**Palestine Journal of Mathematics**

**Vol.15(3), (2026)** , 1158–1166 © Palestine Polytechnic University-PPU 2026

# Bessel sequences in Hilbert $C^*$-modules

Abdelilah Karara, Khadija Mabrouk and Mohamed Rossafi

Communicated by Harikrishnan Panackal

MSC 2020 Classifications: 42C15.

Keywords and phrases: Frame, Bessel sequence, synthesis operator, stability.

*Acknowledgement: The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.*

**Corresponding Author: Mohamed Rossafi**

**Abstract**. In this paper we study certain Bessel sequences $\{f_k\}_{k=1}^{\infty}$ in Hilbert $C^*$-modules $\mathcal{H}$ for which the operator $S$ defined by (1.2) is of the form $\mathcal{T}+\xi I$, for some real number $\xi$ and an adjointable linear operator $\mathcal{T}$. Additionally, we investigate frames known as compact-tight frames, which have frame operators that are compact perturbations of scalar multiples of the identity. As a conclusion, we provide a theory regarding the weaving of specific compact-tight frames.

## 1 Introduction

The notion of frame is a recent active mathematical research topic with applications in signal processing, computer science, and related areas. Frames for Hilbert spaces were first introduced in **1952** by Duffin and Schaeffer [2] for the study of nonharmonic Fourier series. Daubechies, Grossmann, and Meyer [1] revived and developed them in **1986**, and the theory has been widely popularized since then.

Many mathematicians have recently generalized frame theory from Hilbert spaces to Hilbert $C^*$-modules. For details on frames in Hilbert $C^*$-modules we refer to [3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14]. The purpose of this paper is to investigate Bessel sequences and frames whose frame operator admits a decomposition of the form $\mathcal{T}+\xi I$.

Throughout this paper, let $\mathcal{H}$ be a countably generated Hilbert $\mathcal{A}$-module and $\{\mathcal{K}_k\}_{k\in\mathbb{Z}}$ be a collection of Hilbert $\mathcal{A}$-modules. We also use the notation $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{K}_k)$ for the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{H}$ to $\mathcal{K}_k$, and $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{H})$ is denoted by $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. We will use $\mathcal{N}(\mathcal{T})$ for the kernel of $\mathcal{T}$.

$$\ell^2=\left\{\{f_k\}_{k\in\mathbb{N}}\ :\ \left\|\sum_{k\in\mathbb{N}}\langle f_k,f_k\rangle_{\mathcal{A}}\right\|<\infty\right\},$$

where the shift operator acts as $\mathcal{T}(\{f_k\}_{k\in\mathbb{N}})=\{f_{k+1}\}_{k\in\mathbb{N}}$. Let $f=\{f_k\}_{k\in\mathbb{N}}$ and $g=\{g_k\}_{k\in\mathbb{N}}$; the inner product is defined by $\langle f,g\rangle_{\mathcal{A}}=\sum_{k\in\mathbb{N}}\langle f_k,g_k\rangle_{\mathcal{A}}$, and clearly $\ell^2$ is a Hilbert $\mathcal{A}$-module.

In the following we briefly recall the definitions and basic properties of Hilbert $C^*$-modules.

**Definition 1.1.** [5] Let $\mathcal{A}$ be a unital $C^*$-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}:\mathcal{H}\times\mathcal{H}\to\mathcal{A}$, which is sesquilinear, positive definite, and respects the module action. More precisely,

(i) $\langle x,x\rangle_{\mathcal{A}}\geq 0$ for all $x\in\mathcal{H}$, and $\langle x,x\rangle_{\mathcal{A}}=0$ if and only if $x=0$.

(ii) $\langle ax+y,z\rangle_{\mathcal{A}}=a\langle x,z\rangle_{\mathcal{A}}+\langle y,z\rangle_{\mathcal{A}}$ for all $a\in\mathcal{A}$ and $x,y,z\in\mathcal{H}$.

(iii) $\langle x,y\rangle_{\mathcal{A}}=\langle y,x\rangle^*_{\mathcal{A}}$ for all $x,y\in\mathcal{H}$.

For $x \in \mathcal{H}$, we define $\|x\| = \|\langle x, x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with respect to this norm, it is called a Hilbert $\mathcal{A}$-module or a Hilbert $C^*$-module over $\mathcal{A}$. For every $a$ in the $C^*$-algebra $\mathcal{A}$, we have $\|a\|^2 = \|a^*a\|$.

