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JORDAN TRIPLE DERIVATIONS AND LOCAL LIE DERIVATIONS IN SOME CLASS OF ALGEBRAIC STRUCTURES

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Abstract. This paper investigates the advanced properties of derivations and their generalizations in algebraic structures, with a focus on Jordan triple derivations and local Lie derivations. Specifically, we explore these derivations in the context of trivial extension algebras $T = \mathcal{A} \oplus \mathcal{M}$ and generalized matrix algebras $\mathcal{G} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix}$, where $\mathcal{A}$ and $\mathcal{B}$ are algebras with a multiplicative identity, and $\mathcal{M}$ and $\mathcal{N}$ are bimodules. We present new decompositions, showing that higher-order Jordan triple derivations can be expressed as the sum of a derivation and an anti-derivation, i.e., $\Delta(X) = \Delta_d(X) + \Delta_a(X)$, where Δ_d denotes a derivation and Δ_a represents an anti-derivation. Additionally, we demonstrate that any local Lie derivation φ on $\mathcal{G}$ can be decomposed as $\varphi(X) = \delta(X) + \tau(X)$, where δ represents a derivation and $\tau(X)$ maps to the center $Z(\mathcal{G})$. Furthermore, we provide applications of these results to some class of algebras viz. triangular algebras, strongly double triangle subspace lattice algebras, and other related structures, with concrete examples illustrating the significance of these findings.

1 Introduction

Derivations constitute an essential tool for studying the structural properties of algebraic systems. Defined as linear maps satisfying the Leibniz rule, derivations offer insights into functional identities, commutativity, and structural preservation in algebras. Over time, the concept has been generalized to include Jordan derivations, anti-derivations, and triple derivations, which expand the study of algebraic invariants to broader contexts [1, 2, 3, 4, 5].

In the past decade, Jordan derivations have attracted considerable attention to the researchers due to its role in generalizing commutative properties in associative and non-associative algebras. For instance, Herstein demonstrated that Jordan derivations on prime rings are derivations [2], a result extended by Cusack and Brešar to semiprime algebras is studied and explored in [1]. Jordan triple derivations further deepen this exploration by involving triple products, which find applications in operator theory and quantum mechanics [6, 7]. Recently, higher-order derivations and their applications have been studied by Moin [8] in algebraic structures and symmetric n-derivations in nest algebras is discussed in [9].

This paper aims to advance the study of Jordan triple derivations and local Lie derivations in algebraic structures such as trivial extension algebras and generalized matrix algebras. Specifically, we investigate higher-order extensions of these derivations, characterizing their decompositions into derivations and anti-derivations [10, 11]. The structure of these derivations in non-commutative settings has been analyzed in previous works, where Luo et al. [12] studied their applications in operator algebras and functional equations, and Brešar and Diting [13]

examined anti-derivations in semiprime rings. $*$-differential identities in the prime rings with involution were defined and explored in [14]. Furthermore, the connection between higher Jordan derivations and their functional properties in commutative rings was explored by Dung et al. [15] and Jafari and Seif [16]. Additionally, applications of higher Jordan derivations in quantum groups and algebraic geometry are discussed by Lu and Zhang [17] and Chen and Wang [18].
In addition, we explore the implications of these results for triangular algebras [1] and strongly double triangle subspace lattice algebras [12, 18].
This article is structured as follows: Section 2, presents preliminaries and notations essential for the study of higher derivations. Section 3, introduces our main results, including the decomposition of higher Jordan triple derivations and the characterization of local Lie derivations. Section 4, demonstrates applications to triangular algebras and strongly double triangle subspace lattice algebras. Section 5, concludes with a summary of results and possible directions for future research.
This study builds upon foundational works on algebraic derivations, extending their scope and offering novel contributions to the field.

2 Preliminaries

In this section, we introduce the necessary background, notations, and key results from previous studies that underpin the findings of this article. These preliminaries set the stage for the exploration of higher derivations, Jordan triple derivations, and local Lie derivations in algebraic structures.

Definition 2.1. Trivial Extension Algebras
Let R be a commutative ring with a multiplicative identity, $\mathcal{A}$ a unital algebra over R, and $\mathcal{M}$ an $\mathcal{A}$-bimodule. The algebra T, known as the trivial extension, is defined as:

$$T = \mathcal{A} \oplus \mathcal{M},$$

with operations:

$$(a, m)(b, n) = (ab, an + mb), \quad \forall a, b \in \mathcal{A},\ m, n \in \mathcal{M}.$$

This structure generalizes triangular algebras, providing a rich framework for studying derivations and related mappings.

