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A REMARK ON NONLINEAR MIXED $*$-JORDAN TYPE DERIVATIONS ON $*$-ALGEBRAS

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Abstract. Let $\mathcal{R}$ be a $*$-algebra with the unity and a nontrivial projection. In the present paper, we show that under the nondegeneracy assumptions and (2.1) - (2.2), if a map $\Psi : \mathcal{R} \to \mathcal{R}$ satisfies $\Psi(E \diamond K \circ D) = \Psi(E) \diamond K \circ D + E \diamond \Psi(K) \circ D + E \diamond K \circ \Psi(D)$ for all $E, K, D \in \mathcal{R}$, then Ψ is an additive $*$-derivation. As applications, this result is investigated in different algebras.

1 Introduction

Let $\mathcal{R}$ be a $*$-algebra with unity over the field $\mathbb{C}$ of complex numbers. For $E, K \in \mathcal{R}$, $E \circ K = EK + KE$ is called the Jordan product, $[E, K] = EK - KE$ is called the Lie product and similarly $E \bullet K = EK + KE^*$ is the Jordan $*$-product, $[E, K]_* = EK - KE^*$ is the skew Lie $*$-product and $E \diamond K = E^*K + K^*E$ denotes the bi-skew Jordan product of E and K respectively. An additive map $\Psi : \mathcal{R} \to \mathcal{R}$ is called an additive derivation if $\Psi(EK) = \Psi(E)K + E\Psi(K)$ for all $E, K \in \mathcal{R}$. Moreover, if $\Psi(E^*) = \Psi(E)^*$ holds for all $E \in \mathcal{R}$, then Ψ is called an additive $*$-derivation. Let $\Psi : \mathcal{R} \to \mathcal{R}$ be a mapping (not necessarily additive). Then Ψ is called a nonlinear Jordan $*$-derivation if

$$\Psi(E \bullet K) = \Psi(E) \bullet K + E \bullet \Psi(K)$$

holds for all $E, K \in \mathcal{R}$. A map (not necessarily additive) $\Psi : \mathcal{R} \to \mathcal{R}$ is known as a nonlinear Jordan triple $*$-derivation if

$$\Psi(E \bullet K \bullet D) = \Psi(E) \bullet K \bullet D + E \bullet \Psi(K) \bullet D + E \bullet K \bullet \Psi(D)$$

holds for all $E, K, D \in \mathcal{R}$. Also, a map (not necessarily additive) $\Psi : \mathcal{R} \to \mathcal{R}$ is known as a nonlinear mixed Jordan triple derivation if

$$\Psi(E \circ K \bullet D) = \Psi(E) \circ K \bullet D + E \circ \Psi(K) \bullet D + E \circ K \bullet \Psi(D)$$

holds for all $E, K, D \in \mathcal{R}$ (for details see [16]).

Along similar lines, a map (not necessarily additive) $\Psi : \mathcal{R} \to \mathcal{R}$ is called a nonlinear mixed $(\diamond, \circ)$-Jordan triple derivation if

$$\Psi(E \diamond K \circ D) = \Psi(E) \diamond K \circ D + E \diamond \Psi(K) \circ D + E \diamond K \circ \Psi(D)$$

holds for all $E, K, D \in \mathcal{R}$.

From the past few years, the study of Jordan product, Lie product, Jordan $*$-product, skew Jordan product, bi-skew Jordan product, mixed Lie and Jordan triple products has attracted the attention of many algebraists (see references [2, 3, 5, 8, 9, 10, 11, 12, 13, 14, 19, 18, 20, 21, 22,

24, 25]). In 2020, Darvish et al. [4] showed that if $\mathcal{R}$ is a prime $*$-algebra and $\Psi : \mathcal{R} \to \mathcal{R}$ is a map such that

$$\Psi(E \diamond K \diamond D) = \Psi(E) \diamond K \diamond D + E \diamond \Psi(K) \diamond D + E \diamond K \diamond \Psi(D)$$

holds for all $E, K, D \in \mathcal{R}$, then Ψ is an additive $*$-derivation. Here, it is to be noted that all the mixed products mentioned above are non-associative in the order written as, i.e;
$E \bullet K \bullet D \equiv (E \bullet K) \bullet D$; $E \circ K \bullet D \equiv (E \circ K) \bullet D$; $E \diamond K \circ D \equiv (E \diamond K) \circ D$; $E \diamond K \diamond D \equiv (E \diamond K) \diamond D$.
Yaoxian et al. [23] studied the structure of nonlinear mixed Lie triple derivation on factor von Neumann algebras and proved that every nonlinear mixed Lie triple derivation on factor von Neumann algebra is an additive $*$-derivation. Zhou et al. [26], extended their result to prime $*$-algebras and obtained the same conclusion. Furthermore, Rehman et al.[16] demonstrated that any nonlinear Jordan triple derivation on a unital $*$-algebra is an additive $*$-derivation. Very recently, Siddeeque et al. [17] proved that under certain mild conditions, every nonlinear mixed skew triple derivation is an additive $*$-derivation.

In this paper, we examine nonlinear mixed $(\diamond, \circ)$-Jordan triple derivations on unital $*$-algebras under the influence of the aforementioned outcomes. Specifically, we demonstrate that every nonlinear mixed $(\diamond, \circ)$-Jordan triple derivation on $*$-algebra is an additive $*$-derivation under certain conditions. Finally, the mixed $(\diamond, \circ)$- product studied in this paper provides a unified framework for the characterization and investigation of various other single and mixed products, including mixed Lie–Jordan triple products, mixed biskew Jordan products, and skew Lie products, on different algebraic structures. This formulation allows flexible interactions among mixed operations, subsumes earlier results as special cases, and facilitates the study of their algebraic properties in diverse algebraic settings.

2 Main result

To begin with, we introduce the lemma that is crucial for proving our main theorem.

