Palestine Journal of Mathematics

Vol.15(3), (2026) , 1148–1157 © Palestine Polytechnic University-PPU 2026

ON BI-ALGEBRAS UNDER FUZZY POINTS

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Communicated by Madeleine Al Tahan

2020 Mathematics Subject Classification: Primary 03E72 Theory of Fuzzy Sets; Secondary 03G25, 08A72.

Keywords and phrases: BI-Algebras, Fuzzy Set, Fuzzy Point, BI-Ideals, Fuzzy Point Ideal.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract This paper explores the application of fuzzy point theory to BI-algebras. The study introduces a new operation on fuzzy points and shows that it preserves all key properties of a BI-algebra. Several fundamental results are established, including the behavior of fuzzy points under the operation $\odot$ and the verification of both left and right distributive laws. We introduce the concept of a fuzzy point ideal and show that a subset of fuzzy points forms an ideal if and only if it contains 0_φ and satisfies a specific absorption condition. This work extends the theory of BI-algebras by incorporating fuzzy points and examining their algebraic properties.

1 Introduction

The study of logical algebras has produced several important structures, including *BI*-algebras, which were introduced by Saeid et al. [14]. This class occupies a significant position as it provides a relatively simple yet rich structure that connects and generalizes several well-known classes, such as *BCH*-algebras [6] and *B*-algebras [11]. On the other hand, fuzzy set theory, pioneered by Zadeh [20], has become a fundamental tool for handling uncertainty in mathematical systems. A particularly granular approach within this theory is the concept of a fuzzy point, introduced by Wong [19] and developed by Pu and Liu [13].

Fuzzy points facilitate a local, point-wise analysis that offers finer detail than the global perspective of fuzzy subsets. This granularity is especially valuable in algebra because it allows operations and axioms to be examined at the level of individual membership degrees, preserving the precise logical relationships that define the structure. While this point-wise theory and the fuzzification of algebraic structures (e.g., in *B*-algebras [1]) are both well-studied, their intersection remains unexplored: there is no systematic framework applying fuzzy point theory to *BI*-algebras. Recent studies on fuzzy algebraic structures such as fuzzy ideals in TM algebras [3], complex fuzzy soft rings [2], and prime elements in BE-algebras [18], highlight the growing interest in this field and further motivate the present investigation.

This paper fills that gap by constructing a *BI*-algebra structure directly on fuzzy points. The point-wise approach is crucial; it allows algebraic operations to be defined at the most fundamental level, thereby preserving the intrinsic equational axioms in a natural, point-wise form. The primary objectives of this study are: (i) to define a suitable operation $\odot$ on fuzzy points and prove it satisfies all *BI*-algebra axioms; (ii) to establish key properties of this structure; and (iii) to introduce and characterize the novel concept of a fuzzy point ideal.

The successful completion of these objectives accomplishes two main goals. First, it provides a rigorous point-wise fuzzification of *BI*-algebras, thereby extending their applicability to domains involving uncertainty. Second, and more significantly, it moves beyond merely mimicking subset-level results; by working directly with fuzzy points, the framework yields genuinely new structural insights.

2 Preliminaries

This section recalls essential definitions, examples, and propositions regarding *BI*-algebras and fuzzy point theory that form the foundation for the results in subsequent sections.

Definition 2.1. [12] An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BI-algebra* if for all $x, y \in X$,

(BI_1) $x * x = 0$

(BI_2) $x * (y * x) = x$.

Example 2.2. Let $X = \{0, 1, 2, 3\}$ be a set with the binary operation "$*$" defined in the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Let $x = 3$ and $y = 1$. Then $3 * 3 = 0$ and $3 * (1 * 3) = 3 * 0 = 3$. Clearly, conditions (BI_1) and (BI_2) are satisfied. The same holds for all arbitrary values of $x, y \in X$. Therefore, $(X; *, 0)$ is a BI-algebra.

From this point forward, X will denote the BI-algebra $(X; *, 0)$ unless otherwise specified.

Definition 2.3. [12] A *BI*-algebra *X* is said to be *right distributive* if $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.

Definition 2.4. [12] A *BI*-algebra *X* is said to be *left distributive* if $z * (x * y) = (z * x) * (z * y)$ for all $x, y, z \in X$.

Definition 2.5. [12] A subset *I* of *BI*-algebra *X* is called an *ideal* of BI-algebra X if

(I_1) $0 \in I$,

(I_2) $y \in I$ and $x * y \in I$ imply $x \in I$ for any $x, y \in X$.

Definition 2.6. [16] Let X be a nonempty set. A *fuzzy set* μ on X is a function μ: $X \to [0, 1]$. For all $x \in X$, the image $\mu(x)$ of X is called the degree of membership of x in X under μ.

Definition 2.7. [16] Let S be a nonempty set. A mapping μ: $S \to [0, 1]$ is called a *fuzzy subset* μ of the set S.

Example 2.8. Consider the *BI*-Algebra X, let $\mu : X \to [0, 1]$ be defined as follows

$$\mu = \begin{cases} 0.8 & \text{if x=0} \\ 0.5 & \text{if x=2} \\ 0.2 & \text{otherwise} \end{cases}$$

Then μ is a fuzzy subset of the set X.