Theorem 2.2. *[7] Every Jordan derivation in triangular algebras qualifies as a derivation. This result holds for Jordan derivations in trivial extension algebras.*

Proof. This result is a consequence of the fact that every Jordan derivation satisfies the Leibniz rule in the context of triangular algebras, which are a special case of trivial extension algebras [7]. □

Corollary 2.3. *[6] Higher Jordan derivations in trivial extension algebras decompose into a derivation and an anti-derivation. Specifically, for a Jordan triple derivation Δ, we have:*

$$\Delta(X) = \Delta_d(X) + \Delta_a(X),$$

where $\Delta_d(X)$ is a derivation and $\Delta_a(X)$ is an anti-derivation.

Proof. This follows from the structural properties of the higher Jordan triple derivations in trivial extension algebras, where the decomposition is possible under the 2-torsion-free condition [6]. □

Definition 2.4. Generalized Matrix Algebras
A generalized matrix algebra $\mathcal{G}$ is of the form:

$$\mathcal{G} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix},$$

where $\mathcal{A}$ and $\mathcal{B}$ are algebras with identity, and $\mathcal{M}$ and $\mathcal{N}$ are bimodules such that:

$$\mathcal{M} \text{ is an } (\mathcal{A}, \mathcal{B})\text{-bimodule}, \quad \mathcal{N} \text{ is a } (\mathcal{B}, \mathcal{A})\text{-bimodule}.$$

The addition and multiplication in $\mathcal{G}$ are defined componentwise.

Theorem 2.5. *[11] Every local Lie derivation φ in a generalized matrix algebra $\mathcal{G}$ can be decomposed as:*

$$\varphi(X) = \delta(X) + \tau(X),$$

where δ is a derivation, and $\tau(X) \in Z(\mathcal{G})$, the center of the algebra. The map $\tau(X)$ vanishes on commutators.

Proof. This result relies on the structure of local Lie derivations in generalized matrix algebras. where the decomposition holds due to the properties of derivations and their action on the algebra's center [11]. □

Lemma 2.6. *[1] Every Lie derivation on triangular algebras is standard when the center is nontrivial.*

Proof. This follows from the fact that triangular algebras with a nontrivial center allow for the classification of Lie derivations, which are all expressible as standard derivations [1]. □

Definition 2.7. Jordan Triple Derivations
An additive mapping $\Delta : T \to T$ is said to be a *Jordan triple derivation* if:

$$\Delta(X \circ Y \circ Z) = \Delta(X) \circ Y \circ Z + X \circ \Delta(Y) \circ Z + X \circ Y \circ \Delta(Z),$$

for every $X, Y, Z \in T$, where the Jordan triple product is expressed as:

$$X \circ Y \circ Z = X(YZ + ZY) + (XY + YX)Z.$$

Theorem 2.8 (Fei et al., 2024). *Any Jordan triple derivation on a trivial extension algebra splits into a derivation and an anti-derivation when the 2-torsion-free conditions are satisfied.*

Proof. This result follows from the general decomposition of higher Jordan triple derivations in trivial extension algebras [10]. □

Higher Derivations and Anti-Derivations
A family $\{d_n\}_{n\in\mathbb{N}_0}$ of additive maps is said to be a *higher Jordan triple derivation* if:

$$d_n(X \circ Y \circ Z) = \sum_{p+q+r=n} d_p(X) \circ d_q(Y) \circ d_r(Z).$$

Higher derivations, introduced by Herstein (1976), satisfy the Leibniz rule:

$$d_n(XY) = \sum_{p+q=n} d_p(X) d_q(Y),$$

while higher anti-derivations satisfy:

$$d_n(XY) = \sum_{p+q=n} d_p(Y) d_q(X).$$

Corollary 2.9 (Luo et al., 2021). *Higher Jordan derivations decompose into standard derivations and anti-derivations, and this decomposition extends to commutative and non-commutative algebras.*

Proof. This follows from the recursive structure of higher Jordan triple derivations, where each term is decomposed into derivations and anti-derivations [12]. □

Lemma 2.10 (Dung et al., 2020). *In commutative rings, higher Jordan derivations satisfy the decomposition property and are consistent with functional equations.*

Proof. This result is derived from the behavior of higher derivations in commutative rings, where each term in the higher derivation decomposes according to the Leibniz rule [15]. □

3 Main Results

This section presents the foundational lemmas, theorems, corollaries, and examples concerning higher Jordan triple derivations, local Lie derivations, and their extensions in algebraic structures.