Lemma 2.1 (Peirce multiplication table). *Let $\mathcal{R}$ be a unital $*$-algebra and let $P_1, P_2 \in \mathcal{R}$ be two orthogonal projections such that $P_1 + P_2 = I$. For $i, j = 1, 2$, set $\mathcal{R}_{ij} = P_i \mathcal{R} P_j$. Then the following Peirce multiplication relations hold:*

$$\mathcal{R}_{ij}\mathcal{R}_{kl} \subseteq \begin{cases} \mathcal{R}_{il}, & \text{if } j = k, \\ \{0\}, & \text{if } j \neq k. \end{cases}$$

In particular,

$$\begin{aligned} &\mathcal{R}_{11}\mathcal{R}_{11} \subseteq \mathcal{R}_{11}, \qquad \mathcal{R}_{22}\mathcal{R}_{22} \subseteq \mathcal{R}_{22}, \\ &\mathcal{R}_{11}\mathcal{R}_{12} \subseteq \mathcal{R}_{12}, \qquad \mathcal{R}_{12}\mathcal{R}_{22} \subseteq \mathcal{R}_{12}, \\ &\mathcal{R}_{22}\mathcal{R}_{21} \subseteq \mathcal{R}_{21}, \qquad \mathcal{R}_{21}\mathcal{R}_{11} \subseteq \mathcal{R}_{21}, \\ &\mathcal{R}_{12}\mathcal{R}_{21} \subseteq \mathcal{R}_{11}, \qquad \mathcal{R}_{21}\mathcal{R}_{12} \subseteq \mathcal{R}_{22}, \\ &\mathcal{R}_{12}\mathcal{R}_{11} = \mathcal{R}_{22}\mathcal{R}_{12} = \mathcal{R}_{21}\mathcal{R}_{22} = \mathcal{R}_{11}\mathcal{R}_{21} = \{0\}. \end{aligned}$$

Moreover, the algebra $\mathcal{R}$ admits the Peirce decomposition

$$\mathcal{R} = \mathcal{R}_{11} \oplus \mathcal{R}_{12} \oplus \mathcal{R}_{21} \oplus \mathcal{R}_{22}.$$

Our main theorem is as follows:

Theorem 2.2. *Let $\mathcal{R}$ be a unital $*$-algebra with a nontrivial projection P_1 satisfying*

$$K\mathcal{R}P_1 = 0 \quad \Longrightarrow \quad K = 0 \tag{2.1}$$

and

$$K\mathcal{R}(I - P_1) = 0 \quad \Longrightarrow \quad K = 0, \tag{2.2}$$

where K is an element of the unital $*$-algebra $\mathcal{R}$. Suppose that a map $\Psi : \mathcal{R} \to \mathcal{R}$ satisfies the condition

$$\Psi(E \diamond K \circ D) = \Psi(E) \diamond K \circ D + E \diamond \Psi(K) \circ D + E \diamond K \circ \Psi(D)$$

for all $E, K, D \in \mathcal{R}$. *Then* Ψ *is additive. Moreover, if* $\Psi(P_1)$ *is self-adjoint, then* Ψ *is a* $*$*-derivation.*

Proof. Let P_1 be a nontrivial projection in $\mathcal{R}$ and set $P_2 = I - P_1$. For $i, j = 1, 2$, define $\mathcal{R}_{ij} = P_i \mathcal{R} P_j$. Then, by the Peirce decomposition of $\mathcal{R}$, we have $\mathcal{R} = \mathcal{R}_{11} \oplus \mathcal{R}_{12} \oplus \mathcal{R}_{21} \oplus \mathcal{R}_{22}$. Note that any $E \in \mathcal{R}$ can be written as $E = E_{11} + E_{12} + E_{21} + E_{22}$, where $E_{ij} \in \mathcal{R}_{ij}$ for $i, j = 1, 2$. Now to show the additivity of Ψ on $\mathcal{R}$, we use the above partition of $\mathcal{R}$ and establish some lemmas that will show that Ψ is additive on each $\mathcal{R}_{ij}$ for $i, j = 1, 2$. Also the following multiplicative relations are satisfied:
(i) $\mathcal{R}_{ij}\mathcal{R}_{jl} \subseteq \mathcal{R}_{il} \quad (i, j, l = 1, 2)$,
(ii) $\mathcal{R}_{ij}\mathcal{R}_{kl} = 0 \ (k = 1, 2), \ \text{if } j \neq k$.

□

Thus, the above theorem is a consequence of the following lemmas.

Lemma 2.3. $\Psi(0) = 0$.

Proof. It is trivial that
$\Psi(0) = \Psi(0 \diamond 0 \circ 0) = \Psi(0) \diamond 0 \circ 0 + 0 \diamond \Psi(0) \circ 0 + 0 \diamond 0 \circ \Psi(0) = 0$. □

Lemma 2.4. Ψ *is additive.*

We will complete the proof of Lemma 2.4 in following steps.

Step 2.1. Let $E_{12} \in \mathcal{R}_{12}$ and $E_{21} \in \mathcal{R}_{21}$. Then $\Psi(E_{12} + E_{21}) = \Psi(E_{12}) + \Psi(E_{21})$.

Proof. Let $K = \Psi(E_{12} + E_{21}) - \Psi(E_{12}) - \Psi(E_{21})$. We have

$$\begin{aligned} P_2 \diamond E_{12} \circ I &= \{P_2^* E_{12} + E_{12}^* P_2\} \circ I \\ &= \{P_2 E_{12} + P_2^* E^* P_1^* P_2\} \circ I \\ &= \{P_2 P_1 E P_2 + P_2 E^* P_1 P_2\} \circ I \\ &= \{0 + 0\} \circ I \\ &= 0. \end{aligned}$$

Invoking Lemma 2.3, we have

$$\begin{aligned} \Psi(P_2 \diamond (E_{12} + E_{21}) \circ I) &= \Psi(P_2 \diamond E_{12} \circ I) + \Psi(P_2 \diamond E_{21} \circ I) \\ &= \Psi(P_2) \diamond E_{12} \circ I + P_2 \diamond \Psi(E_{12}) \circ I + P_2 \diamond E_{12} \circ \Psi(I) \\ &+ \Psi(P_2) \diamond E_{21} \circ I + P_2 \diamond \Psi(E_{21}) \circ I + P_2 \diamond E_{21} \circ \Psi(I). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \Psi(P_2 \diamond (E_{12} + E_{21}) \circ I) &= \Psi(P_2) \diamond (E_{12} + E_{21}) \circ I + P_2 \diamond \Psi(E_{12} + E_{21}) \circ I \\ &+ P_2 \diamond (E_{12} + E_{21}) \circ \Psi(I). \end{aligned}$$