Definition 2.9. [11] A fuzzy set μ in X is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at $\mu(x)$ is α, $(0 < \alpha \leq 1)$, then denote this fuzzy point by x_α, where the point x is called its support. The set of all fuzzy points on X is denoted by $FP(X)$.

Proposition 2.10. *[12]* Let *X* be a *BI*-algebra. Then for all $x, y \in X$,

(i) $x * 0 = x$,

(ii) $0 * x = 0$, and

(iii) $x * y = (x * y) * y$.

Proposition 2.11. *[12]* Let X be a right *BI*-algebra. Then for all $x, y, z \in X$,

(i) $y * x \leq y$,

(ii) $(y * x) * x \leq y$,

(iii) $(x * z) * (y * z) \leq x * y$, and

(iv) $(x * y) * z \leq x * (y * z)$.

3 Fuzzy Points in BI-algebras

This section defines a binary operation $\odot$ on the set of fuzzy points $FP(X)$ and examines when $(FP(X); \odot, 0_\sigma)$ forms a BI-algebra, including left and right distributivity and various identities. The point-wise operation $\odot$ defined below allows us to examine BI-algebra axioms at the level of individual membership degrees.

Theorem 3.1. Let X be a BI-algebra and $FP(X)$ be the set of all fuzzy points on X. Define the operation $\odot$ on $FP(X)$ by $x_\alpha \odot y_\beta = (x * y)_{\min\{\alpha,\beta\}}$ for all $x_\alpha, y_\beta \in FP(X)$. Then $\odot$ is a *binary operation* on $FP(X)$.

Proof. Let $x_\alpha, y_\beta \in FP(X)$. Since X is a BI-algebra, it is closed under $*$, so $(x * y) \in X$. Moreover, $\min\{\alpha, \beta\} \in (0, 1]$, hence $(x * y)_{\min\{\alpha,\beta\}}$ is a fuzzy point on X. Thus $x_\alpha \odot y_\beta \in FP(X)$. Additionally, since $\min\{\alpha, \beta\}$ lies in the interval $(0, 1]$, $(x * y)_{\min\{\alpha,\beta\}}$ is also a fuzzy point on X. Hence, $x_\alpha \odot y_\beta \in FP(X)$. The operation "$\odot$" is independent of the choice of representatives x and y. For any $x, x' \in FP(X)$ and $y, y' \in FP(X)$ such that $x = x'$ and $y = y'$, $(x * y)_{\min\{\alpha,\beta\}} = (x' * y')_{\min\{\alpha,\beta\}}$ for all $\alpha, \beta \in (0, 1]$. Consequently, $x_\alpha \odot y_\beta = x'_\alpha \odot y'_\beta$, and thus $\odot$ is well-defined. □

The following example can be utilized in subsequent examples of the results unless another example is mentioned.

Example 3.2. Consider the *BI*-Algebra X in Example 2.2 and the fuzzy set in Example 2.8. The fuzzy set μ yields four fuzzy points: $0_{0.8}$, $1_{0.2}$, $2_{0.5}$, and $3_{0.2}$. Moreover, these four points are elements of $FP(X)$. These fuzzy points can be operated as defined in Theorem 3.1 in the following table of operation:

$\odot$	$0_{0.8}$	$1_{0.2}$	$2_{0.5}$	$3_{0.2}$
$0_{0.8}$	$0_{0.8}$	$0_{0.2}$	$0_{0.5}$	$0_{0.2}$
$1_{0.2}$	$1_{0.2}$	$0_{0.2}$	$1_{0.2}$	$0_{0.2}$
$2_{0.5}$	$2_{0.5}$	$2_{0.2}$	$0_{0.5}$	$0_{0.2}$
$3_{0.2}$	$3_{0.2}$	$2_{0.2}$	$1_{0.2}$	$0_{0.2}$

Computing $2_{0.5} \odot 0_{0.8}$ yields $2_{0.5}$, while $1_{0.2} \odot 3_{0.2} = 0_{0.2}$. Notice that $0_{0.2}$ is not included in the list of four fuzzy points as mentioned in Example 2.8. Thus, it is possible that a new fuzzy point is obtained by operating two distinct fuzzy points under the binary operation "$\odot$", and it is still a fuzzy point in the *BI*-algebra *X*.

The following proposition shows that the binary operation "$\odot$" almost satisfies the two conditions (BI_1) and (BI_2) of Definition 2.1.

Proposition 3.3. Let *X* be a *BI*-algebra and $FP(X)$ be the set of all fuzzy points on *X*. Then for all $x, y \in X$ and $\alpha, \beta \in (0, 1]$,

$(BI_{1'})$ $x_\alpha \odot x_\alpha = 0_\alpha$; and

$(BI_{2'})$ $x_\alpha \odot (y_\beta \odot x_\alpha) = x_{\min\{\alpha,\beta\}}$.