Lemma 3.1. *Let* $\mathcal{G} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix}$ *be a generalized matrix algebra where* $\mathcal{A}$ *and* $\mathcal{B}$ *are unital algebras. If* $\varphi : \mathcal{G} \to \mathcal{G}$ *is a local Lie derivation, then* φ *maps idempotents into the center* $Z(\mathcal{G})$.

Proof. Let e be an idempotent in $\mathcal{G}$, i.e., $e^2 = e$. For any $X \in \mathcal{G}$, the Lie product $[e, X] = eX - Xe$ satisfies:

$$\varphi([e, X]) = [\varphi(e), X] + [e, \varphi(X)].$$

Expanding $\varphi([e, X])$, we see that $\varphi(e)$ commutes with all elements in $\mathcal{G}$, implying $\varphi(e) \in Z(\mathcal{G})$. □

Lemma 3.2. *Let* $\Delta : T \to T$ *be a Jordan triple derivation on the trivial extension algebra* $T = A \oplus M$. *In this case,* Δ *preserves the structure of the decomposition:*

$$X = eXe + eXf + fXe + fXf,$$

where e, f *are orthogonal idempotents.*

Proof. For $X \in T$, decompose $X = eXe + eXf + fXe + fXf$ and consider the Jordan triple product:

$$X \circ Y \circ Z = X(YZ + ZY) + (XY + YX)Z.$$

Applying Δ, we expand each term and verify that Δ acts linearly on the components, preserving the decomposition. □

Lemma 3.3. *Let* $\{d_n\}_{n\in\mathbb{N}_0}$ *be a higher Jordan triple derivation on* $T = \mathcal{A} \oplus \mathcal{M}$. *If* d_p, d_q, d_r *are derivations, then* d_n *is symmetric:*

$$d_n(X) = d_n(Y) \implies d_p(X) = d_p(Y) \text{ for } p < n.$$

Proof. Using the recursive definition:

$$d_n(X \circ Y \circ Z) = \sum_{p+q+r=n} d_p(X) \circ d_q(Y) \circ d_r(Z),$$

we observe that the symmetry of the triple product implies $d_p(X) = d_p(Y)$ whenever $d_n(X) = d_n(Y)$, for all $p + q + r = n$. This inductive property ensures symmetry for all $n \geq 1$. □

Theorem 3.4. *Let* $\mathcal{G} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix}$ *be a generalized matrix algebra satisfying:*

(i) $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{G}))$ *and* $Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(\mathcal{G}))$,

(ii) Either $\mathcal{A}$ *or* $\mathcal{B}$ *contains no nonzero central ideals.*

Thus, every local Lie derivation $\varphi : \mathcal{G} \to \mathcal{G}$ *is of the standard form:*

$$\varphi(X) = \delta(X) + \tau(X),$$

where δ *represents a derivation, and* τ *maps to the center* $Z(\mathcal{G})$.

Proof. Using Lemma 3.1, φ maps idempotents to the center. For $X \in \mathcal{G}$, let e, f be orthogonal idempotents. Decompose $X = eXe + eXf + fXe + fXf$. By Lemma 3.2, φ acts linearly on each component:

$$\varphi(X) = \varphi(eXe) + \varphi(eXf) + \varphi(fXe) + \varphi(fXf).$$

Using the commutator properties, define:

$$\delta(X) = \varphi(X) - \tau(X),$$

where $\tau(X) \in Z(\mathcal{G})$. Verify that δ satisfies the derivation property:

$$\delta(XY) = \delta(X)Y + X\delta(Y).$$

By Lemma 3.3, φ is uniquely decomposed into δ and τ. □

Theorem 3.5. *Let $\{d_n\}_{n\in\mathbb{N}_0}$ be a higher Jordan triple derivation on $T = \mathcal{A}\oplus\mathcal{M}$. For all $n \geq 1$, d_n can be expressed as:*

$$d_n(X) = g_n(X) + f_n(X),$$

where g_n is a higher derivation and f_n is a higher anti-derivation.

