. 84].

Lemma 2.1. *[6, Theorem 2.1, p. 94] A ring extension $R \subseteq S$ is a Prüfer extension if and only if $R \subseteq S$ is a flat epimorphism (in the category of rings) and every finitely generated S-regular R-submodule of S is S-invertible.*

Let $R \subseteq S$ be a ring extension, and let X be an indeterminate over S. For any ring A between R and S, and any $g \in S[X]$, we denote by $c_A(g)$ the A-submodule of S generated by the coefficients of g. In particular, when $A = S$, then $c_S(g)$ is the ideal of S generated by the coefficients of g. Let $T = \{g \in S[X] : c_S(g) = S\}$. Then T is a multiplicative system in $S[X]$ containing only regular elements [6, Proposition 33.1]. The quotient $(S[X])_T$ is denoted by $S(X)$. More properties of the ring $S(X)$ can be found in [6, p. 410].

Lemma 2.2. *[6, Theorem 1.1, p. 126] Let $R \subseteq S$ be a ring extension, and let X be an indeterminate over S. For $f, g \in S[X]$ and $n \geq \deg(f)$, we have the Dedekind-Mertens formula:*

$$c_R(fg)c_R(g)^n = c_R(f)c_R(g)^{n+1}.$$

Remark 2.3. [9, Remark 1.3, p. 3] Let S be a ring, and let X be an indeterminate over S. Let $\frac{f}{g} \in S(X)$ with $f, g \in S[X]$ and $c_S(g) = S$. Then $\frac{f}{g} = 0$ if and only if $f = 0$.

We now define what is a pullback diagram, which is a key tool in this paper.

Definition 2.4. Let $R \subseteq S$ and $L \subseteq T$ be two ring extensions, and consider the following commutative diagram:

$$\begin{array}{ccc} R & \longrightarrow & S \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\Psi} \\ L & \longrightarrow & T \end{array}$$

such that

a) $\ker \Psi$ is an ideal of R,

b) $\Psi : S \to T$ is surjective,

c) The restriction $\alpha : R \to L$ of Ψ is also surjective, and the horizontal mappings are inclusions.

If $\ker \Psi$ is a maximal ideal of S, the previous commutative diagram is called a pullback diagram of type $\square$.

Pullback diagrams of type $\square$ are studied by Gabelli and Houston in [3].

3 Results

In this section, we study the classical Kronecker function rings on the "$D + M$" constructions in a pullback diagram. Let T be a valuation domain of $k + M$, where M is a maximal ideal of T and k is a subring of T canonically isomorphic to the field T/M, and D be a subring of k. Set $R := D + M$, which we assume to be an integrally closed domain.

The $R = D + M$ construction is a special case of a pullback diagram of type $\square$. Let R be the pullback obtained from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle\Psi} \\ D \cong R/M & \longrightarrow & k \cong T/M \end{array}$$

For a prime ideal p of R and an R-submodule I of S, we denote by I_p the localization of I at p.

Lemma 3.1. *Let $R \subseteq S$ be a ring extension, and let $f = \sum_{k=0}^{n} a_k X^k$ and $g = \sum_{k=0}^{m} b_k X^k$, with $a_k \in S$ and $b_k \in S$. If there exists an integer $0 \leq l \leq m$ such that $c_R(g)_p^{-1} = b_l R_p$ for any prime ideal p of R, then $c_R(fg)_p = c_{R_p}(f)b_l$.*

Proof. We show that $c_R(fg)_p = c_{R_p}(fg) = c_{R_p}(fb_l) = c_{R_p}(f)b_l$.

Let $\alpha \in c_R(fg)_p$. Then $\alpha = \frac{\sum_{j=0}^{n+m} r_j c_j}{t} = \sum_{j=0}^{n+m} \left(\frac{r_j}{t}\right) c_j$, where $c_j = \sum_{q=0}^{j} a_q b_{j-q}$, with $r_j \in R$, $t \in R \setminus p$. Thus, $\alpha \in c_{R_p}(fg)$, and hence, $c_R(fg)_p \subseteq c_{R_p}(fg)$.