Example 3.4. Consider the set of all fuzzy points $FP(X)$ on X defined in the binary operation "$\odot$" in Example 3.2. Let $x_\alpha = 0_{0.8}$ and $y_\beta = 1_{0.2}$. Then $0_{0.8} \odot 0_{0.8} = 0_{0.8}$, and also it follows that $0_{0.8} \odot (1_{0.2} \odot 0_{0.8}) = 0_{0.8} \odot 1_{0.2} = 0_{0.2}$. Clearly (BI'_1) and (BI'_2) of Proposition 3.3 are satisfied. The same is true for all arbitrary values of $x_\alpha, y_\beta \in FP(X)$.

Consider the triple $(FP(X); \odot, 0_\sigma)$ where $\sigma \in (0, 1]$. Does it follow that $(FP_\sigma(X); \odot, 0_\sigma)$ is a *BI*-algebra? Let $x_\alpha \in FP(X)$ and $\alpha \neq \sigma$. Observe that from (BI_1) of Definition 2.1, $x_\alpha \odot x_\alpha = (x * x)_{\min\{\alpha,\alpha\}} = 0_\alpha \neq 0_\sigma$. Thus, the condition $(BI_{1'})$ in Proposition 3.3 is not satisfied. Consequently, $(FP(X); \odot, 0_\sigma)$ is not a BI-algebra. However, if all fuzzy points in $FP(X)$ have the same membership value, then by Proposition 3.3, $(FP(X); \odot, 0_\sigma)$ becomes a BI-algebra. The following corollary records this observation.

Corollary 3.5. If X is a *BI*-algebra and $FP_\sigma(X) \subseteq FP(X)$ is the set of all fuzzy points of X in which every fuzzy point has the membership value $\sigma \in (0,1]$, then $(FP_\sigma(X); \odot, 0_\sigma)$ is a *BI*-algebra.

We now present some properties of *BI*-algebras using the concept of fuzzy points.

Proposition 3.6. Let X be a *BI*-algebra and $FP(X)$ be the set of all fuzzy points on X. Then $(x_\alpha \odot y_\beta) \odot z_\gamma = (x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma)$ is *right distributive* for all $x, y, z \in X$ and $\alpha, \beta, \gamma \in (0,1]$.

Proof. Let $x_\alpha, y_\beta, z_\gamma \in FP(X)$. Then by Theorem 3.1 and Definition 2.3,

$$\begin{aligned}(x_\alpha \odot y_\beta) \odot z_\gamma &= (x * y)_{\min\{\alpha,\beta\}} \odot z_\gamma \\ &= ((x * y) * z)_{\min\{\alpha,\beta,\gamma\}} \\ &= ((x * z) * (y * z))_{\min\{\alpha,\beta,\gamma\}} \\ &= ((x_\alpha * z_\gamma)_{\min\{\alpha,\gamma\}}) \odot ((y_\beta * z_\gamma)_{\min\{\beta,\gamma\}}) \\ &= (x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma).\end{aligned}$$

□

Proposition 3.7. Let X be a *BI*-algebra and $FP(X)$ be the set of all fuzzy points on X. Then $z_\gamma \odot (x_\alpha \odot y_\beta) = (z_\gamma \odot x_\alpha) \odot (z_\gamma \odot y_\beta)$ is *left distributive* for all $x, y, z \in X$ and $\alpha, \beta, \gamma \in (0,1]$.

Proof. Let $x_\alpha, y_\beta, z_\gamma \in FP(X)$. Then by Theorem 3.1 and Definition 2.4,

$$\begin{aligned}z_\gamma \odot (x_\alpha \odot y_\beta) &= z_\gamma \odot (x * y)_{\min\{\alpha,\beta\}} \\ &= (z * (x * y))_{\min\{\alpha,\beta,\gamma\}} \\ &= ((z * x) * (z * y))_{\min\{\alpha,\beta,\gamma\}} \\ &= (z_\gamma \odot x_\alpha) \odot (z_\gamma \odot y_\beta).\end{aligned}$$

□

Proposition 3.8. Let X be a *BI*-algebra and $FP(X)$ be the set of all fuzzy points on X. Then for any $x_\alpha \in FP(X)$,

(i) $x_\alpha \odot 0_\delta = x_{\min\{\alpha,\delta\}}$; and

(ii) $0_\delta \odot x_\alpha = 0_{\min\{\delta,\alpha\}}$.

Proof. Let $x_\alpha \in FP(X)$ and $\delta \in (0,1]$ such that $0_\delta \in FP(X)$.

(i) By Theorem 3.1 and Proposition 2.10 (*i*), $x_\alpha \odot 0_\delta = (x * 0)_{\min\{\alpha,\delta\}} = x_{\min\{\alpha,\delta\}}$.

(ii) By Theorem 3.1 and Proposition 2.10 (*ii*), $0_\delta \odot x_\alpha = (0 * x)_{\min\{\delta,\alpha\}} = 0_{\min\{\delta,\alpha\}}$.

□

Proposition 3.9. Let X be a *BI*-algebra and $FP(X)$ be the set of all fuzzy points on X. Then for any $x_\alpha, y_\beta \in FP(X)$, $(x_\alpha \odot y_\beta) = (x_\alpha \odot y_\beta) \odot y_\beta$.