From the last two relations, we infer that $P_2 \diamond K \circ I = 0$, i.e., $P_2 K + K^* P_2 = 0$. Multiplying the previous relation by P_1 from right, we get $P_2 K P_1 = 0$. Analogously, interchanging P_2 by P_1 and E_{12} by E_{21} and solve as above, we get $P_1 \diamond K \circ I = 0$, i.e., $P_1 K + K^* P_1 = 0$. Multiplying the previous relation by P_2 from right, we get $P_1 K P_2 = 0$.
Now, again $I \diamond (P_2 - P_1) \circ E_{21} = 0$ and invoking Lemma 2.3, we have

$$\begin{aligned} \Psi(I \diamond (P_2 - P_1) \circ (E_{12} + E_{21})) &= \Psi(I \diamond (P_2 - P_1) \circ E_{12}) + \Psi(I \diamond (P_2 - P_1) \circ E_{21}) \\ &= \Psi(I) \diamond (P_2 - P_1) \circ E_{12} + I \diamond \Psi(P_2 - P_1) \circ E_{12} \\ &+ I \diamond (P_2 - P_1) \circ \Psi(E_{12}) + \Psi(I) \diamond (P_2 - P_1) \circ E_{21} \\ &+ I \diamond \Psi(P_2 - P_1) \circ E_{21} + I \diamond (P_2 - P_1) \circ \Psi(E_{21}). \end{aligned}$$

On the other hand, we have

$$\Psi(I \diamond (P_2 - P_1) \circ (E_{12} + E_{21})) = \Psi(I) \diamond (P_2 - P_1) \circ (E_{12} + E_{21}) + I \diamond \Psi(P_2 - P_1) \circ (E_{12} + E_{21}) + I \diamond (P_2 - P_1) \circ \Psi(E_{12} + E_{21}).$$

From the last two relations, we find that $I \diamond (P_2 - P_1) \circ K = 0$, i.e., $2P_2K - 2P_1K + 2KP_2 - 2KP_1 = 0$. Multiplying the previous relation by P_1 from both left and right, we get $P_1KP_1 = 0$. Analogously, multiplying the previous relation by P_2 from both left and right, we get $P_2KP_2 = 0$. Hence, $K = 0$, i.e., $\Psi(E_{12} + E_{21}) = \Psi(E_{12}) + \Psi(E_{21})$. □

Step 2.2. For every $E_{11} \in \mathcal{R}_{11}, E_{12} \in \mathcal{R}_{12}$, $E_{21} \in \mathcal{R}_{21}$ and $E_{22} \in \mathcal{R}_{22}$, we have

(i) $\Psi(E_{11} + E_{12} + E_{21}) = \Psi(E_{11}) + \Psi(E_{12}) + \Psi(E_{21})$;

(ii) $\Psi(E_{12} + E_{21} + E_{22}) = \Psi(E_{12}) + \Psi(E_{21}) + \Psi(E_{22})$.

Proof. Let $K = \Psi(E_{11} + E_{12} + E_{21}) - \Psi(E_{11}) - \Psi(E_{12}) - \Psi(E_{21})$. On one hand, we have

$$\begin{aligned}\Psi(P_2 \diamond (E_{11} + E_{12} + E_{21}) \circ I) &= \Psi(P_2) \diamond (E_{11} + E_{12} + E_{21}) \circ I \\ &\quad + P_2 \diamond \Psi(E_{11} + E_{12} + E_{21}) \circ I \\ &\quad + P_2 \diamond (E_{11} + E_{12} + E_{21}) \circ \Psi(I).\end{aligned}$$

On the other hand, invoking Step 2.1 and using $P_2 \diamond E_{11} \circ I = 0$, we have

$$\begin{aligned}\Psi(P_2 \diamond (E_{11} + E_{12} + E_{21}) \circ I) &= \Psi(P_2 \diamond E_{11} \circ I) + \Psi(P_2 \diamond E_{12} \circ I) + \Psi(P_2 \diamond E_{21} \circ I) \\ &= \Psi(P_2) \diamond E_{11} \circ I + P_2 \diamond \Psi(E_{11}) \circ I + P_2 \diamond E_{11} \circ \Psi(I) \\ &\quad + \Psi(P_2) \diamond E_{12} \circ I + P_2 \diamond \Psi(E_{12}) \circ I + P_2 \diamond E_{12} \circ \Psi(I) \\ &\quad + \Psi(P_2) \diamond E_{21} \circ I + P_2 \diamond \Psi(E_{21}) \circ I + P_2 \diamond E_{21} \circ \Psi(I).\end{aligned}$$

From the last two relations, we find that $P_2 \diamond K \circ I = 0$. This gives us that $P_2K + K^*P_2 = 0$. Multiplying the previous relation by P_1 from right, we find that $P_2KP_1 = 0$. Analogously, we can show that $P_1KP_2 = 0$.
Since

$$\begin{aligned}(P_2 - P_1) \diamond \frac{I}{2} \circ E_{12} &= \{(P_2 - P_1)\frac{I}{2} + \frac{I}{2}(P_2 - P_1)\} \circ E_{12} \\ &= (P_2 - P_1) \circ E_{12} \\ &= (P_2 - P_1)E_{12} + E_{12}(P_2 - P_1) \\ &= -E_{12} + E_{12} \\ &= 0\end{aligned}$$

and similarly, $(P_2 - P_1) \diamond \frac{I}{2} \circ E_{21} = 0$. Invoking Lemma 2.3, we obtain