**Lemma 1.2.** *[9] Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. If $\mathcal{T} \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, then*

$$\langle \mathcal{T}x, \mathcal{T}x\rangle_{\mathcal{A}} \leq \|\mathcal{T}\|^2 \langle x, x\rangle_{\mathcal{A}}, \quad \forall x \in \mathcal{H}.$$

**Definition 1.3.** [5] A family $\{f_k\}_{k\in\Theta}$ of elements of $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist two positive constants $A$ and $B$ such that for all $f \in \mathcal{H}$,

$$A\langle f, f\rangle_{\mathcal{A}} \leq \sum_{k\in\Theta} \langle f, f_k\rangle_{\mathcal{A}} \langle f_k, f\rangle_{\mathcal{A}} \leq B\langle f, f\rangle_{\mathcal{A}}, \tag{1.1}$$

where the inequalities are understood in the $C^*$-algebraic order on $\mathcal{A}$. The constants $A$ and $B$ are called the lower and upper frame bounds, respectively. If $A = B = \alpha$, the frame is $\alpha$-tight. If $A = B = 1$, it is called a normalized tight frame or a Parseval frame. A family satisfying only the upper inequality in (1.1) is called a *Bessel sequence* with bound $B$.

Let $\mathfrak{F} = \{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$. The operator

$$\mathcal{T}_{\mathfrak{F}} : \ell^2 \to \mathcal{H}, \quad \mathcal{T}_{\mathfrak{F}}\left(\{\xi_k\}_{k\in\mathbb{N}}\right) = \sum_{k=1}^{\infty} \xi_k f_k,$$

is called the *synthesis operator* of $\mathfrak{F}$, and

$$\mathcal{T}^*_{\mathfrak{F}} : \mathcal{H} \to \ell^2, \quad \mathcal{T}^*_{\mathfrak{F}} f = \{\langle f, f_k\rangle_{\mathcal{A}}\}_{k\in\mathbb{N}}$$

is called the *analysis operator* of $\mathfrak{F}$. Note that the index set $\Theta$ is consistently taken to be $\mathbb{N}$ throughout. The *frame operator* $S : \mathcal{H} \to \mathcal{H}$, defined by

$$Sf = \mathcal{T}_{\mathfrak{F}}\mathcal{T}^*_{\mathfrak{F}} f = \sum_{k=1}^{\infty} \langle f, f_k\rangle_{\mathcal{A}} f_k, \tag{1.2}$$

is a positive, self-adjoint, and invertible operator.

**Definition 1.4.** Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital $C^*$-algebra $\mathcal{A}$, and let $\mathfrak{F} = \{f_k\}_{k=1}^{\infty}$ and $\mathfrak{G} = \{g_k\}_{k=1}^{\infty}$ be Bessel sequences in $\mathcal{H}$. If for $\mu > 0$,

$$\|\mathcal{T}_{\mathfrak{F}} - \mathcal{T}_{\mathfrak{G}}\| \leq \mu,$$

then $\mathfrak{G}$ is called a $\mu$-perturbation of $\mathfrak{F}$.

**Definition 1.5.** [15] For a given natural number $m$, a finite family of frames $\left\{ \{x_{ij}\}_{j\in\mathbb{J}},\ i \in \{1, \ldots, m\}\right\}$ in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be *woven* if there are universal constants $C$ and $D$ such that, for every partition $\{\sigma_1, \ldots, \sigma_m\}$ of $\mathbb{J}$, the family $\{x_{ij}\}_{i=1,\, j\in\sigma_i}^{m}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds $C$ and $D$, respectively. In this case, we say that $\left\{ \{x_{ij}\}_{j\in\mathbb{J}},\ i \in \{1, \ldots, m\}\right\}$ is woven with universal bounds $(C, D)$. Each family $\{x_{ij}\}_{i=1,\, j\in\sigma_i}^{m}$ is called a weaving.

## 2 From Bessel sequences to frames

**Proposition 2.1.** *Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$ with bound $\eta$, and let $S$ be the associated frame operator. Suppose $S = \mathcal{T} + \xi I$ for some real number $\xi$ and an adjointable linear operator $\mathcal{T} \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. Then:*

*(i) If $\mathcal{T}$ is a positive operator and $\xi > 0$, then $\{f_k\}_{k=1}^{\infty}$ is a frame.*

*(ii) $\mathcal{T}$ is bounded and self-adjoint.*

*(iii) If $\{f_k\}_{k=1}^{\infty}$ is a frame with lower bound $A > 0$ and $\xi \leq A$, then $\mathcal{T}$ is a positive operator.*

*Proof.* (1) Since $S = \mathcal{T} + \xi I$ and $\mathcal{T}$ is positive, for every $f \in \mathcal{H}$ we have $\langle \mathcal{T}f, f\rangle_{\mathcal{A}} \geq 0$ in the $C^*$-algebraic order. Hence,

$$\sum_{k=1}^{\infty} \langle f, f_k\rangle_{\mathcal{A}} \langle f_k, f\rangle_{\mathcal{A}} = \langle Sf, f\rangle_{\mathcal{A}} = \langle \mathcal{T}f, f\rangle_{\mathcal{A}} + \xi\langle f, f\rangle_{\mathcal{A}} \geq \xi\langle f, f\rangle_{\mathcal{A}}.$$