Proof. Let $x_\alpha, y_\beta \in FP(X)$. By Theorem 3.1 and Proposition 2.10 (*iii*),

$$\begin{aligned}(x_\alpha \odot y_\beta) &= (x * y)_{\min\{\alpha,\beta\}} \\ &= ((x * y) * y)_{\min\{\alpha,\beta\}} \\ &= (x_\alpha \odot y_\beta) \odot y_\beta.\end{aligned}$$

□

The following example can be used for the subsequent results in generating further examples from this point forward.

Example 3.10. Consider the *BI*-Algebra X in Example 2.2. Let $\mu : X \to [0,1]$ be defined as follows:

$$\mu = \begin{cases} 0.3 & \text{if } x = 0 \\ 0.4 & \text{otherwise} \end{cases}$$

Then μ is a fuzzy set on X. By Definition 2.9, the fuzzy points are $0_{0.3}$, $1_{0.4}$, $2_{0.4}$, and $3_{0.4}$. By Theorem 3.1, the following table of operations will be generated:

$\odot$	$0_{0.3}$	$1_{0.4}$	$2_{0.4}$	$3_{0.4}$
$0_{0.3}$	$0_{0.3}$	$0_{0.3}$	$0_{0.3}$	$0_{0.3}$
$1_{0.4}$	$1_{0.3}$	$0_{0.4}$	$1_{0.4}$	$0_{0.4}$
$2_{0.4}$	$2_{0.3}$	$2_{0.4}$	$0_{0.4}$	$0_{0.4}$
$3_{0.4}$	$3_{0.3}$	$2_{0.4}$	$1_{0.4}$	$0_{0.4}$

Proposition 3.11. If *BI*-algebra X is left distributive and $FP(X)$ be the set of all fuzzy points on X, then $FP(X) = \{0_\alpha\}$.

Proof. Let $x_\alpha \in FP(X)$. Then, by Theorem 3.1 and Proposition 3.3 $(B_{1'})$,

$$\begin{aligned} x_\alpha &= x_\alpha \odot (x_\alpha \odot x_\alpha) \\ &= x_\alpha \odot (x * x)_{\min\{\alpha,\alpha\}} \\ &= (x * (x * x))_{\min\{\alpha,\alpha,\alpha\}} \\ &= ((x * x) * (x * x))_{\min\{\alpha,\alpha,\alpha,\alpha\}} \\ &= (0 * 0)_{\min\{\alpha,\alpha,\alpha,\alpha\}} \\ &= 0_\alpha. \end{aligned}$$

Therefore, $FP(X) = \{0_\alpha\}$. □

Proposition 3.12. Let X be a *BI*-algebra and $FP(X)$ be the set of all fuzzy points on X. If $y_\beta \odot x_\alpha = x_{\min\{\beta,\alpha\}}$ for all $x_\alpha, y_\beta \in FP(X)$ where $\alpha \leq \beta$, then $FP(X) = \{0_\alpha\}$.

Proof. Let $y_\beta \odot x_\alpha = x_{\min\{\beta,\alpha\}}$ and $x_\alpha, y_\beta \in FP(X)$. Now, by Proposition 3.3 $(BI_{2'})$ and Proposition 3.3 $(BI_{1'})$, for any $x_\alpha \in FP(X)$,

$$\begin{aligned} x_\alpha = x_{\min\{\alpha,\beta\}} &= x_\alpha \odot (y_\beta \odot x_\alpha) \\ &= x_\alpha \odot x_{\min\{\beta,\alpha\}} \\ &= x_\alpha \odot x_\alpha \\ &= 0_\alpha. \end{aligned}$$

Hence $FP(X) = \{0_\alpha\}$. □

Proposition 3.13. Let X be a *BI*-algebra and $FP(X)$ be the set of all fuzzy points on X. If $x_\alpha \odot (y_\beta \odot x_\alpha) = (y_\beta \odot y_\beta) \odot x_\alpha$ for all $x_\alpha, y_\beta \in FP(X)$ and $\alpha \leq \beta$, then $FP(X) = \{0_\alpha\}$.

Proof. Let $x_\alpha \odot (y_\beta \odot x_\alpha) = (y_\beta \odot y_\beta) \odot x_\alpha$ and $x_\alpha, y_\beta \in FP(X)$. By Proposition 3.3 $(BI_{2'})$, Proposition 3.3 $(BI_{1'})$, and Proposition 3.9 (ii), for any $x_\alpha \in FP(X)$,

$$\begin{aligned} x_\alpha = x_{\min\{\alpha,\beta\}} &= x_\alpha \odot (y_\beta \odot x_\alpha) \\ &= (y_\beta \odot y_\beta) \odot x_\alpha \\ &= 0_\beta \odot x_\alpha \\ &= 0_{\min\{\alpha,\beta\}} \\ &= 0_\alpha. \end{aligned}$$

Hence $FP(X) = \{0_\alpha\}$. □

Propositions 3.11, 3.12, and 3.13 show that left distributivity and certain related identities force $FP(X) = \{0_\alpha\}$. These results directly mirror classical results in BI-algebras [14], where left distributivity, $y * x = x$, and $x * (y * z) = y * (x * z)$ each imply $X = \{0\}$. Thus, our fuzzy point formulation preserves these structural constraints of the classical theory.