Conversely, let $\alpha \in c_{R_p}(fg)$. Then $\alpha = \sum_{j=0}^{n+m} \left(\frac{r_j}{t_j}\right) c_j = \sum_{j=0}^{n+m} \frac{r_j\left(\prod_{k\neq j}^{n+m} t_k\right)c_j}{t}$, where $t = \prod_{k=0}^{n+m} t_k$. Hence, $\alpha = \frac{\sum_{j=0}^{n+m} r_j\left(\prod_{k\neq j}^{n+m} t_k\right)c_j}{t} \in c_R(fg)_p$. Therefore, $c_{R_p}(fg) \subseteq c_R(fg)_p$.

It follows that $c_R(fg)_p = c_{R_p}(fg)$.

Next, if $\alpha \in c_{R_p}(fg)$, we have:

$$\alpha = \sum_{j=0}^{n+m} \left(\frac{r_j}{t_j}\right) c_j = \sum_{j=0}^{n+m} \left(\frac{r_j}{t_j}\right) \sum_{q=0}^{j} a_q b_{j-q} = \sum_{j=0}^{n+m} \sum_{q=0}^{j} \left(\frac{r_j}{t_j}\right) a_q b_{j-q}.$$

Since $b_{j-q} \in c_R(g)_p^{-1} = b_l R_p$, we have $b_{j-q} = b_l \beta_{j-q}$, where $\beta_{j-q} \in R_p$. Thus,

$$\alpha = \sum_{j=0}^{n+m} \sum_{q=0}^{j} \left(\frac{r_j}{t_j}\beta_{j-q}\right) (a_q b_l) = \sum_{j=0}^{n+m} \sum_{q=0}^{j} \left(\frac{r_j}{t_j}\beta_{j-q}\right) (a_q b_l)$$

is an element of $\sum_{q=0}^{n} R_p(a_q b_l)$ because $a_q = 0$ for $q > n$. Hence, $\alpha \in c_{R_p}(fb_l)$, and thus, $c_{R_p}(fg) \subseteq c_{R_p}(fb_l)$.

Conversely, if $\alpha \in c_{R_p}(fb_l)$, we have $\alpha \in \sum_{j=0}^{n} R_p(a_j b_l) \subseteq \sum_{j=0}^{n+m} R_p c_j = c_{R_p}(fg)$. Therefore, $c_{R_p}(fb_l) \subseteq c_{R_p}(fg)$.

Thus, $c_{R_p}(fg) = c_{R_p}(fb_l)$. □

Proposition 3.2. *Consider a pullback diagram of type □. If $R \subseteq T$ is a Prüfer extension, then $MR^{b_R} = MT(X)$.*

Proof. Let $R^{b_R} = \left\{ \frac{f}{g} \mid f, g \in R[X],\ c_R(f)^{b_R} \subseteq c_R(g)^{b_R} \right\}$.

Let T be a valuation domain. Since T is a Prüfer domain, we have $T(X) = T^{b_T}$. Since $R \subseteq T$, by Proposition [2, Proposition 4, p. 267], we know that $R^{b_R} \subseteq T^{b_T}$. Hence, $R^{b_R} \subseteq T(X)$, and the inclusion $R^{b_R} \hookrightarrow T(X)$ holds. Thus, $MR^{b_R} \subseteq MT(X)$.

For the containment $MT(X) \subseteq MR^{b_R}$, consider an element of the form $a = \sum_{i=1}^{r} m_i \frac{f_i}{g_i}$, where $m_i \in M$ and $\frac{f_i}{g_i} \in T(X)$.

For such an element, write $f_i = \sum_{k=0}^{n} a_k^i X^k$ and $g_i = \sum_{l=0}^{m} b_l^i X^l$, with $a_k^i, b_l^i \in T$ for $0 \leq k \leq n$ and $0 \leq l \leq m$. Since $c_T(g_i) = T$, we have $c_R(g_i)T = T$. Since $R \subseteq T$ is a Prüfer extension, by Lemma 2.1, the R-submodule $c_R(g_i)$ is T-invertible. Hence, $c_R(g_i)c_R(g_i)^{-1} = R$, and $c_R(g_i)^{-1} = [R : c_R(g_i)]$.