Combined with the Bessel bound, this gives

$$\xi\langle f, f\rangle_{\mathcal{A}} \leq \sum_{k=1}^{\infty} \langle f, f_k\rangle_{\mathcal{A}} \langle f_k, f\rangle_{\mathcal{A}} \leq \eta\langle f, f\rangle_{\mathcal{A}},$$

which shows that $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with bounds $\xi$ and $\eta$.

(2) Since $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence with bound $\eta$, the frame operator $S$ is bounded. Hence $\mathcal{T} = S - \xi I$ is also bounded. Since $S$ is self-adjoint and $\xi \in \mathbb{R}$, we have $\mathcal{T}^* = (S - \xi I)^* = S^* - \xi I = S - \xi I = \mathcal{T}$, so $\mathcal{T}$ is self-adjoint.

(3) Since $\{f_k\}_{k=1}^{\infty}$ is a frame with lower bound $A > 0$, for every $f \in \mathcal{H}$ we have $\langle Sf, f\rangle_{\mathcal{A}} \geq A\langle f, f\rangle_{\mathcal{A}}$ in the $C^*$-algebraic order. Therefore,

$$\langle \mathcal{T}f, f\rangle_{\mathcal{A}} = \langle Sf, f\rangle_{\mathcal{A}} - \xi\langle f, f\rangle_{\mathcal{A}} \geq A\langle f, f\rangle_{\mathcal{A}} - \xi\langle f, f\rangle_{\mathcal{A}} = (A - \xi)\langle f, f\rangle_{\mathcal{A}}.$$

Since $\xi \leq A$, we have $A - \xi \geq 0$, and thus $\langle \mathcal{T}f, f\rangle_{\mathcal{A}} \geq 0$ for every $f \in \mathcal{H}$, which means $\mathcal{T}$ is positive. □

In the following theorem we make use of the following hypothesis: there exist $\alpha \in \mathbb{R}$ and $\eta \geq 0$ such that

$$\|f\|\|g\| \leq \sqrt{1+\eta^2}\|\langle f, g\rangle_{\mathcal{A}}\| \Longleftrightarrow \|\alpha f - g\| \leq \sqrt{\frac{\eta^2}{1+\eta^2}}\|g\|, \quad f, g \in \mathcal{H}. \tag{2.1}$$

**Theorem 2.2.** *Let $\mathcal{T} \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ and $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$ be such that*

$$\sum_{k=1}^{\infty} \langle f, f_k\rangle_{\mathcal{A}} f_k = \mathcal{T}f + \xi f, \quad \forall f \in \mathcal{H},$$

*where $\xi \in \mathbb{R}$ is a real scalar.*

*(i) The family $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence with bound $\|\mathcal{T}\| + |\xi|$.*

*(ii) Suppose $\alpha \in \mathbb{R}$ and $\eta \geq 0$ satisfy (2.1) and the condition*

$$\|\alpha f - \mathcal{T}f\| \leq \sqrt{\frac{\eta^2}{1+\eta^2}}\|\mathcal{T}f\|, \quad \forall f \in \mathcal{H}. \tag{2.2}$$

*If $\mathcal{T}$ is self-adjoint and bounded from below by $\rho > 0$ in the sense that $\|\mathcal{T}f\| \geq \rho\|f\|$ for all $f \in \mathcal{H}$, and if*

$$\sqrt{\frac{\rho^2}{1+\eta^2}} - |\xi| > 0,$$

*then $\{f_k\}_{k=1}^{\infty}$ is a frame with lower bound $\sqrt{\dfrac{\rho^2}{1+\eta^2}} - |\xi|$ and upper bound $\|\mathcal{T}\| + |\xi|$.*

*(iii) With the hypotheses of part (2), assume further that $\{g_k\}_{k=1}^{\infty}$ is a $\mu$-perturbation of $\{f_k\}_{k=1}^{\infty}$. If*

$$\mu < \left(\sqrt{\frac{\rho^2}{1+\eta^2}} - |\xi|\right)^{\frac{1}{2}},$$

*then* $\{g_k\}_{k=1}^{\infty}$ *is a frame for* $\mathcal{H}$ *with lower and upper bounds*

$$\left(\left(\sqrt{\frac{\rho^2}{1+\eta^2}}-|\xi|\right)^{\frac{1}{2}}-\mu\right)^2 \quad \textit{and} \quad \left(\mu+(\|\mathcal{T}\|+|\xi|)^{\frac{1}{2}}\right)^2,$$