$$\begin{aligned}\Psi((P_2 - P_1) \diamond \frac{I}{2} \circ (E_{11} + E_{12} + E_{21})) &= \Psi((P_2 - P_1) \diamond \frac{I}{2} \circ E_{11}) + \Psi((P_2 - P_1) \diamond \frac{I}{2} \circ E_{12}) \\ &\quad + \Psi((P_2 - P_1) \diamond \frac{I}{2} \circ E_{21}) = \Psi(P_2 - P_1) \diamond \frac{I}{2} \circ E_{11} + (P_2 - P_1) \diamond \Psi(\frac{I}{2}) \circ E_{11} \\ &\quad + (P_2 - P_1) \diamond \frac{I}{2} \circ \Psi(E_{11}) + \Psi(P_2 - P_1) \diamond \frac{I}{2} \circ E_{12} + (P_2 - P_1) \diamond \Psi(\frac{I}{2}) \circ E_{12} \\ &\quad + (P_2 - P_1) \diamond \frac{I}{2} \circ \Psi(E_{12}) + \Psi(P_2 - P_1) \diamond \frac{I}{2} \circ E_{21} + (P_2 - P_1) \diamond \Psi(\frac{I}{2}) \circ E_{21} \\ &\quad + (P_2 - P_1) \diamond \frac{I}{2} \circ \Psi(E_{21}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Psi((P_2-P_1)\diamond\frac{I}{2}\circ(E_{11}+E_{12}+E_{21}))&=\Psi(P_2-P_1)\diamond\frac{I}{2}\circ(E_{11}+E_{12}+E_{21})\\&+(P_2-P_1)\diamond\Psi(\frac{I}{2})\circ(E_{11}+E_{12}+E_{21})\\&+(P_2-P_1)\diamond\frac{I}{2}\circ\Psi(E_{11}+E_{12}+E_{21}).\end{aligned}$$

From the last two relations, we obtain $(P_2-P_1)\diamond\frac{I}{2}\circ K=0$. This implies that $P_2K-P_1K+KP_2-KP_1=0$. That yields $P_1KP_1=P_2KP_2=0$. Hence, $K=0$, i.e.,

$$\Psi(E_{11}+E_{12}+E_{21})=\Psi(E_{11})+\Psi(E_{12})+\Psi(E_{21}).$$

Analogously, we can show

$$\Psi(E_{12}+E_{21}+E_{22})=\Psi(E_{12})+\Psi(E_{21})+\Psi(E_{22}).$$

□

Step 2.3. For any $E_{ij}\in\mathcal{R}_{ij}, 1\leq i,j\leq 2$, we have

$$\Psi(\sum_{i,j=1}^{2}E_{ij})=\sum_{i,j=1}^{2}\Psi(E_{ij}).$$

Proof. Let $K=\Psi(E_{11}+E_{12}+E_{21}+E_{22})-\Psi(E_{11})-\Psi(E_{12})-\Psi(E_{21})-\Psi(E_{22})$.
On one hand, we have

$$\begin{aligned}\Psi(P_2\diamond I\circ(E_{11}+E_{12}+E_{21}+E_{22}))&=\Psi(P_2)\diamond I\circ(E_{11}+E_{12}+E_{21}+E_{22})\\&+P_2\diamond\Psi(I)\circ(E_{11}+E_{12}+E_{21}+E_{22})\\&+P_2\diamond I\circ\Psi(E_{11}+E_{12}+E_{21}+E_{22}).\end{aligned}$$

On the other hand, since $P_2\diamond I\circ E_{11}=0$, invoking Lemma 2.3 and steps used in the proof of Lemma 2.4, we have

$$\begin{aligned}&\Psi(P_2\diamond I\circ(E_{11}+E_{12}+E_{21}+E_{22}))=\Psi(P_2\diamond I\circ E_{11})+\Psi(P_2\diamond I\circ E_{12})+\Psi(P_2\diamond I\circ E_{21})\\&+\Psi(P_2\diamond I\circ E_{22})=\Psi(P_2)\diamond I\circ E_{11}+P_2\diamond\Psi(I)\circ E_{11}+P_2\diamond I\circ\Psi(E_{11})\\&+\Psi(P_2)\diamond I\circ E_{12}+P_2\diamond\Psi(I)\circ E_{12}+P_2\diamond I\circ\Psi(E_{12})+\Psi(P_2)\diamond I\circ E_{21}\\&+P_2\diamond\Psi(I)\circ E_{21}+P_2\diamond I\circ\Psi(E_{21})+\Psi(P_2)\diamond I\circ E_{22}+P_2\diamond\Psi(I)\circ E_{22}\\&+P_2\diamond I\circ\Psi(E_{22}).\end{aligned}$$

From the last two relations, we arrive at $P_2\diamond I\circ K=0$. Hence $P_1KP_2=P_2KP_1=P_2KP_2=0$. Analogously, we can show that $P_1KP_1=0$. Thus $K=0$, i.e.,

$$\Psi(E_{11}+E_{12}+E_{21}+E_{22})=\Psi(E_{11})+\Psi(E_{12})+\Psi(E_{21})+\Psi(E_{22}).$$

□

Step 2.4. For any $E_{ij},N_{ij}\in\mathcal{R}_{ij}$ with $i\neq j,\Psi(E_{ij}+N_{ij})=\Psi(E_{ij})+\Psi(N_{ij})$.

Proof. Let $N=\Psi(E_{ij}+N_{ij})-\Psi(E_{ij})-\Psi(N_{ij})$. Since

$$\begin{aligned}&I\diamond(\frac{P_i+E_{ij}+E_{ij}^*}{2})\circ(P_j+N_{ij})\\&=\{I(\frac{P_i+E_{ij}+E_{ij}^*}{2})+(\frac{P_i+E_{ij}+E_{ij}^*}{2})I\}\circ(P_j+N_{ij})\\&=(P_i+E_{ij}+E_{ij}^*)\circ(P_j+N_{ij})\\&=(P_i+E_{ij}+E_{ij}^*)(P_j+N_{ij})+(P_j+N_{ij})(P_i+E_{ij}+E_{ij}^*)\\&=N_{ij}+E_{ij}+E_{ij}^*N_{ij}+E_{ij}^*+N_{ij}E_{ij}^*.\end{aligned}$$

Invoking step 2.3, we have

$$\begin{aligned}\Psi(N_{ij}+E_{ij})+\Psi(E_{ij}^*N_{ij})+\Psi(E_{ij}^*)+\Psi(N_{ij}E_{ij}^*) &= \Psi(I\diamond(\frac{P_i+E_{ij}+E_{ij}^*}{2})\circ(P_j+N_{ij}))\\ &= \Psi(I)\diamond(\frac{P_i+E_{ij}+E_{ij}^*}{2})\circ(P_j+N_{ij})+\ I\diamond\Psi(\frac{P_i+E_{ij}+E_{ij}^*}{2})\circ(P_j+N_{ij})\\ &+\ I\diamond(\frac{P_i+E_{ij}+E_{ij}^*}{2})\circ\Psi(P_j+N_{ij})\\ &= \Psi(I\diamond\frac{P_i}{2}\circ P_j)+\ \Psi(I\diamond\frac{P_i}{2}\circ N_{ij})+\ \Psi(I\diamond(\frac{E_{ij}+E_{ij}^*}{2})\circ P_j)\\ &+\ \Psi(I\diamond(\frac{E_{ij}+E_{ij}^*}{2})\circ N_{ij})\\ &= \Psi(N_{ij})+\Psi(E_{ij}+E_{ij}^*)+\Psi(E_{ij}^*N_{ij}+N_{ij}E_{ij}^*)\\ &= \Psi(N_{ij})+\Psi(E_{ij})+\Psi(E_{ij}^*)+\Psi(E_{ij}^*N_{ij})+\Psi(N_{ij}E_{ij}^*).\end{aligned}$$

Therefore, $\Psi(E_{ij}+N_{ij})=\Psi(E_{ij})+\Psi(N_{ij})$.