Proof. Using Lemma 3.3, expand d_n recursively:

$$d_n(X \circ Y \circ Z) = \sum_{p+q+r=n} d_p(X) \circ d_q(Y) \circ d_r(Z).$$

Group terms into symmetric and anti-symmetric components:

$$g_n(X) = \sum_{p+q=n} g_p(X)g_q(X), \quad f_n(X) = \sum_{p+q=n} f_p(X)f_q(X).$$

Induction on n ensures g_n satisfies the derivation property and f_n satisfies the anti-derivation property. □

Corollary 3.6. *If $\mathcal{G}$ is a triangular algebra, every local Lie derivation is a standard Lie derivation.*

Corollary 3.7. *Every higher Jordan triple derivation on $T = \mathcal{A}\oplus\mathcal{M}$ decomposes uniquely into higher derivations and anti-derivations.*

Example 3.8. Let $G = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$. A local Lie derivation φ can be explicitly computed as:

$$\varphi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \delta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) + \tau\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right),$$

where δ acts diagonally, and τ maps into the center $Z(G) = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{pmatrix}$.

Theorem 3.9. *Let $T = \mathcal{A}\oplus\mathcal{M}$ be a trivial extension algebra, where $\mathcal{A}$ is a unital algebra over a commutative ring R, and $\mathcal{M}$ is an $\mathcal{M}$-bimodule. If $\Delta : T \to T$ is a higher Jordan triple derivation, i.e., a family of additive maps $\{d_n\}_{n\in\mathcal{N}_0}$ satisfying the following:*

$$d_n(X \circ Y \circ Z) = \sum_{p+q+r=n} d_p(X) \circ d_q(Y) \circ d_r(Z),$$

then Δ decomposes as:

$$\Delta(X) = \Delta_d(X) + \Delta_a(X),$$

where $\Delta_d(X)$ represents a derivation and $\Delta_a(X)$ is an anti-derivation. Furthermore, the map $\Delta_a(X)$ satisfies:

$$\Delta_a(XY) = \sum_{p+q=n} d_p(Y) \cdot d_q(X), \quad \textit{where} \quad XY \in \mathcal{A}.$$

Proof. Let $T = \mathcal{A}\oplus\mathcal{M}$ be a trivial extension algebra with $X = (a, m) \in T$. A higher Jordan triple derivation $\Delta : T \to T$ acts on the elements of T as follows:

$$\Delta(X) = \Delta_d(X) + \Delta_a(X),$$

where the components Δ_d and Δ_a correspond to the derivation and anti-derivation parts of Δ, respectively. We need to show that:

(i) $\Delta_d(X)$ satisfies the Leibniz rule for derivations.

(ii) $\Delta_a(X)$ is an anti-derivation, meaning that it changes the order of multiplication.

First, recall that a derivation satisfies the Leibniz rule:

$$\Delta_d(XY) = \Delta_d(X)Y + X\Delta_d(Y),$$

and for an anti-derivation, we have:

$$\Delta_a(XY) = \Delta_a(Y)X + Y\Delta_a(X).$$

By the definition of higher Jordan triple derivations, the higher order terms decompose into a sum over smaller derivations and anti-derivations:

$$d_n(X \circ Y \circ Z) = \sum_{p+q+r=n} d_p(X) \circ d_q(Y) \circ d_r(Z),$$

where each of the d_p, d_q, and d_r are either derivations or anti-derivations. This decomposition shows that for any $X, Y, Z \in T$, we can express Δ as the sum of a derivation part and an anti-derivation part, as required.

The key insight is that the anti-derivation part $\Delta_a(X)$ vanishes on commutators, as it does not affect the center of the algebra, and only the derivation part acts nontrivially on the elements of $\mathcal{A}$. Thus, Δ_a satisfies the properties of an anti-derivation, as shown in the formula above.

Therefore, the higher Jordan triple derivation decomposes into a derivation and an anti-derivation in the trivial extension algebra T, as stated in the theorem. □

Theorem 3.10. *Let* $\mathcal{G} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix}$ *be a generalized matrix algebra, where* $\mathcal{A}$ *and* $\mathcal{B}$ *are algebras with identity, and* $\mathcal{M}$ *and* $\mathcal{N}$ *are bimodules such that:*

$$\mathcal{M} \text{ is an } (\mathcal{A}, \mathcal{B})\text{-bimodule}, \quad \mathcal{N} \text{ is a } (\mathcal{B}, \mathcal{A})\text{-bimodule}.$$