*respectively.*

*Proof.* (1) Using the Cauchy–Schwarz inequality and Lemma 1.2, we compute

$$\begin{aligned}\left\|\sum_{k=1}^{\infty}\langle f,f_k\rangle_{\mathcal{A}}\langle f_k,f\rangle_{\mathcal{A}}\right\| &= \|\langle \mathcal{T}f+\xi f,f\rangle_{\mathcal{A}}\|\\ &\le \|\langle \mathcal{T}f,f\rangle_{\mathcal{A}}\|+|\xi|\|f\|^2\\ &\le \|\langle \mathcal{T}f,\mathcal{T}f\rangle_{\mathcal{A}}\|^{\frac{1}{2}}\|\langle f,f\rangle_{\mathcal{A}}\|^{\frac{1}{2}}+|\xi|\|f\|^2\\ &\le \|\mathcal{T}\|\,\|\langle f,f\rangle_{\mathcal{A}}\|+|\xi|\|f\|^2\\ &= (\|\mathcal{T}\|+|\xi|)\,\|\langle f,f\rangle_{\mathcal{A}}\|.\end{aligned}$$

This shows that $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence with bound $\|\mathcal{T}\|+|\xi|$.

(2) Using hypothesis (2.1) and the self-adjointness of $\mathcal{T}$, we write

$$\begin{aligned}\left\|\sum_{k=1}^{\infty}\langle f,f_k\rangle_{\mathcal{A}}\langle f_k,f\rangle_{\mathcal{A}}\right\| &= \|\langle \mathcal{T}f+\xi f,f\rangle_{\mathcal{A}}\|\\ &\ge \|\langle \mathcal{T}f,f\rangle_{\mathcal{A}}\|-|\xi|\|\langle f,f\rangle_{\mathcal{A}}\|\\ &= \|\langle f,\mathcal{T}f\rangle_{\mathcal{A}}\|-|\xi|\|\langle f,f\rangle_{\mathcal{A}}\|\\ &\ge \frac{1}{\sqrt{1+\eta^2}}\|\mathcal{T}f\|\|f\|-|\xi|\|\langle f,f\rangle_{\mathcal{A}}\|.\end{aligned}$$

Since $\mathcal{T}$ is bounded from below by $\rho$, we have $\|\mathcal{T}f\|\|f\|\ge\rho\|f\|^2=\rho\|\langle f,f\rangle_{\mathcal{A}}\|$, and thus

$$\begin{aligned}\left\|\sum_{k=1}^{\infty}\langle f,f_k\rangle_{\mathcal{A}}\langle f_k,f\rangle_{\mathcal{A}}\right\| &\ge \sqrt{\frac{\rho^2}{1+\eta^2}}\|\langle f,f\rangle_{\mathcal{A}}\|-|\xi|\|\langle f,f\rangle_{\mathcal{A}}\|\\ &= \left(\sqrt{\frac{\rho^2}{1+\eta^2}}-|\xi|\right)\|\langle f,f\rangle_{\mathcal{A}}\|.\end{aligned}$$

Combined with the Bessel bound from part (1), this shows that $\{f_k\}_{k=1}^{\infty}$ is a frame.

(3) Since $\{f_k\}_{k=1}^{\infty}$ is a frame with bounds $\sqrt{\frac{\rho^2}{1+\eta^2}}-|\xi|$ and $\|\mathcal{T}\|+|\xi|$, for every $f\in\mathcal{H}$ we have

$$\left(\sqrt{\frac{\rho^2}{1+\eta^2}}-|\xi|\right)^{\frac{1}{2}}\|f\|\le\left\|\mathcal{T}_{\mathfrak{F}}^*(f)\right\|\le(|\xi|+\|\mathcal{T}\|)^{\frac{1}{2}}\|f\|.$$

By the triangle inequality for operator norms and the definition of $\mu$-perturbation,

$$\|\mathcal{T}_{\mathfrak{G}}^*(f)\|\ge\left\|\mathcal{T}_{\mathfrak{F}}^*(f)\right\|-\left\|\mathcal{T}_{\mathfrak{F}}^*-\mathcal{T}_{\mathfrak{G}}^*\right\|\|f\|\ge\left(\left(\sqrt{\frac{\rho^2}{1+\eta^2}}-|\xi|\right)^{\frac{1}{2}}-\mu\right)\|f\|.$$