By Remark [6, Remark 1.10, p. 90], let $v_0^i, \ldots, v_l^i \in T$ such that $c_R(g_i)^{-1} = \sum_{k=0}^{l} R v_k^i$, and let $h_i = \sum_{l=0}^{l} v_l^i X^l$. By Lemma 2.2, we have $c_R(g_i h_i)c_R(h_i)^e = c_R(g_i)c_R(h_i)^{e+1}$ for some positive integer $e \geq m$, and $c_R(g_i h_i)c_R(h_i)^e T = c_R(g_i)c_R(h_i)^{e+1}T$.

Since $c_R(g_i)T = T$ and $c_R(g_i)c_R(g_i)^{-1} = c_R(g_i)c_R(h_i) = R$, it follows that $c_R(h_i)T = T$, and hence $c_R(g_i h_i)T = T$. Let p be a maximal ideal of R. By Proposition [6, Proposition 2.3, p. 97], we have $c_R(g_i)_p^{-1} = v_k^i R_p$ for some $1 \leq k \leq l$. By Lemma 3.1, we have $c_R(g_i h_i)_p = c_{R_p}(g_i)v_k^i$, and since $c_R(g_i)c_R(g_i)^{-1} = R$, we have $R_p = c_R(g_i)_p c_R(g_i)_p^{-1} = c_{R_p}(g_i)v_k^i$.

Thus, $c_R(g_i h_i)_p = R_p$, and by Lemma [6, Lemma 1.1, p. 85], we have $c_R(g_i h_i) = R$. Hence, $c_R(1)^{b_R} \subseteq c_R(g_i h_i)^{b_R}$, and so $\frac{1}{g_i h_i} \in R^{b_R}$. Therefore,

$$a = \sum_{i=1}^{r} m_i f_i h_i \frac{1}{g_i h_i} \in M[X]R^{b_R}.$$

This shows that $MT(X) \subseteq M[X]R^{b_R}$.

Now, let $h = \sum_{k=0}^{d} c_k X^k \in M[X]$ with $c_k \in M$, and let $\frac{f}{g} \in R^{b_R}$. Since $c_R(fX^k)^{b_R} = c_R(f)^{b_R} \subseteq c_R(g)^{b_R}$, we have $\frac{fX^k}{g} \in R^{b_R}$ for all k. Thus,

$$h \cdot \frac{f}{g} = \sum_{k=0}^{d} c_k \frac{fX^k}{g} \in MR^{b_R}.$$

This proves that $M[X]R^{b_R} \subseteq MR^{b_R}$, and so $MT(X) \subseteq MR^{b_R}$. Since the reverse inclusion $MR^{b_R} \subseteq MT(X)$ is obvious, we conclude that $MR^{b_R} = MT(X)$. □

Proposition 3.3. *Consider a pullback diagram of type □, and let $*$ be the map from $F(D)$ to $F(D)$ defined in the following way: for any $A \in F(D)$, $A^* = \Psi((A+M)^{b_R})$. Then:*

(i) $*$ is a star operation e.a.b. on D.

(ii) For each $A \in F(R)$, $\Psi(A)^* = \Psi(A^{b_R})$.

Proof. 1) If $A \in F(D)$, then $A + M \in F(R)$, so $(A+M)^{b_R} \in F(R)$ and $(A+M)^{b_R} = B + M$ with $B \in F(D)$, and so $(A+M)^{b_R} \subseteq T$, ($B \subseteq k \Rightarrow B + M \subseteq k + M = T$), then $\Psi((A+M)^{b_R}) \subseteq \Psi(T) = k$, so $A^* = \Psi((A+M)^{b_R}) \in F(D)$. Clearly, if $A = B$, then $A^* = B^*$. This shows that $*$ is well defined.