Step 2.5. For any $E_{ii},N_{ii}\in\mathcal{R}_{ii}$, $1\leq i\leq 2$, we have

$$\Psi(E_{ii}+N_{ii})=\Psi(E_{ii})+\Psi(N_{ii}).$$

Proof. Let $Q=\Psi(E_{ii}+N_{ii})-\Psi(E_{ii})-\Psi(N_{ii})$. Since $P_j\diamond I\circ E_{ii}=0$ for $i\neq j$, invoking Lemma 2.3, we have

$$\begin{aligned}\Psi(P_j\diamond I\circ(E_{ii}+N_{ii})) &= \Psi(P_j\diamond I\circ E_{ii})+\Psi(P_j\diamond I\circ N_{ii})\\ &= \Psi(P_j)\diamond I\circ E_{ii}+P_j\diamond\Psi(I)\circ E_{ii}+P_j\diamond I\circ\Psi(E_{ii})\\ &\quad+\Psi(P_j)\diamond I\circ N_{ii}+P_j\diamond\Psi(I)\circ N_{ii}+P_j\diamond I\circ\Psi(N_{ii}).\end{aligned}$$

On the other way, we also have

$$\begin{aligned}\Psi(P_j\diamond I\circ(E_{ii}+N_{ii})) &= \Psi(P_j)\diamond I\circ(E_{ii}+N_{ii})+P_j\diamond\Psi(I)\circ(E_{ii}+N_{ii})\\ &\quad+P_j\diamond I\circ\Psi(E_{ii}+N_{ii}).\end{aligned}$$

From the last two relations, we arrive at $P_j\diamond I\circ Q=0$. It follows that $P_jQP_j=P_jQP_i=P_iQP_j=0$.
Next, for any $X_{ij}\in\mathcal{R}_{ij}$ with $i\neq j$, we have

$$\begin{aligned}\Psi(P_i\diamond X_{ij}\circ(E_{ii}+N_{ii})) &= \Psi(P_i)\diamond X_{ij}\circ(E_{ii}+N_{ii})+P_i\diamond\Psi(X_{ij})\circ(E_{ii}+N_{ii})\\ &\quad+P_i\diamond X_{ij}\circ\Psi(E_{ii}+N_{ii}).\end{aligned}$$

On the other hand, using steps 2.3 and 2.4, we have

$$\begin{aligned}\Psi(P_i\diamond X_{ij}\circ(E_{ii}+N_{ii})) &= \Psi(X_{ij}^*E_{ii}+X_{ij}^*N_{ii}+E_{ii}X_{ij}+N_{ii}X_{ij})\\ &= \Psi(X_{ij}^*E_{ii}+E_{ii}X_{ij})+\Psi(X_{ij}^*N_{ii}+N_{ii}X_{ij})\\ &= \Psi(X_{ij}^*E_{ii})+\Psi(E_{ii}X_{ij})+\Psi(X_{ij}^*N_{ii})+\Psi(N_{ii}X_{ij})\\ &= \Psi(P_i\diamond X_{ij}\circ E_{ii})+\Psi(P_i\diamond X_{ij}\circ N_{ii})\\ &= \Psi(P_i)\diamond X_{ij}\circ E_{ii}+P_i\diamond\Psi(X_{ij})\circ E_{ii}+P_i\diamond X_{ij}\circ\Psi(E_{ii})\\ &\quad+\Psi(P_i)\diamond X_{ij}\circ N_{ii}+P_i\diamond\Psi(X_{ij})\circ N_{ii}+P_i\diamond X_{ij}\circ\Psi(N_{ii}).\end{aligned}$$

From the last two relations, we conclude that $P_i\diamond X_{ij}\circ Q=0$. It follows that $P_i\diamond X_{ij}\circ Q_{ii}=0$. Now solving this, we get

$$X_{ij}^*Q_{ii}+Q_{ii}X_{ij}=0.$$

Multiplying the previous relation from the right by P_j, we get $Q_{ii}X_{ij}=0$. Using (2.1) and (2.2), it follows that $Q_{ii}=0$. Thus $Q=0$. □

Step 2.6. For any $E, N \in \mathcal{R}$, we write $E = E_{11} + E_{12} + E_{21} + E_{22}$ and $N = N_{11} + N_{12} + N_{21} + N_{22}$. Invoking steps 2.3-2.5, we get

$$\begin{aligned}\Psi(E+N) &= \Psi(E_{11}+E_{12}+E_{21}+E_{22}+N_{11}+N_{12}+N_{21}+N_{22})\\ &= \Psi(E_{11}+N_{11})+\Psi(E_{12}+N_{12})+\Psi(E_{21}+N_{21})+\Psi(E_{22}+N_{22})\\ &= \Psi(E_{11})+\Psi(N_{11})+\Psi(E_{12})+\Psi(N_{12})+\Psi(E_{21})+\Psi(N_{21})\\ &+ \Psi(E_{22})+\Psi(N_{22})\\ &= \Psi(E_{11}+E_{12}+E_{21}+E_{22})+\Psi(N_{11}+N_{12}+N_{21}+N_{22})\\ &= \Psi(E)+\Psi(N).\end{aligned}$$

Now, it follows from above all steps that Ψ is additive, proving Lemma 2.4. □

Now, in the rest of the paper, we show that Ψ is a $*$-derivation.