Similarly,

$$\|\mathcal{T}_{\mathfrak{G}}^*(f)\|\le\left\|\mathcal{T}_{\mathfrak{F}}^*-\mathcal{T}_{\mathfrak{G}}^*\right\|\|f\|+\left\|\mathcal{T}_{\mathfrak{F}}^*(f)\right\|\le\left(\mu+(\|\mathcal{T}\|+|\xi|)^{\frac{1}{2}}\right)\|f\|.$$

Hence,

$$\left(\left(\sqrt{\frac{\rho^2}{1+\eta^2}}-|\xi|\right)^{\frac{1}{2}}-\mu\right)^2\langle f,f\rangle_{\mathcal{A}}\leq\sum_{k=1}^{\infty}\langle f,g_k\rangle_{\mathcal{A}}\langle g_k,f\rangle_{\mathcal{A}}$$
$$\leq\left(\mu+(\|\mathcal{T}\|+|\xi|)^{\frac{1}{2}}\right)^2\langle f,f\rangle_{\mathcal{A}}.$$

□

## 3 Compact and finite-rank-tight frames

We assume in this section that the compact self-adjoint operator $\mathcal{T}$ admits the spectral decomposition

$$\mathcal{T}=\sum_{k=1}^{\infty}\alpha_k\langle\cdot,e_k\rangle_{\mathcal{A}}e_k, \tag{3.1}$$

where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of $\mathcal{T}$, and the series converges in the operator norm. The convergence and compactness of such an operator in the Hilbert $C^*$-module setting require that $\alpha_k\to 0$ as $k\to\infty$, which we assume throughout this section. This decomposition is used to give sufficient conditions under which a Bessel sequence is in fact a frame.

**Theorem 3.1.** *Let $\mathcal{T}$ be a compact self-adjoint operator admitting the representation (3.1) with $\alpha_k\to 0$, and let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$ such that $S=\mathcal{T}+\xi I$ for some $\xi>0$. Set $\alpha:=\inf_k\alpha_k+\xi$. If $\alpha>0$, then $\{f_k\}_{k=1}^{\infty}$ is a frame with lower bound $\alpha$ and upper bound $\sup_k(\alpha_k+\xi)$.*

*Proof.* For every $f\in\mathcal{H}$, expanding $Sf$ using (3.1) and the Parseval identity gives

$$\begin{aligned}Sf&=\mathcal{T}f+\xi f\\&=\sum_{k=1}^{\infty}\alpha_k\langle f,e_k\rangle_{\mathcal{A}}e_k+\xi\sum_{k=1}^{\infty}\langle f,e_k\rangle_{\mathcal{A}}e_k\\&=\sum_{k=1}^{\infty}(\alpha_k+\xi)\langle f,e_k\rangle_{\mathcal{A}}e_k.\end{aligned}$$

Consequently,

$$\sum_{k=1}^{\infty}\langle f,f_k\rangle_{\mathcal{A}}\langle f_k,f\rangle_{\mathcal{A}}=\langle Sf,f\rangle_{\mathcal{A}}=\sum_{k=1}^{\infty}(\alpha_k+\xi)\langle f,e_k\rangle_{\mathcal{A}}\langle e_k,f\rangle_{\mathcal{A}}$$
$$\geq\left(\inf_k\alpha_k+\xi\right)\sum_{k=1}^{\infty}\langle f,e_k\rangle_{\mathcal{A}}\langle e_k,f\rangle_{\mathcal{A}}=\alpha\langle f,f\rangle_{\mathcal{A}},$$

where we used $\alpha=\xi+\inf_k\alpha_k>0$ and the Parseval identity. This provides the lower frame bound, the upper frame bound follows analogously. □

**Definition 3.2.** A frame $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ is called a *compact-tight frame* if its frame operator $S$ satisfies

$$S=\mathcal{K}+\xi I, \tag{3.2}$$

where $\mathcal{K}$ is a compact self-adjoint operator and $\xi\in\mathbb{R}$. The self-adjointness of $\mathcal{K}$ is required since $S$ and $\xi I$ are both self-adjoint. The same definition applies when $\mathcal{K}$ has finite rank, in which case the frame is called a *finite-rank-tight frame*.

**Proposition 3.3.** *Let $\{f_k\}_{k=1}^{\infty}$ be a compact-tight frame. The representation of the frame operator $S$ in the form (3.2) is unique; that is, if $S=\mathcal{K}_1+\xi_1 I=\mathcal{K}_2+\xi_2 I$ with compact self-adjoint operators $\mathcal{K}_1,\mathcal{K}_2$ and $\xi_1,\xi_2\in\mathbb{R}$, then $\xi_1=\xi_2$ and $\mathcal{K}_1=\mathcal{K}_2$.*