Since $R = D + M$, for every $I \in F(R)$, there exists $I' \in F(D)$ such that $I = I' + M$.. Let $\alpha \in k - 0$ and $I' \in F(D)$, we have $(\alpha I')^* = \Psi((\alpha I' + M)^{b_R}) = \Psi((\alpha(I' + M))^{b_R}) = \Psi(\alpha(I' + M)^{b_R}) = \Psi(\alpha)\Psi(I' + M)^{b_R}) = \alpha(I')^*$; ($\forall \alpha \in k; \Psi(\alpha) = \alpha$), then $(\alpha I')^* = \alpha(I')^*$.

Trivially, $D^* = \Psi((D+M)^{b_R}) = \Psi(R^{b_R}) = \Psi(R) = D$.

If $A \subseteq B$ with $A, B \in F(D)$, then $A + M \subseteq B + M$, so $(A+M)^{b_R} \subseteq (B+M)^{b_R}$, so $\Psi((A+M)^{b_R}) \subseteq \Psi((B+M)^{b_R})$, then $A^* \subseteq B^*$.

Let $A \in F(D)$, we have $A + M \subseteq (A+M)^{b_R}$, so $\Psi(A+M) \subseteq \Psi((A+M)^{b_R})$, then $A \subseteq A^*$.

Let $A \in F(D)$, we have $A^* = \Psi((A+M)^{b_R})$, since $(A+M)^{b_R} \in F(R)$, then $(A+M)^{b_R} = B + M$ with $B \in F(D)$, so $A^* = \Psi((A+M)^{b_R}) = \Psi(B+M) = B$, since $((A+M)^{b_R})^{b_R} = (A+M)^{b_R}$, then $(A+M)^{b_R} = (B+M)^{b_R} = (A^*+M)^{b_R}$, then $A^* = \Psi((A+M)^{b_R}) = \Psi((A^*+M)^{b_R}) = (A^*)^*$, then $(A^*)^* = A^*$.

The above shows that $*$ is a (Well defined) star operation on integral domain D.

For any finitely generated $A, B, C \in F(D)$ such that $(AB)^* \subseteq (AC)^*$, then $A+M, B+M, C+M$ are finitely generated, since $(AB)^* \subseteq (AC)^*$, then $\Psi((AB+M)^{b_R}) \subseteq \Psi((AC+M)^{b_R})$, so $\Psi((A+M)(B+M))^{b_R}) \subseteq \Psi((A+M)(C+M))^{b_R})$. Since $Ker(\Psi) = M \subseteq (A+M)(C+M))^{b_R}$, then by Remark [9, Remark 2.5,p.7], $(A+M)(B+M))^{b_R} \subseteq (A+M)(C+M))^{b_R}$, then $(B+M)^{b_R} \subseteq (C+M)^{b_R}$, because b_R is e.a.b., then $\Psi((B+M))^{b_R}) \subseteq \Psi((C+M))^{b_R})$, so $B^* \subseteq C^*$. It follows that $*$ is e.a.b.

2) Let $A \in F(R)$, we have $A = I + M \mid I \in F(D)$, then $\Psi(A)^* = \Psi(I+M)^* = I^* = \Psi((I+M)^{b_R}) = \Psi(A^{b_R})$. □

Let $\overline{\Psi} : T(X) \to (T/M)(X)$, $\frac{f}{g} \mapsto \frac{\overline{f}}{\overline{g}}$ be the extension of the canonical map $\Psi : T \to k = T/M$.

Let $*$ be a star operation on D, and let b_R be a b-operation on R.

We denote by $\overline{\alpha} : R^{b_R} \to D^*$, $\frac{f}{g} \mapsto \frac{\overline{f}}{\overline{g}}$ as the restriction of $\overline{\Psi}$ to R^{b_R}.

The vertical maps $R^{b_R} \hookrightarrow T(X)$ and $D^* \hookrightarrow (T/M)(X)$ are inclusion maps.

Proposition 3.4. *Let T be a Prüfer extension of R and $*$ be a star operation on D defined in Proposition 3.3. Then:*

(i) $\overline{\Psi}$ and $\overline{\alpha}$ are well-defined.

(ii) $\overline{\Psi}$ and $\overline{\alpha}$ are surjective.