Lemma 2.5. (i) $P_1\Psi(P_1)P_2 = -P_1\Psi(P_2)P_2$;
(ii) $P_2\Psi(P_1)P_1 = -P_2\Psi(P_2)P_1$;
(iii) $P_1\Psi(P_2)P_1 = P_2\Psi(P_1)P_2 = 0$.

Proof. (i) It follows from $P_1 \diamond P_1 \circ P_2 = 0$ and Lemma 2.3, that

$$\begin{aligned}0 &= \Psi(P_1 \diamond P_1 \circ P_2)\\ &= \Psi(P_1) \diamond P_1 \circ P_2 + P_1 \diamond \Psi(P_1) \circ P_2 + P_1 \diamond P_1 \circ \Psi(P_2)\\ &= 2P_1\Psi(P_1)P_2 + 2P_2\Psi(P_1)^*P_1 + 2P_1\Psi(P_2) + 2\Psi(P_2)P_1.\end{aligned}$$

Multiplying the previous relation by P_1 from left, we get

$$2P_1\Psi(P_1)P_2 + 2P_1\Psi(P_2) + 2P_1\Psi(P_2)P_1 = 0.$$

Again, multiplying the last obtained relation by P_2 from right, we deduce that

$$2P_1\Psi(P_1)P_2 + 2P_1\Psi(P_2)P_2 = 0.$$

Thus

$$P_1\Psi(P_1)P_2 = -P_1\Psi(P_2)P_2.$$

(ii) Since $P_2 \diamond P_2 \circ P_1 = 0$, applying Lemma 2.3, we get

$$\begin{aligned}0 &= \Psi(P_2 \diamond P_2 \circ P_1)\\ &= \Psi(P_2) \diamond P_2 \circ P_1 + P_2 \diamond \Psi(P_2) \circ P_1 + P_2 \diamond P_2 \circ \Psi(P_1)\\ &= 2P_2\Psi(P_2)P_1 + 2P_1\Psi(P_2)^*P_2 + 2P_2\Psi(P_1) + 2\Psi(P_1)P_2.\end{aligned}$$

Multiplying the previous relation by P_2 from left, we get

$$2P_2\Psi(P_2)P_1 + 2P_2\Psi(P_1) + 2P_2\Psi(P_1)P_2 = 0.$$

Again, multiplying the last obtained relation by P_1 from right, we arrive at

$$2P_2\Psi(P_2)P_1 + 2P_2\Psi(P_1)P_1 = 0.$$

Therefore

$$P_2\Psi(P_2)P_1 = -P_2\Psi(P_1)P_1.$$

(iii) From (i), we have

$$0 = 2P_1\Psi(P_1)P_2 + 2P_2\Psi(P_1)^*P_1 + 2P_1\Psi(P_2) + 2\Psi(P_2)P_1. \tag{2.3}$$

Multiplying (2.3) by P_1 from both right and left, we get $P_1\Psi(P_2)P_1 = 0$. Analogously, from (ii), we have

$$0 = 2P_2\Psi(P_2)P_1 + 2P_1\Psi(P_2)^*P_2 + 2P_2\Psi(P_1) + 2\Psi(P_1)P_2. \tag{2.4}$$

Multiplying (2.4) by P_2 from both right and left, we get

$$P_2\Psi(P_1)P_2 = 0.$$

□

Lemma 2.6. *Let $\Psi(P_1)^*$ is self-adjoint. Then $P_1\Psi(P_1)P_1 = P_2\Psi(P_2)P_2 = 0$.*

Proof. For every $E_{12} \in \mathcal{R}_{12}$, applying Lemma 2.4 , we get

$$\Psi(P_1 \diamond P_1 \circ E_{12}) = 2\Psi(E_{12}).$$

On the other hand, we have

$$\begin{aligned}\Psi(P_1 \diamond P_1 \circ E_{12}) &= \Psi(P_1) \diamond P_1 \circ E_{12} + P_1 \diamond \Psi(P_1) \circ E_{12} + P_1 \diamond P_1 \circ \Psi(E_{12}) \\ &= 2\Psi(P_1)^* E_{12} + 2P_1\Psi(P_1)E_{12} + 2E_{12}\Psi(P_1)^* P_1 \\ &+ 2P_1\Psi(E_{12}) + 2\Psi(E_{12})P_1.\end{aligned}$$

From the last two relations, we conclude that

$$\Psi(P_1)^* E_{12} + P_1\Psi(P_1)E_{12} + E_{12}\Psi(P_1)^* P_1 + P_1\Psi(E_{12}) + \Psi(E_{12})P_1 - \Psi(E_{12}) = 0.$$

By the given hypothesis $\Psi(P_1)$ is self-adjoint. Therefore, from last relation, we have

$$\Psi(P_1)E_{12} + P_1\Psi(P_1)E_{12} + E_{12}\Psi(P_1)P_1 + P_1\Psi(E_{12}) + \Psi(E_{12})P_1 - \Psi(E_{12}) = 0.$$

Multiplying the previous relation by P_2 from right, we obtain

$$\Psi(P_1)E_{12} + P_1\Psi(P_1)E_{12} + P_1\Psi(E_{12})P_2 - \Psi(E_{12})P_2 = 0.$$

Again, multiplying the last obtained relation by P_1 from left, we get

$$P_1\Psi(P_1)E_{12} + P_1\Psi(P_1)E_{12} + P_1\Psi(E_{12})P_2 - P_1\Psi(E_{12})P_2 = 0.$$

Hence, we have $P_1\Psi(P_1)E_{12} = 0$, i.e., $P_1\Psi(P_1)P_1EP_2 = 0$ for all $E \in \mathcal{R}$. It follows from (2.1) and (2.2) that $P_1\Psi(P_1)P_1 = 0$. Analogously, we can prove that $P_2\Psi(P_2)P_2 = 0$. □

Lemma 2.7. (i) $\Psi(P_1) = P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1, \Psi(P_2) = P_1\Psi(P_2)P_2 + P_2\Psi(P_2)P_1$.
(ii) $\Psi(I) = 0$.