Proposition 3.14. Let X be a right distributive *BI*-algebra and $FP(X)$ be the set of all fuzzy points on X. Then for all $x_\alpha, y_\beta \in FP(X)$,

(i) $y_\beta \odot x_\alpha \leq y_{\min\{\beta,\alpha\}}$

(ii) $(y_\beta \odot x_\alpha) \odot x_\alpha \leq y_{\min\{\beta,\alpha\}}$.

Proof.

(i) By Theorem 3.1 and Proposition 2.11 (*i*), $y_\beta \odot x_\alpha = (y * x)_{\min\{\beta,\alpha\}} \leq y_{\min\{\beta,\alpha\}}$.

(ii) By Theorem 3.1 and Proposition 2.11 (*ii*), $(y_\beta \odot x_\alpha) \odot x_\alpha = ((y*x)*x)_{\min\{\beta,\alpha\}} \leq y_{\min\{\beta,\alpha\}}$.

□

Proposition 3.15. Let X be a right distributive *BI*-algebra and $FP(X)$ be the set of all fuzzy points on X. Then for all $x_\alpha, y_\beta, z_\gamma \in \mathrm{FP}(X)$, $(x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma) \leq (x * y)_{\min\{\alpha,\beta,\gamma\}}$.

Proof. By Theorem 3.1 and Proposition 2.11 (*iii*),

$$\begin{aligned}(x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma) &= (x * z)_{\min\{\alpha,\gamma\}} \odot (y * z)_{\min\{\beta,\gamma\}} \\ &= ((x * z) * (y * z))_{\min\{\alpha,\beta,\gamma\}} \\ &\leq (x * y)_{\min\{\alpha,\beta,\gamma\}}.\end{aligned}$$

□

Proposition 3.16. Let X be a right distributive *BI*-algebra and $FP(X)$ be the set of all fuzzy points on X. For all $x_\alpha, y_\beta, z_\gamma \in \mathrm{FP}(X)$, $(x_\alpha \odot y_\beta) \odot z_\gamma \leq x_\alpha \odot (y_\beta \odot z_\gamma)$.

Proof. Let $x_\alpha, y_\beta, z_\gamma \in \mathrm{FP}(X)$. By Theorem 3.1 and Proposition 2.11 (*iv*),

$$\begin{aligned}(x_\alpha \odot y_\beta) \odot z_\gamma &= ((x * y) * z)_{\min\{\alpha,\beta,\gamma\}} \\ &\leq (x * (y * z))_{\min\{\alpha,\beta,\gamma\}} \\ &= x_\alpha \odot (y_\beta \odot z_\gamma).\end{aligned}$$

□

4 Fuzzy Points in Ideals of BI-Algebra

Having established the BI-algebra structure on $FP(X)$, we now investigate ideals within this framework. We introduce the notion of a fuzzy point ideal and examine its basic properties.

Definition 4.1. Let X be a BI-algebra. A non-empty subset $FP(I) \subseteq FP(X)$ is called a *fuzzy point ideal* of $FP(X)$ if it satisfies:

($\mathrm{I}_{1'}$) $0_\varphi \in FP(I)$, where $\varphi = \min\{\alpha, \beta\}$ for some $\alpha, \beta \in (0, 1]$; and

($\mathrm{I}_{2'}$) for any $x_\alpha, y_\beta \in FP(X)$, if $y_\beta \in FP(I)$ and $x_\alpha \odot y_\beta \in FP(I)$, then $x_\alpha \in FP(I)$.

Example 4.2. Let X be the BI-algebra from Example 2.2 and consider $FP(X)$ with the operation $\odot$ as defined in Theorem 3.1. Take the fuzzy set μ from Example 3.10 so that we can have $FP(X) = \{0_{0.3}, 1_{0.4}, 2_{0.4}, 3_{0.4}\}$.

(i) Consider the set $\{0_{0.3}\}$. The only $y_\beta \in \{0_{0.3}\}$ is $y_\beta = 0_{0.3}$. For any $x_\alpha \in FP(X)$, if $x_\alpha \odot 0_{0.3} \in \{0_{0.3}\}$, then $x_\alpha \odot 0_{0.3} = 0_{0.3}$, which occurs only when $x_\alpha = 0_{0.3}$. Since $0_{0.3} \in \{0_{0.3}\}$, condition ($\mathrm{I}_{2'}$) holds. All other x_α yield $x_\alpha \odot 0_{0.3} \notin \{0_{0.3}\}$, so the implication is vacuously true. Thus $\{0_{0.3}\}$ is an ideal (the trivial zero ideal).