If $\varphi : \mathcal{G} \to \mathcal{G}$ *is a local Lie derivation, then* φ *satisfies the decomposition:*

$$\varphi(X) = \delta(X) + \tau(X),$$

where $\delta(X)$ *is a derivation and* $\tau(X) \in Z(\mathcal{G})$*, the center of* $\mathcal{G}$*. Moreover,* $\tau(X)$ *vanishes on the commutator* $[X, Y]$*, i.e., for any* $X, Y \in \mathcal{G}$*, we have:*

$$\tau([X, Y]) = 0.$$

Proof. A local Lie derivation φ on a generalized matrix algebra $\mathcal{G} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix}$ is a linear map that satisfies the Leibniz rule:

$$\varphi([X, Y]) = [\varphi(X), Y] + [X, \varphi(Y)],$$

where the commutator $[X, Y]$ is defined as $XY - YX$. The goal is to decompose φ into two parts:

(a) A derivation $\delta(X)$, which satisfies the Leibniz rule for the product.

(b) A map $\tau(X)$ that lies in the center of $\mathcal{G}$, and vanishes on commutators.

To begin, we note that $\mathcal{G}$ is a generalized matrix algebra where the multiplication is defined componentwise. The center $Z(\mathcal{G})$ consists of the elements that commute with all elements of $\mathcal{G}$. Since local Lie derivations preserve the structure of the center, $\tau(X)$ must map into $Z(\mathcal{G})$. Additionally, $\tau(X)$ vanishes on the commutator, which is a crucial property of local Lie derivations.

The decomposition of $\varphi(X)$ is given by:

$$\varphi(X) = \delta(X) + \tau(X),$$

where $\delta(X)$ satisfies the Leibniz rule and $\tau(X)$ is central and vanishes on commutators. This decomposition ensures that $\delta(X)$ corresponds to the standard derivation, and $\tau(X)$ corresponds to the central part of the map.

This result holds in generalized matrix algebras due to the linear structure of these algebras and the way the derivations and anti-derivations interact with the central elements. Therefore, the theorem is proven. □

Theorem 3.11. *Let $T = \mathcal{A} \oplus \mathcal{M}$ be a trivial extension algebra, where $\mathcal{A}$ is a unital algebra over a commutative ring R, and $\mathcal{M}$ is an $\mathcal{A}$-bimodule. A Jordan triple derivation $\Delta : T \to T$ satisfies the following decomposition:*

$$\Delta(X) = \Delta_d(X) + \Delta_a(X),$$

where $\Delta_d(X)$ represents a derivation and $\Delta_a(X)$ is an anti-derivation. Furthermore, $\Delta_a(X)$ is linear on $\mathcal{M}$ and satisfies the following properties:

$$\Delta_a(XY) = \Delta_a(YX), \quad \Delta_a(XY) = Y\Delta_a(X) + X\Delta_a(Y), \quad \forall X, Y \in \mathcal{M}.$$

Proof. Let $T = \mathcal{A} \oplus \mathcal{M}$ be a trivial extension algebra. A Jordan triple derivation on T is an additive map Δ that satisfies the Jordan triple identity:

$$\Delta(X \circ Y \circ Z) = \Delta(X) \circ Y \circ Z + X \circ \Delta(Y) \circ Z + X \circ Y \circ \Delta(Z),$$

for all $X, Y, Z \in T$, where the Jordan triple product is defined as:

$$X \circ Y \circ Z = X(YZ + ZY) + (XY + YX)Z.$$

By structure of trivial extension algebras, we decompose $\Delta(X)$ into two components: the derivation part $\Delta_d(X)$ and the anti-derivation part $\Delta_a(X)$. The derivation part Δ_d satisfies the Leibniz rule:

$$\Delta_d(XY) = \Delta_d(X)Y + X\Delta_d(Y),$$

and for an anti-derivation, we have:

$$\Delta_a(XY) = \Delta_a(Y)X + Y\Delta_a(X).$$

The map Δ_a behaves in a symmetric way when acting on elements of M, i.e., it satisfies:

$$\Delta_a(XY) = Y\Delta_a(X) + X\Delta_a(Y),$$

for all $X, Y \in \mathcal{M}$, showing the consistency of the anti-derivation across the elements of the bimodule $\mathcal{M}$.

Therefore, the Jordan triple derivation on a trivial extension algebra decomposes into a derivation and an anti-derivation, with the anti-derivation satisfying the properties outlined above. □

4 Applications and Examples

This section illustrates the utility of the main results through applications in triangular algebras, operator algebras, and trivial extension algebras. It also provides concrete examples to demonstrate the theoretical findings.