(iii) $T(X)/MT(X) \cong (T/M)(X)$ and $D^* \cong R^{b_R}/MR^{b_R}$.

Proof. 1) Let $f = \sum_{k=0}^{n} a_k X^k$ and $g = \sum_{k=0}^{m} b_k X^k$ with $a_k, b_k \in T$. Write $\overline{f} = \sum_{k=0}^{n} \overline{a_k} X^k$, $\overline{g} = \sum_{k=0}^{m} \overline{b_k} X^k$.

If $\frac{f}{g} \in T(X)$, then $c_T(g) = T$, so $\Psi(c_T(g)) = \Psi(T)$, and we have $\Psi(b_0 T + \cdots + b_m T) = \Psi(T)$. This implies $\Psi(b_0)\Psi(T) + \cdots + \Psi(b_m)\Psi(T) = T/M$, so $\overline{b_0}T/M + \cdots + \overline{b_m}T/M = T/M$. Hence, $c_{T/M}(\overline{g}) = T/M$, and therefore $\frac{\overline{f}}{\overline{g}} \in (T/M)(X)$. It is clear that if $\frac{f}{g} = \frac{f'}{g'}$ with $f, g, f', g' \in T[X]$, then $\frac{\overline{f}}{\overline{g}} = \frac{\overline{f'}}{\overline{g'}}$. This shows that $\overline{\Psi}$ is well-defined.

Let $\frac{f}{g} \in R^{b_R}$. We have $f, g \in R[X]$ and $c_R(f)^{b_R} \subseteq c_R(g)^{b_R}$. Then $\overline{f}, \overline{g} \in D[X]$ and $\Psi(c_R(f)^{b_R}) \subseteq \Psi(c_R(g)^{b_R})$. Thus, $\frac{\overline{f}}{\overline{g}} \in D^*$, and it is clear that if $\frac{f}{g} = \frac{f'}{g'}$ with $f, g, f', g' \in R[X]$, then $\frac{\overline{f}}{\overline{g}} = \frac{\overline{f'}}{\overline{g'}}$. This shows that $\overline{\alpha}$ is well-defined.

2) Let $\frac{u}{v} \in k(X) = (T/M)(X)$ with $u = \sum_{i=0}^{n} \overline{a_i} X^i$ and $v = \sum_{j=0}^{m} \overline{b_j} X^j \in (T/M)[X]$, and $c_k(v) = k$. Write $f = \sum_{i=0}^{n} a_i X^i$ and $g = \sum_{j=0}^{m} b_j X^j$, where $a_i, b_j \in T$. Then, $c_D(v)k = (\overline{b_0}D + \cdots + \overline{b_m}D)k = \overline{b_0}k + \cdots + \overline{b_m}k = k$, since $\overline{1} \in k = T/M$.

Thus, $\overline{1} = \sum_{i=1}^{l} y_i \overline{t_i}$ where $y_i \in c_D(v)$ and $t_i \in T$. Since $y_i \in c_D(v)$, we have $y_i = \sum_{k=0}^{m} \overline{b_k} \overline{r_k^i}$, with $b_k \in T$ and $r_k^i \in R$. Thus, $\overline{1} = \overline{\sum_{i=1}^{l} \sum_{k=0}^{m} b_k r_k^i t_i}$. Then, $1 - \sum_{i=1}^{l} \sum_{k=0}^{m} b_k r_k^i t_i \in M$, so $\sum_{i=1}^{l} \sum_{k=0}^{m} b_k r_k^i t_i$ is a unit in T. Therefore, there exists $x \in T$ such that $(\sum_{i=1}^{l} \sum_{k=0}^{m} b_k r_k^i t_i)x = 1$. This shows $c_R(g)T = T$, implying $c_T(g) = T$.

Thus, $\frac{f}{g} \in T(X)$, and since $\overline{\Psi}(\frac{f}{g}) = \frac{\overline{f}}{\overline{g}} = \frac{u}{v}$, we conclude that $\overline{\Psi}$ is surjective.