Proof. (i) By Peirce decomposition, we have

$$\Psi(P_1) = P_1\Psi(P_1)P_1 + P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1 + P_2\Psi(P_1)P_2.$$

In view of Lemma 2.5 and Lemma 2.6, it follows that $\Psi(P_1) = P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1$. Analogously, we can show that $\Psi(P_2) = P_1\Psi(P_2)P_2 + P_2\Psi(P_2)P_1$.
(ii) Invoking Lemmas 2.4- 2.6, we have

$$\begin{aligned}\Psi(I) = \Psi(P_1 + P_2) = \Psi(P_1) + \Psi(P_2) &= P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1 + P_1\Psi(P_2)P_2 \\ &+ P_2\Psi(P_2)P_1 = 0.\end{aligned}$$

□

Lemma 2.8. *Ψ preserves '$*$', i.e., $\Psi(E^*) = \Psi(E)^*$ for all $E \in \mathcal{R}$.*

Proof. By using Lemma 2.4, i.e.,Ψ is additive, we have

$$\begin{aligned}I \diamond E \circ I &= \{I^*E + E^*I\} \circ I \\ I \diamond E \circ I &= 2E + 2E^* \\ \Psi(I \diamond E \circ I) &= \Psi(2E + 2E^*) \\ \Psi(I \diamond E \circ I) &= \Psi(2E) + \Psi(2E^*) \\ \Psi(I \diamond E \circ I) &= \Psi(E + E) + \Psi(E^* + E^*) \\ \Psi(I \diamond E \circ I) &= \Psi(E) + \Psi(E) + \Psi(E^*) + \Psi(E^*) \\ &= 2\Psi(E) + 2\Psi(E^*).\end{aligned}$$

On the other hand, using Lemma 2.7, we have

$$\Psi(I \diamond E \circ I) = I \diamond \Psi(E) \circ I = 2\Psi(E) + 2\Psi(E)^*.$$

Comparing the above two relations, we arrive at $\Psi(E^*) = \Psi(E)^*$ for all $E \in \mathcal{R}$. Thus, we explicitly note that the above computation relies solely on the additivity of Ψ and that no assumption of $\mathbb{C}$-linearity is used at this stage. □

Lemma 2.9. (i) $\Psi(iI) = 0$;
(ii) $\Psi(-iI) = 0$, *where* i *is the imaginary unit.*

Proof. (i) Applying Lemmas 2.4 and 2.7, we have

$$\Psi(iI \diamond iI \circ I) = \Psi(iI) \diamond iI \circ I + iI \diamond \Psi(iI) \circ I$$

Thus, $\Psi(iI)^* = \Psi(iI)$. Also, we have

$$\Psi(iI \diamond I \circ I) = \Psi(iI) \diamond I \circ I$$

Thus, $\Psi(iI)^* = -\Psi(iI)$. Since iI is skew-adjoint, the two identities imply $\Psi(iI) = 0$.
(ii) Analogously, we can show that $\Psi(-iI) = 0$. □

Lemma 2.10. (i) $\Psi(-iE) = -i\Psi(E)$;
(ii) $\Psi(iE) = i\Psi(E)$, *where* i *is the imaginary unit.*

Proof. (i) Since $(-iE) \diamond I \circ I = E \diamond iI \circ I$, we have

$$\Psi((-iE) \diamond I \circ I) = \Psi(E \diamond iI \circ I).$$

Invoking Lemmas 2.7 and 2.9, we have

$$\Psi(-iE)^* + \Psi(-iE) = i\Psi(E)^* - i\Psi(E). \tag{2.5}$$

Also, since $(-iE) \diamond iI \circ I = (-I) \diamond E \circ I$, utilizing Lemma 2.3, we have

$$\Psi((-iE) \diamond iI \circ I) = \Psi(-I \diamond E \circ I).$$

Now by Lemmas 2.7 and 2.9, we have

$$i\Psi(-iE)^* - i\Psi(-iE) = -\Psi(E) - \Psi(E)^*. \tag{2.6}$$

Multiplying (2.6) from left by iI, we get

$$\begin{aligned} i^2\Psi(-iE)^* - i^2\Psi(-iE) &= -i\Psi(E) - i\Psi(E)^* \\ -\Psi(-iE)^* + \Psi(-iE) &= -i\Psi(E) - i\Psi(E)^*. \end{aligned} \tag{2.7}$$

By adding (2.5) and (2.7), we get
$\Psi(-iE) = -i\Psi(E)$ for all $E \in \mathcal{R}$.
(ii) Analogously, we can show that $\Psi(iE) = i\Psi(E)$. □

Lemma 2.11. Ψ *is a derivation.*

Proof. For every E, K $\in \mathcal{R}$, we have $(E \diamond K^* \circ I) = 2(E^*K^* + KE)$. Now, applying Lemmas 2.7 - 2.8, we have

$$\begin{aligned} 2\ \Psi(E^*K^* + KE) &= \Psi(E \diamond K^* \circ I) \\ &= \Psi(E) \diamond K^* \circ I + E \diamond \Psi(K)^* \circ I + 0 \\ &= \Psi(E) \diamond K^* \circ I + E \diamond \Psi(K)^* \circ I \\ &= \{\Psi(E)^*K^* + K\Psi(E)\} \circ I + \{E^*\Psi(K)^* + \Psi(K)E\} \circ I \\ &= \{\Psi(E)^*K^* + K\Psi(E)\}I + I\{\Psi(E)^*K^* + K\Psi(E)\} \\ &+ \{E^*\Psi(K)^* + \Psi(K)E\}I + I\{E^*\Psi(K)^* + \Psi(K)E\} \\ &= \Psi(E)^*K^* + K\Psi(E) + \Psi(E)^*K^* + K\Psi(E) \\ &+ E^*\Psi(K)^* + \Psi(K)E + E^*\Psi(K)^* + \Psi(K)E \\ &= 2\Psi(E)^*K^* + 2K\Psi(E) + 2E^*\Psi(K)^* + 2\Psi(K)E. \end{aligned}$$

Therefore,

$$2\Psi(KE + E^*K^*) = 2\Psi(E)^*K^* + 2K\Psi(E) + 2E^*\Psi(K)^* + 2\Psi(K)E. \tag{2.8}$$

Also, we have $(K^* \diamond iE \circ I) = 2\ i\ (KE - E^*K^*)$.