(ii) Consider the set $FP(I_1) = \{0_{0.3}, 1_{0.4}\}$. Since $0_{0.3} \in FP(I_1)$, ($I_{1'}$) holds. Pick $y_\beta = 0_{0.3}$. For each $x_\alpha \in FP(X)$, the only x_α with $x_\alpha \odot 0_{0.3} \in FP(I_1)$ is $x_\alpha = 0_{0.3}$, and it follows that $0_{0.3} \in FP(I_1)$. Pick $y_\beta = 1_{0.4}$. For each $x_\alpha \in FP(X)$, the only x_α with $x_\alpha \odot 1_{0.4} \in FP(I_1)$ is $x_\alpha = 0_{0.3}$, and $0_{0.3} \in FP(I_1)$. Thus $FP(I_1)$ is an ideal of $FP(X)$.

(iii) Consider the set $FP(I_2) = \{0_{0.3}, 2_{0.4}\}$. Since $0_{0.3} \in FP(I_2)$, ($I_{1'}$) holds. Pick $y_\beta = 0_{0.3}$. The only x_α with $x_\alpha \odot 0_{0.3} \in FP(I_2)$ is $x_\alpha = 0_{0.3}$, and $0_{0.3} \in FP(I_2)$. Now pick $y_\beta = 2_{0.4}$. For each $x_\alpha \in FP(X)$, the only x_α with $x_\alpha \odot 2_{0.4} \in FP(I_2)$ is $x_\alpha = 0_{0.3}$, and $0_{0.3} \in FP(I_2)$. Thus $FP(I_2)$ is also an ideal of $FP(X)$. By symmetry, $\{0_{0.3}, 3_{0.4}\}$ is likewise an ideal.

(iv) Consider the set $FP(J) = \{0_{0.3}, 1_{0.4}, 2_{0.4}\}$. Since $0_{0.3} \in FP(J)$, condition ($I_{1'}$) holds. However, $FP(J)$ is not a fuzzy point ideal because $3_{0.4} \odot 1_{0.4} = 2_{0.4} \in FP(J)$ but we know that $3_{0.4} \notin FP(J)$. This violates condition ($I_{2'}$).

(v) Consider the set $FP(I_4) = FP(X)$. If $y_\beta \in FP(I_4)$ and $x_\alpha \odot y_\beta \in FP(I_4)$, then trivially $x_\alpha \in FP(I_4)$ because $FP(I_4) = FP(X)$ contains every x_α. Thus $FP(I_4)$ is an ideal.

The table below summarizes the ideal structure of $FP(X)$ for this concrete example.

Subset	Ideal?	Reason
$\{0_{0.3}\}$	Yes	Trivial zero ideal
$\{0_{0.3}, 1_{0.4}\}$	Yes	Exhaustive verification passes
$\{0_{0.3}, 2_{0.4}\}$	Yes	Exhaustive verification passes
$\{0_{0.3}, 3_{0.4}\}$	Yes	By symmetry with $\{0_{0.3}, 2_{0.4}\}$
$\{0_{0.3}, 0_{0.3}\}$	Yes	Contains only zero fuzzy points
$\{0_{0.3}, 1_{0.4}, 2_{0.4}\}$	No	$3_{0.4} \odot 1_{0.4} = 2_{0.4} \in FP(J)$ but $3_{0.4} \notin FP(J)$
$\{0_{0.3}, 1_{0.4}, 3_{0.4}\}$	No	By symmetry
$\{0_{0.3}, 2_{0.4}, 3_{0.4}\}$	No	By symmetry
$FP(X)$	Yes	Trivial whole ideal

This example demonstrates that not every subset containing 0_φ is an ideal. The ideal condition is restrictive, and membership values alone do not determine ideal status.

Theorem 4.3. *If $\{FP(I_i)\}_{i\in\Lambda}$ is a family of ideals of $FP(X)$, then $\bigcap_{i\in\Lambda} FP(I_i)$ is an ideal of $FP(X)$.*

Proof. Let $\{FP(I_i)\}_{i\in\Lambda}$ be a family of ideals of $FP(X)$. For each $i \in \Lambda$, since $FP(I_i)$ is an ideal, we have $0_\varphi \in FP(I_i)$. Hence $0_\varphi \in \bigcap_{i\in\Lambda} FP(I_i)$. Now suppose $y_\beta \in \bigcap_{i\in\Lambda} FP(I_i)$ and $x_\alpha \odot y_\beta \in \bigcap_{i\in\Lambda} FP(I_i)$ for some $x_\alpha, y_\beta \in FP(X)$. Then for every $i \in \Lambda$, we have $y_\beta \in FP(I_i)$ and $x_\alpha \odot y_\beta \in FP(I_i)$. Since each $FP(I_i)$ is an ideal, Definition 4.1($I_{2'}$) implies $x_\alpha \in FP(I_i)$ for all $i \in \Lambda$. Therefore $x_\alpha \in \bigcap_{i\in\Lambda} FP(I_i)$. Thus $\bigcap_{i\in\Lambda} FP(I_i)$ satisfies both conditions of Definition 4.1 and is therefore an ideal of $FP(X)$. □

Now, for any $x_\alpha, y_\beta \in FP(X)$, define

$$\begin{aligned} A(x_\alpha, y_\beta) &:= \{t_\delta \in FP(X) : (t_\delta \odot x_\alpha) \odot y_\beta = 0_\varphi\} \\ &:= \{t_\delta \in FP(X) : ((t * x) * y)_{\min\{\delta,\alpha,\beta\}} = 0_\varphi\} \end{aligned}$$

such that $\varphi = \min\{\delta, \alpha, \beta\}$.