Let $\frac{u}{v} \in D^*$ with $u = \sum_{i=0}^{n} \overline{a_i} X^i$ and $v = \sum_{j=0}^{m} \overline{b_j} X^j \in D[X]$, with $a_i, b_j \in R$. Write $f = \sum_{i=0}^{n} a_i X^i$ and $v = \sum_{j=0}^{m} b_j X^j$. Since $c_{T/M}(v) = T/M$, we have $c_{R/M}(v)(T/M) = T/M$, and so $\overline{1} = \sum_{i=1}^{l} y_i \overline{t_i}$ where $y_i \in c_D(v)$ and $t_i \in T$.

Thus, $\overline{1} = \overline{\sum_{i=1}^{l}\sum_{k=0}^{m} b_k r_k^i t_i}$, which implies $1 - \sum_{i=1}^{l}\sum_{k=0}^{m} b_k r_k^i t_i \in M$. Therefore, $\sum_{i=1}^{l}\sum_{k=0}^{m} b_k r_k^i t_i$ is a unit in T. Hence, $c_R(g)T = T$, which implies $c_T(g) = T$.

Thus, $\frac{f}{g} \in R^{b_R}$. Since $\Psi(c_R(f)) = c_D(u)$ and $\Psi(c_R(g)) = c_D(v)$, and $(c_D(u))^* \subseteq (c_D(v))^*$, by Proposition 3.3(ii), we obtain $\Psi(c_R(f)^{b_R}) \subseteq \Psi(c_R(g)^{b_R})$. Since Ψ is surjective, we conclude that $\overline{\alpha}$ is surjective.

3) Since $\overline{\Psi}$ is surjective, we have $T(X)/\ker(\overline{\Psi}) \cong (T/M)(X)$. Let $\frac{f}{g} = \sum_{i=1}^{l} u_i \frac{f_i}{g_i} \in MT(X)$, where $u_i \in M$ and $\frac{f_i}{g_i} \in T(X)$. We have $\overline{\Psi}(\frac{f}{g}) = \sum_{i=1}^{l} \overline{\Psi}(u_i \frac{f_i}{g_i})$, and since $u_i a_k^i \in M$, we conclude that $\overline{\Psi}(\frac{u_i f_i}{g_i}) = \overline{0}$.

Thus, $\overline{\Psi}(\frac{f}{g}) = \overline{0}$, which implies $\frac{f}{g} \in \ker(\overline{\Psi})$, and hence $MT(X) \subseteq \ker(\overline{\Psi})$. Similarly, it can be shown that $\ker(\overline{\Psi}) \subseteq MT(X)$, so $\ker(\overline{\Psi}) = MT(X)$. Therefore, $T(X)/MT(X) \cong (T/M)(X)$.

Let $\frac{f}{g} \in R^{b_R}$ such that $\overline{\alpha}(\frac{f}{g}) = \overline{0}$. Then $\overline{\alpha}(\frac{f}{g}) = \overline{0}$ implies that $f = \sum_{i=0}^{n} a_i X^i$ and $g = \sum_{j=0}^{n} b_j X^j$, where $a_i, b_j \in R$. Since $\overline{f} = \overline{0}$ and R/M is a domain, it follows that $a_i \in M$. Therefore, $\frac{f}{g} \in MT(X)$, so $MR^{b_R} \subseteq \ker(\overline{\alpha})$.

Finally, since $\overline{\alpha}$ is surjective, we conclude that $R^{b_R}/MR^{b_R} \cong D^*$. □

Theorem 3.5. *Consider a pullback diagram of type □, and suppose that $R \subseteq T$ is a Prüfer extension. Let $*$ be the star operation of D defined in Proposition 3.4. Then the diagram*

$$\begin{array}{ccc} R^{b_R} & \xrightarrow{\ i_1\ } & T(X) \\ \downarrow{\scriptstyle \overline{\alpha}} & & \downarrow{\scriptstyle \overline{\Psi}} \\ D^* & \xrightarrow{\ i_2\ } & (T/M)(X) \end{array}$$

is a pullback diagram of type □.