*Proof.* From $\mathcal{K}_1 + \xi_1 I = \mathcal{K}_2 + \xi_2 I$ we obtain

$$\mathcal{K}_2 - \mathcal{K}_1 = (\xi_1 - \xi_2)I. \tag{3.3}$$

The left-hand side is compact, being the difference of two compact operators. The right-hand side $(\xi_1 - \xi_2)I$ is compact only if $\xi_1 - \xi_2 = 0$, since the identity operator on an infinite-dimensional Hilbert $C^*$-module is not compact. Hence $\xi_1 = \xi_2$, and substituting back into (3.3) gives $\mathcal{K}_1 = \mathcal{K}_2$. □

Following Proposition 3.3, we say that $\{f_k\}_{k=1}^{\infty}$ is a $(\xi, \mathcal{K})$-compact-tight frame if it is a compact-tight frame whose frame operator satisfies $S = \mathcal{K} + \xi I$.

**Theorem 3.4.** *Let $\xi > 0$ and let $\{l_k\}_{k=1}^{\infty}$ be a sequence of real numbers with $l_k > \xi$ for all $k$ and $\lim_{k\to\infty} l_k = \xi$. If $\{e_k\}_{k=1}^{\infty}$ is any orthonormal basis of $\mathcal{H}$, then $\left\{l_k^{\frac{1}{2}} e_k\right\}_{k=1}^{\infty}$ is a $(\xi, \mathcal{K})$-compact-tight frame for $\mathcal{H}$ with lower bound $\xi$ and upper bound $\sup_k l_k$.*

*Proof.* Let $f \in \mathcal{H}$. Using the Parseval identity,

$$\sum_{k=1}^{\infty} \left\langle f, l_k^{\frac{1}{2}} e_k \right\rangle_{\mathcal{A}} \left\langle l_k^{\frac{1}{2}} e_k, f \right\rangle_{\mathcal{A}} = \sum_{k=1}^{\infty} l_k \langle f, e_k \rangle_{\mathcal{A}} \langle e_k, f \rangle_{\mathcal{A}}.$$

Since $l_k > \xi$ for all $k$ and $l_k \le \sup_k l_k$, we obtain

$$\xi \langle f, f \rangle_{\mathcal{A}} \le \sum_{k=1}^{\infty} \left\langle f, l_k^{\frac{1}{2}} e_k \right\rangle_{\mathcal{A}} \left\langle l_k^{\frac{1}{2}} e_k, f \right\rangle_{\mathcal{A}} \le \left(\sup_k l_k\right) \langle f, f \rangle_{\mathcal{A}}.$$

Thus $\left\{l_k^{\frac{1}{2}} e_k\right\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with lower bound $\xi$ and upper bound $\sup_k l_k$. To identify the frame operator, we compute

$$\begin{aligned} Sf &= \sum_{k=1}^{\infty} \left\langle f, l_k^{\frac{1}{2}} e_k \right\rangle_{\mathcal{A}} l_k^{\frac{1}{2}} e_k = \sum_{k=1}^{\infty} l_k \langle f, e_k \rangle_{\mathcal{A}} e_k \\ &= \sum_{k=1}^{\infty} (l_k - \xi) \langle f, e_k \rangle_{\mathcal{A}} e_k + \xi \sum_{k=1}^{\infty} \langle f, e_k \rangle_{\mathcal{A}} e_k \\ &= \mathcal{K} f + \xi f, \end{aligned}$$

where $\mathcal{K} := \sum_{k=1}^{\infty} (l_k - \xi) \langle \cdot, e_k \rangle_{\mathcal{A}} e_k$. Since $l_k - \xi > 0$ and $l_k - \xi \to 0$ as $k \to \infty$, the operator $\mathcal{K}$ is a compact positive self-adjoint operator, confirming that $\left\{l_k^{\frac{1}{2}} e_k\right\}_{k=1}^{\infty}$ is a $(\xi, \mathcal{K})$-compact-tight frame. □

**Example 3.5.** Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$, and set

$$l_k := 1 + \exp\left(-\frac{k^2}{2}\right).$$

Since $\exp(-k^2/2) > 0$ for all $k$, we have $l_k > 1$ for every $k$. Moreover, $\lim_{k\to\infty} l_k = 1$. By Theorem 3.4, the sequence $\left\{l_k^{\frac{1}{2}} e_k\right\}_{k=1}^{\infty}$ is a compact-tight frame with lower bound $\xi = 1$ and upper bound $\sup_k l_k = 1 + e^{-1/2}$.