So, invoking Lemmas 2.7, 2.8 and 2.10, we have

$$\begin{aligned} 2\ \Psi(i(KE - E^*K^*)) &= \Psi(K^* \diamond iE \circ I) \\ &= \Psi(K)^* \diamond iE \circ I + K^* \diamond \Psi(iE) \circ I \\ &= 2i\Psi(K)E + 2iK\Psi(E) - 2i\Psi(E)^*K^* - 2iE^*\Psi(K)^*. \end{aligned}$$

Therefore

$$2i\Psi(KE - E^*K^*) = 2i\Psi(K)E + 2iK\Psi(E) - 2i\Psi(E)^*K^* - 2iE^*\Psi(K)^*.$$

Multiply above relation from left by iI, to get

$$2\Psi(KE - E^*K^*) = 2\Psi(K)E + 2K\Psi(E) - 2\Psi(E)^*K^* - 2E^*\Psi(K)^*. \tag{2.9}$$

Adding (2.8) and (2.9) and using additivity of Ψ, we get

$$\begin{aligned} 2\Psi(KE) + 2\Psi(E^*K^*) + 2\Psi(KE) - 2\Psi(E^*K^*) &= 2\Psi(E)^*K^* + 2K\Psi(E) + 2E^*\Psi(K)^* \\ &\quad + 2\Psi(K)E + 2\Psi(K)E + 2K\Psi(E) \\ &\quad - 2\Psi(E)^*K^* - 2E^*\Psi(K)^* \\ 4\Psi(KE) &= 4\Psi(K)E + 4K\Psi(E). \end{aligned}$$

Thus

$$\Psi(KE) = \Psi(K)E + K\Psi(E).$$

Hence Ψ is a derivation. So by Lemmas 2.4, 2.8 and 2.11, we get Ψ is a $*$-derivation. This completes the proof of Theorem 2.2 □

Remark 2.12. Some special classes of $*$-algebras in which conditions (2.1) and (2.2) hold:
1:- If $\mathcal{R}$ is a prime algebra, then by definition, $\mathcal{R}$ satisfies conditions (2.1) and (2.2).
2:- As shown in [15, 1], any von Neumann algebra $\mathcal{Z}$ without central summands of type I_1 satisfies conditions (2.1), (2.2).
3:- Moreover, conditions (2.1)- (2.2) hold in each corollary, for further details, see corollaries 3.1–3.3.

3 Corollaries

Let $\mathcal{R}$ be an algebra, it is said to be prime if for each E, K $\in \mathcal{R}$, $E\mathcal{R}K = 0$, implies either $E = 0$ or $K = 0$. Therefore, from the definition of a prime $*$-algebra, the conditions (2.1) and (2.2) are satisfied as shown below.

For any $K, P_1 \in \mathcal{R}$ such that $K\mathcal{R}P_1 = 0$. By using the definition of primeness, either $K = 0$ or $P_1 = 0$. Since P_1 is a nontrivial projection, we have $P_1 \neq 0$, hence $K = 0$. Similarly, the second condition follows. Thus the following corollary follows:

Corollary 3.1. *Suppose $\mathcal{R}$ is a unital prime $*$-algebra with a nontrivial projection P_1 such that (2.1) -(2.2) follows from the primeness of $\mathcal{R}$. If $\Psi : \mathcal{R} \to \mathcal{R}$ satisfies*

$$\Psi(E \diamond K \circ D) = \Psi(E) \diamond K \circ D + E \diamond \Psi(K) \circ D + E \diamond K \circ \Psi(D)$$

for all $E, K, D \in \mathcal{R}$. Then Ψ is additive. Moreover, if $\Psi(P_1)$ is self-adjoint, then Ψ is a $*$-derivation.

Consider $\mathcal{H}$, as a complex Hilbert space. Assume $\mathcal{B}(\mathcal{H})$ and $\mathbb{T}(\mathcal{H})$ denote the algebra of all bounded linear operators and its subalgebra consisting of finite rank operators respectively. It is well known that $\mathbb{T}(\mathcal{H})$ forms a $*$-closed ideal of $\mathcal{B}(\mathcal{H})$. A subalgebra $\mathcal{F}$ of $\mathcal{B}(\mathcal{H})$ is called a standard operator algebra if $\mathbb{T}(\mathcal{H}) \subseteq \mathcal{F}$. As a result, we have the following immediate corollary.

Corollary 3.2. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{F}$ be a unital standard operator algebra on $\mathcal{H}$ such that $\mathcal{F}$ is closed under adjoint operation. Suppose that $\Psi : \mathcal{F} \to \mathcal{F}$ is a map satisfying*

$$\Psi(E \diamond K \circ D) = \Psi(E) \diamond K \circ D + E \diamond \Psi(K) \circ D + E \diamond K \circ \Psi(D)$$

for all $E, K, D \in \mathcal{F}$. Then Ψ is additive. Moreover, if $\Psi(P_1)$ is self-adjoint for some nontrivial projection P_1, then Ψ is a $*$-derivation.

Proof. A standard operator algebra, denoted as $\mathcal{F}$, which is a conventional operator algebra is a prime algebra that follows directly from the Hahn-Banach theorem [7]. Since this algebra $\mathcal{F}$ inherently satisfies the conditions (2.1) and (2.2). Therefore, we deduce, by Theorem 2.2 that Ψ is an additive $*$-derivation. □

A von Neumann algebra $\mathcal{Z} \subseteq \mathcal{B}(\mathcal{H})$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. Also it is well known that if a von Neumann algebra $\mathcal{Z}$ has no central summands of type I_1, then $\mathcal{Z}$ satisfies conditions (2.1) and (2.2) of Theorem 2.2. As a result, the immediate corollary is as follows:

Corollary 3.3. *Let $\mathcal{Z}$ be a von Neumann algebra with no central summands of type I_1 and consider the map $\Psi : \mathcal{Z} \to \mathcal{Z}$ satisfying*

$$\Psi(E \diamond K \circ D) = \Psi(E) \diamond K \circ D + E \diamond \Psi(K) \circ D + E \diamond K \circ \Psi(D)$$

for all $E, K, D \in \mathcal{Z}$. Then Ψ is additive. Moreover, if $\Psi(P_1)$ is self-adjoint for some nontrivial projection P_1, then Ψ is a $*$-derivation.

Proof. Let $\mathcal{Z}$ be a von Neumann algebra. As shown in [15, 1, 6], any von Neumann algebra $\mathcal{Z}$ without central summands of type I_1 satisfies conditions (2.1) - (2.2). Consequently, by Theorem 2.2, it follows that the previously defined map is an additive $*$-derivation. □

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