Theorem 4.4. If $FP(X)$ is a right distributive BI-algebra under fuzzy points, then for any $x_\alpha, y_\beta \in FP(X)$, the set $A(x_\alpha, y_\beta)$ is an ideal of $FP(X)$.

Proof. Assume $FP(X)$ is right distributive. We show that $A(x_\alpha, y_\beta)$ satisfies Definition 4.1. For any $\delta \in (0, 1]$, $(0_\delta \odot x_\alpha) \odot y_\beta = 0_{\min\{\delta,\alpha\}} \odot y_\beta = 0_\varphi \odot y_\beta = 0_\varphi$, where $\varphi = \min\{\delta, \alpha, \beta\}$. Hence $0_\delta \in A(x_\alpha, y_\beta)$, so $A(x_\alpha, y_\beta)$ is nonempty and contains the zero fuzzy point. Let $t_{\delta'} \in A(x_\alpha, y_\beta)$

and suppose $z_\gamma \odot t_{\delta'} \in A(x_\alpha, y_\beta)$ for some $z_\gamma \in FP(X)$. We must show $z_\gamma \in A(x_\alpha, y_\beta)$. Since $z_\gamma \odot t_{\delta'} \in A(x_\alpha, y_\beta)$, $((z_\gamma \odot t_{\delta'}) \odot x_\alpha) \odot y_\beta = 0_\varphi$. Using right distributivity,

$$\begin{aligned}((z_\gamma \odot t_{\delta'}) \odot x_\alpha) \odot y_\beta &= ((z_\gamma \odot x_\alpha) \odot (t_{\delta'} \odot x_\alpha)) \odot y_\beta \\ &= (z_\gamma \odot x_\alpha) \odot ((t_{\delta'} \odot x_\alpha) \odot y_\beta).\end{aligned}$$

Since $t_{\delta'} \in A(x_\alpha, y_\beta)$, we have $(t_{\delta'} \odot x_\alpha) \odot y_\beta = 0_\varphi$. Thus

$$((z_\gamma \odot t_{\delta'}) \odot x_\alpha) \odot y_\beta = (z_\gamma \odot x_\alpha) \odot 0_\varphi = 0_\varphi.$$

Therefore $z_\gamma \in A(x_\alpha, y_\beta)$. Hence $A(x_\alpha, y_\beta)$ satisfies ($I_{2'}$). By Definition 4.1, $A(x_\alpha, y_\beta)$ is an ideal of $FP(X)$. □

Proposition 4.5. Let X be a BI-algebra. Then for all $x_\alpha, y_\beta \in FP(X)$, $A(x_\alpha, 0_\varphi) \subseteq A(x_\alpha, y_\beta)$.

Proof. Let $x_\alpha, y_\beta \in FP(X)$ and let $t_\delta \in A(x_\alpha, 0_\varphi)$. Then by definition, $(t_\delta \odot x_\alpha) \odot 0_\varphi = 0_\varphi$, which implies $t_\delta \odot x_\alpha = 0_\varphi$. Now, $(t_\delta \odot x_\alpha) \odot y_\beta = 0_\varphi \odot y_\beta = 0_\varphi$, so $t_\delta \in A(x_\alpha, y_\beta)$. Hence $A(x_\alpha, 0_\varphi) \subseteq A(x_\alpha, y_\beta)$. □

Proposition 4.6. Let X be a BI-algebra. Then for all $x_\alpha, y_\beta \in FP(X)$,

$$A(x_\alpha, 0_\varphi) = \bigcap_{y_\beta \in FP(X)} A(x_\alpha, y_\beta).$$

Proof. We prove the equality by showing both inclusions. Let $t_\delta \in A(x_\alpha, 0_\varphi)$. Then by definition, $(t_\delta \odot x_\alpha) \odot 0_\varphi = 0_\varphi$, which implies $t_\delta \odot x_\alpha = 0_\varphi$. Now for any $y_\beta \in FP(X)$, $(t_\delta \odot x_\alpha) \odot y_\beta = 0_\varphi \odot y_\beta = 0_\varphi$. Thus $t_\delta \in A(x_\alpha, y_\beta)$ for every $y_\beta \in FP(X)$. Hence $t_\delta \in \bigcap_{y_\beta \in FP(X)} A(x_\alpha, y_\beta)$. Let $t_\delta \in \bigcap_{y_\beta \in FP(X)} A(x_\alpha, y_\beta)$. Then $t_\delta \in A(x_\alpha, y_\beta)$ for every $y_\beta \in FP(X)$. In particular, taking $y_\beta = 0_\varphi$, we have $t_\delta \in A(x_\alpha, 0_\varphi)$. Since both inclusions hold, we conclude that for all $x_\alpha \in FP(X)$,

$$A(x_\alpha, 0_\varphi) = \bigcap_{y_\beta \in FP(X)} A(x_\alpha, y_\beta).$$

□

Theorem 4.7. Let $FP(I)$ be a non-empty subset of $FP(X)$. Then $FP(I)$ is an ideal of $FP(X)$ if and only if $A(x_\alpha, y_\beta) \subseteq FP(I)$ for all $x_\alpha, y_\beta \in FP(I)$.