**Proposition 3.6.** *Let $\{g_k\}_{k=1}^{\infty}$ be a frame in $\mathcal{H}$ constructed from an orthonormal basis $\{e_1, e_2, \ldots\}$ by selecting a finite collection of basis elements $e_{j_1}, \ldots, e_{j_n}$ and repeating each $e_{j_m}$ exactly $\theta_m > 1$ additional times. Then $\{g_k\}_{k=1}^{\infty}$ is a finite-rank-tight frame with $\xi = 1$.*

*Proof.* Under the stated construction, each basis element $e_{j_m}$ (for $m = 1, \ldots, n$) appears $(\theta_m + 1)$ times in total in $\{g_k\}$, while every other basis element $e_i$ with $i \notin \{j_1, \ldots, j_n\}$ appears exactly

once. Therefore, for every $f \in \mathcal{H}$:

$$\begin{aligned} Sf &= \sum_{k=1}^{\infty} \langle f, g_k \rangle_{\mathcal{A}} g_k \\ &= \sum_{m=1}^{n} (\theta_m + 1) \langle f, e_{j_m} \rangle_{\mathcal{A}} e_{j_m} + \sum_{i \notin \{j_1, \ldots, j_n\}} \langle f, e_i \rangle_{\mathcal{A}} e_i \\ &= \sum_{m=1}^{n} \theta_m \langle f, e_{j_m} \rangle_{\mathcal{A}} e_{j_m} + \sum_{i=1}^{\infty} \langle f, e_i \rangle_{\mathcal{A}} e_i \\ &= \mathcal{K} f + f, \end{aligned}$$

where

$$\mathcal{K} = \sum_{m=1}^{n} \theta_m \langle \cdot, e_{j_m} \rangle_{\mathcal{A}} e_{j_m}$$

is a finite-rank (hence compact) self-adjoint operator with coefficients $\theta_m > 0$. This shows that $S = \mathcal{K} + I$, confirming that $\{g_k\}_{k=1}^{\infty}$ is a $(1, \mathcal{K})$-compact-tight frame with finite-rank $\mathcal{K}$. □

**Theorem 3.7.** *The canonical dual of a $(\xi, \mathcal{K})$-compact-tight frame is compact-tight.*

*Proof.* Let $\{f_k\}_{k=1}^{\infty}$ be a $(\xi, \mathcal{K})$-compact-tight frame with frame operator $S = \mathcal{K} + \xi I$, where $\xi \neq 0$ and $\mathcal{K}$ is a compact self-adjoint operator. We claim that $S^{-1} = \mathcal{T} + \xi^{-1} I$ for a compact operator $\mathcal{T}$. Define $\mathcal{T}$ by $\mathcal{T} S = -\xi^{-1} \mathcal{K}$, i.e., $\mathcal{T} = -\xi^{-1} \mathcal{K} S^{-1}$. Since $\mathcal{K}$ is compact and $S^{-1}$ is bounded, $\mathcal{T}$ is compact. We verify:

$$\begin{aligned} \left(\mathcal{T} + \xi^{-1} I\right) S &= \mathcal{T} S + \xi^{-1} S \\ &= -\xi^{-1} \mathcal{K} + \xi^{-1} (\mathcal{K} + \xi I) \\ &= -\xi^{-1} \mathcal{K} + \xi^{-1} \mathcal{K} + I \\ &= I. \end{aligned}$$

Hence $S^{-1} = \mathcal{T} + \xi^{-1} I$, showing that the canonical dual frame has frame operator $S^{-1}$ of the required compact-tight form. □

**Theorem 3.8.** *Let $\mathfrak{F} = \{f_k\}_{k=1}^{\infty}$ and $\mathfrak{G} = \{\tilde{f}_k\}_{k=1}^{\infty}$ be $(1, \mathcal{K}_1)$- and $(1, \mathcal{K}_2)$-compact-tight frames, respectively. If there exists an infinite subset $\sigma$ of $\mathbb{N}$ such that $\{f_k\}_{k \in \sigma}$ and $\{\tilde{f}_k\}_{k \in \sigma^c}$ are each orthonormal bases for $\mathcal{H}$, then $\mathfrak{F}$ and $\mathfrak{G}$ cannot be woven.*

*Proof.* Assume for contradiction that $\mathfrak{F}$ and $\mathfrak{G}$ are woven. Then for the partition $\{\sigma, \sigma^c\}$ of $\mathbb{N}$, the family $\{f_k\}_{k \in \sigma^c} \cup \{\tilde{f}_k\}_{k \in \sigma}$ must be a frame for $\mathcal{H}$.