Proof. We show first that the diagram commutes. Let $\frac{f}{g} \in R^{b_R} \subseteq T(X)$, we have $\overline{\Psi}(\frac{f}{g}) = \frac{\overline{f}}{\overline{g}} \in (T/M)(X)$ and $\overline{\alpha}(\frac{f}{g}) = \frac{\overline{f}}{\overline{g}} \in D^* \subseteq (T/M)(X)$, then for all $\frac{f}{g} \in R^{b_R}$:

$$\overline{\Psi}(\frac{f}{g}) = \overline{\alpha}(\frac{f}{g}).$$

Since for all $\frac{f}{g} \in R^{b_R}$, $i_1(\frac{f}{g}) = \frac{f}{g}$, then $\overline{\Psi}(i_1(\frac{f}{g})) = \overline{\alpha}(\frac{f}{g})$. Since for all $\frac{f}{g} \in R^{b_R}$, $\overline{\alpha}(\frac{f}{g}) \in D^*$, then for all $\frac{f}{g} \in R^{b_R}$, we have $i_2(\overline{\alpha}(\frac{f}{g})) = \overline{\alpha}(\frac{f}{g})$. Therefore, for all $\frac{f}{g} \in R^{b_R}$,

$$\overline{\Psi} \circ i_1(\frac{f}{g}) = i_2 \circ \overline{\alpha}(\frac{f}{g}).$$

Thus, the diagram commutes.

Now, since the extension $R \subseteq T$ is Prüfer, it follows from Proposition 3.2 and Proposition 3.4 that $MT(X)$ is a common ideal of R^{b_R} and $T(X)$. Since M is a maximal ideal of T, by [4, Proposition 33.1], $MT(X)$ is a maximal ideal of $T(X)$. By Proposition 3.4, we have $D^* \cong R^{b_R}/MT(X)$. Therefore, we have a pullback diagram of type □. □

Example 3.6. Let T be a valuation domain of the form $k + M$, where $M = 2\mathbb{Z}$ is the maximal ideal of T and $k = \mathbb{Q}$ is a subring of T canonically isomorphic to the field T/M, and $D = \mathbb{Z}$ is a subring of k with field of quotients $\mathrm{Frac}(D) = k = \mathbb{Q}$. Let $R := D + M$.

- $D = \mathbb{Z}$ is a Prüfer domain and the field of fractions of D is $\mathrm{Frac}(D) = \mathbb{Q}$.
- $M = 2\mathbb{Z}$ is a maximal ideal of T.
- $R = D + M = \mathbb{Z} + 2\mathbb{Z}$.
- $T = k + M = \mathbb{Q} + 2\mathbb{Z}$ and $T/M \cong \mathbb{Q}$.

Since D is a Prüfer domain, by [6, Example 5.17, p. 55], we have $D = R/M \subseteq k = T/M$ is a Prüfer extension. So by [6, Proposition 5.8, p. 52], $R \subseteq T$ is a Prüfer extension.

Let $*$ be the map from $F(\mathbb{Z})$ to $F(\mathbb{Z})$ defined in the following way: for any $A \in F(\mathbb{Z})$, $A^* = \Psi((A + 2\mathbb{Z})^{b_R})$.

We have $*$ is a star operation on $\mathbb{Z}$. Then the diagram

$$\begin{array}{ccc} (\mathbb{Z}+2\mathbb{Z})^{b_R} & \xrightarrow{\ i_1\ } & (\mathbb{Q}+2\mathbb{Z})(X) \\ \Big\downarrow{\scriptstyle \overline{\alpha}} & & \Big\downarrow{\scriptstyle \overline{\Psi}} \\ \mathbb{Z}^* & \xrightarrow{\ i_2\ } & \mathbb{Q}(X) \end{array}$$

is a pullback diagram of type $\square$.

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Author information

Elhousseine EDDAMANE [1,2], [1] Laboratoire d'Analyse Mathématique, Algèbre et Applications, Faculté des Sciences Aïn Chock, Hassan II University of Casablanca, Morocco [2] Equipe STEF, Centre Régional des Métiers de l'Education et de la Formation Casablanca-Settat, Morocco.
E-mail: eddamane1979@gmail.com

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