Proof. Assume $FP(I)$ is an ideal of $FP(X)$ and let $x_\alpha, y_\beta \in FP(I)$. Suppose $t_\delta \in A(x_\alpha, y_\beta)$. Then $(t_\delta \odot x_\alpha) \odot y_\beta = 0_\varphi \in FP(I)$. Since $FP(I)$ is an ideal and $x_\alpha, y_\beta \in FP(I)$, Definition 4.1($I_{2'}$) implies $t_\delta \in FP(I)$. Hence $A(x_\alpha, y_\beta) \subseteq FP(I)$ for all $x_\alpha, y_\beta \in FP(I)$. Conversely, suppose $A(x_\alpha, y_\beta) \subseteq FP(I)$ for all $x_\alpha, y_\beta \in FP(I)$. First, we show $0_\varphi \in FP(I)$. For any $x_\alpha, y_\beta \in FP(I)$, $(0_\varphi \odot x_\alpha) \odot y_\beta = 0_\varphi \odot y_\beta = 0_\varphi$, so $0_\varphi \in A(x_\alpha, y_\beta) \subseteq FP(I)$. Thus $0_\varphi \in FP(I)$. Now let $w_\omega \in FP(I)$ and $z_\gamma \odot w_\omega \in FP(I)$. Observe that

$$(z_\gamma \odot w_\omega) \odot (z_\gamma \odot w_\omega) = 0_\varphi,$$

which implies $z_\gamma \in A(w_\omega, z_\gamma \odot w_\omega)$. Since $w_\omega, z_\gamma \odot w_\omega \in FP(I)$, the hypothesis gives $A(w_\omega, z_\gamma \odot w_\omega) \subseteq FP(I)$, and therefore $z_\gamma \in FP(I)$. Thus $FP(I)$ satisfies Definition 4.1 and is an ideal of $FP(X)$. □

Theorem 4.8. If $FP(I)$ is an ideal of $FP(X)$, then $FP(I) = \bigcup_{x_\alpha, y_\beta \in FP(I)} A(x_\alpha, y_\beta)$.

Proof. Let $FP(I)$ be an ideal of $FP(X)$ and $t_\delta \in FP(I)$. Since $(t_\delta \odot t_\delta) \odot 0_\varphi = 0_\varphi \odot 0_\varphi = 0_\varphi$, we have $t_\delta \in A(t_\delta, 0_\varphi)$. As $t_\delta, 0_\varphi \in FP(I)$, it follows that $t_\delta \in \bigcup_{x_\alpha, y_\beta \in FP(I)} A(x_\alpha, y_\beta)$. Hence $FP(I) \subseteq \bigcup_{x_\alpha, y_\beta \in FP(I)} A(x_\alpha, y_\beta)$. Next, let $t_\delta \in \bigcup_{x_\alpha, y_\beta \in FP(I)} A(x_\alpha, y_\beta)$. Then there exist $x_\alpha, y_\beta \in FP(I)$ such that $t_\delta \in A(x_\alpha, y_\beta)$. By Theorem 4.7, since $FP(I)$ is an ideal and $x_\alpha, y_\beta \in FP(I)$, we have $A(x_\alpha, y_\beta) \subseteq FP(I)$. Thus $t_\delta \in FP(I)$. Therefore,

$$\bigcup_{x_\alpha, y_\beta \in FP(I)} A(x_\alpha, y_\beta) \subseteq FP(I).$$

Since both inclusions hold, $FP(I) = \bigcup_{x_\alpha, y_\beta \in FP(I)} A(x_\alpha, y_\beta)$. □

The following corollary is a direct implication of Theorem 4.8 for every $0_\varphi, x_\alpha \in FP(I)$.

Corollary 4.9. If $FP(I)$ is an ideal of $FP(X)$, then

$$FP(I) = \bigcup_{x_\alpha \in FP(I)} A(0_\varphi, x_\alpha).$$

5 Conclusion

This paper introduced a binary operation $\odot$ on $FP(X)$ of a BI-algebra X, and established that $(FP_\sigma(X); \odot, 0_\sigma)$ forms a BI-algebra when all fuzzy points share the same membership value σ. The structure satisfies left and right distributivity, and conditions forcing triviality mirror classical BI-algebra theory. The main contribution is the characterization of fuzzy point ideals: $FP(I) \subseteq FP(X)$ is an ideal iff it contains 0_φ and satisfies $x_\alpha \in FP(I)$ whenever $y_\beta \in FP(I)$ and $x_\alpha \odot y_\beta \in FP(I)$. Future work includes defining congruence relations induced by fuzzy point ideals and examining quotient structures, obtaining a full classification of fuzzy point ideals for finite BI-algebras and extending this framework to other logical algebras.

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Received: 2025-02-11
Accepted: 2026-03-01