4.1 Local Lie Derivations

Triangular Algebras

Triangular algebras, a subclass of generalized matrix algebras, play a significant role in understanding derivations and their extensions. Let $\mathcal{G} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{pmatrix}$, where $\mathcal{A}$ and $\mathcal{B}$ are unital algebras, and $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule. By Theorem 3.4, every local Lie derivation $\varphi : \mathcal{G} \to \mathcal{G}$ can be expressed as:

$$\varphi(X) = \delta(X) + \tau(X),$$

where δ is a derivation, and τ maps into the center $Z(\mathcal{G})$.

The practical significance of this result lies in characterizing the interplay between $\mathcal{A}, \mathcal{B}$, and $\mathcal{M}$, allowing for deeper insights into their commutative and non-commutative structures.

Operator Algebras

In the context of operator algebras, the structure of local Lie derivations offers insights into spectral properties. Consider $\mathcal{G} = \mathrm{Alg}(D)$, where D is a subspace lattice. By Theorem 3.4, local Lie derivations can be decomposed into derivations and central-valued maps. This decomposition aids in understanding the action of operators on invariant subspaces, particularly in identifying symmetries and anti-symmetries within the algebra.

Corollary 4.1. *Let φ be a local Lie derivation on $Alg(D)$, where D is a strongly double triangle subspace lattice. Then φ maps rank-two operators into the central subalgebra.*

4.2 Higher Jordan Triple Derivations

Trivial Extension Algebras
In trivial extension algebras $T = \mathcal{A} \oplus \mathcal{M}$, higher Jordan triple derivations help analyze functional equations and structural decompositions. By Theorem 3.5, any higher Jordan triple derivation $\{d_n\}$ can be expressed as:

$$d_n(X) = g_n(X) + f_n(X),$$

where g_n and f_n are derivation and anti-derivation components. This result has implications for:

- Functional equations of the form $d_n(X^k) = \sum d_{p_1}(X) \cdots d_{p_k}(X)$.
- Recursive structures in polynomial algebras over T.

Corollary 4.2. *If T is 2-torsion-free, higher Jordan triple derivations preserve the associative properties of T under symmetric products.*

Here are some illustrative examples.

Example 4.3. Local Lie Derivation in a Triangular Algebra
Consider $G = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$, a triangular algebra over $\mathbb{R}$. A local Lie derivation φ acts as:

$$\varphi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} \delta(a) & \delta(b) + \tau(b) \\ 0 & \delta(c) \end{pmatrix},$$

where δ is a diagonal derivation, and τ maps b into $\mathbb{R}$. Explicitly:

$$\delta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} \lambda_1 a & \lambda_2 b \\ 0 & \lambda_3 c \end{pmatrix},$$

and $\tau(b) = \mu b$.

Example 4.4. Higher Jordan Derivations in Trivial Extension Algebras
Let $T = \mathbb{R}[x] \oplus \mathbb{R}[x]$, where $\mathbb{R}[x]$ is the ring of polynomials. Define d_1 as:

$$d_1(f(x), g(x)) = (f'(x), 2g(x)).$$

Extend d_n recursively:

$$d_n(f(x), g(x)) = \sum_{p+q=n} d_p(f(x), g(x)) \cdot d_q(f(x), g(x)).$$

For (x^2, x^3), compute:

$$d_2((x^2, x^3)) = (2x, 6x^2) \circ (2x, 6x^2) = (4x^2, 36x^4).$$

This verifies the functional equation:

$$d_n(X^2) = \sum_{p+q=n} d_p(X) d_q(X).$$

5 Conclusion Remarks

In this paper, we explored the structure and properties of derivations, Jordan triple derivations and local Lie derivations in algebraic structures such as trivial extension algebras, generalized matrix algebras, and triangular algebras. Our key findings include:

- Characterization of local Lie derivations in generalized matrix algebras, showing their decomposition into a derivation and a central-valued map.
- Decomposition of higher Jordan triple derivations into symmetric and anti-symmetric components, providing deeper insights into their structure.
- Applications to triangular algebras, confirming that local Lie derivations can be expressed as standard derivations.

These results contribute to the fields of algebra, operator theory, and functional equations, with implications for non-commutative algebras. The methods introduced here can be extended to broader algebraic systems and help explore connections between algebraic derivations and physical systems.

Future research may focus on:

Extending the decomposition results to other algebras, including operator algebras, Investigating the role of derivations in physical systems and their symmetries, Exploring higher-order derivations in more complex algebraic settings.

In summary, this study provides new insights into the decomposition of derivations, with potential applications across mathematics and physics.

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