Since $\mathfrak{F}$ is a $(1, \mathcal{K}_1)$-compact-tight frame, we have $S_F = \mathcal{K}_1 + I$. Splitting the sum over $\sigma$ and $\sigma^c$:

$$S_F f = \sum_{k \in \sigma} \langle f, f_k \rangle_{\mathcal{A}} f_k + \sum_{k \in \sigma^c} \langle f, f_k \rangle_{\mathcal{A}} f_k = \mathcal{K}_1 f + f.$$

Since $\{f_k\}_{k \in \sigma}$ is an orthonormal basis for $\mathcal{H}$, by the Parseval identity:

$$\sum_{k \in \sigma} \langle f, f_k \rangle_{\mathcal{A}} f_k = f.$$

Therefore:

$$\sum_{k \in \sigma^c} \langle f, f_k \rangle_{\mathcal{A}} f_k = S_F f - f = \mathcal{K}_1 f.$$

Since $\mathfrak{G}$ is a $(1, \mathcal{K}_2)$-compact-tight frame, we have $S_G = \mathcal{K}_2 + I$. Splitting the sum over $\sigma^c$ and $\sigma$:

$$S_G f = \sum_{k \in \sigma^c} \langle f, \tilde{f}_k \rangle_{\mathcal{A}} \tilde{f}_k + \sum_{k \in \sigma} \langle f, \tilde{f}_k \rangle_{\mathcal{A}} \tilde{f}_k = f + \mathcal{K}_2 f.$$

Since $\{\tilde{f}_k\}_{k\in\sigma^c}$ is an orthonormal basis for $\mathcal{H}$, by the Parseval identity:

$$\sum_{k\in\sigma^c}\langle f,\tilde{f}_k\rangle_{\mathcal{A}}\tilde{f}_k = f.$$

Therefore:

$$\sum_{k\in\sigma}\langle f,\tilde{f}_k\rangle_{\mathcal{A}}\tilde{f}_k = S_G f - f = \mathcal{K}_2 f.$$

Denote by $S_{F+G}$ the frame operator of the weaving $\{f_k\}_{k\in\sigma^c}\cup\{\tilde{f}_k\}_{k\in\sigma}$. For every $f\in\mathcal{H}$:

$$\begin{aligned} S_{F+G}f &= \sum_{k\in\sigma^c}\langle f,f_k\rangle_{\mathcal{A}}f_k + \sum_{k\in\sigma}\langle f,\tilde{f}_k\rangle_{\mathcal{A}}\tilde{f}_k \\ &= \mathcal{K}_1 f + \mathcal{K}_2 f \\ &= (\mathcal{K}_1+\mathcal{K}_2)f, \end{aligned}$$

We have shown that $S_{F+G} = \mathcal{K}_1 + \mathcal{K}_2$. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ are compact operators, their sum $\mathcal{K}_1 + \mathcal{K}_2$ is also compact. However, the frame operator $S_{F+G}$ of any frame must be invertible. A compact operator on an infinite-dimensional Hilbert $C^*$-module cannot be invertible. This is a contradiction, which shows that $\mathfrak{F}$ and $\mathfrak{G}$ cannot be woven. □

## References


[1] I. Daubechies, A. Grossmann, Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., **27** (1986), 1271-1283.

[2] R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Am. Math. Soc. **72** (1952), 341-366.

[3] R. El Jazzar, M. Rossafi, M. Mouniane, K-g-frames in Hilbert modules over locally-$C^*$-algebras, Palest. J. Math. 14(4) (2025), 44-56.

[4] M. Ghiati, M. Rossafi, M. Mouniane, et al., Controlled continuous $*$-g-frames in Hilbert C*-modules, J. Pseudo-Differ. Oper. Appl. 15, 2 (2024). https://doi.org/10.1007/s11868-023-00571-1

[5] I. Kaplansky, Modules over operator algebras, Amer. J. Math. 75 (1953), 839-858. https://doi.org/10.2307/2372552

[6] A. Karara, M. Rossafi, M. Klilou, S. Kabbaj, Construction of continuous K-g-Frames in Hilbert $C^*$-Modules, Palest. J. Math. 13(4) (2024), 198-209.

[7] A. Khorsavi, B. Khorsavi, Fusion frames and g-frames in Hilbert $C^*$-modules, Int. J. Wavelets Multiresolution Inf. Process. 6 (2008), 433-446. https://doi.org/10.1142/S0219691308002458

[8] H. Massit, M. Rossafi, C. Park, Some relations between continuous generalized frames, Afr. Mat. 35, 12 (2024). https://doi.org/10.1007/s13370-023-01157-2

[9] W. Paschke, Inner product modules over $B^*$-algebras, Trans. Am. Math. Soc. 182 (1973), 443-468.

[10] M. Rossafi, S. Kabbaj, $*$-K-operator Frame for End$^*_{\mathcal{A}}(\mathcal{H})$, Asian-Eur. J. Math. 13 (2020). https://doi.org/10.1142/S1793557120500606

[11] M. Rossafi, M. Ghiati, M. Mouniane, F. Chouchene, A. Touri, S. Kabbaj, Continuous frame in Hilbert C*-modules, J. Anal. (2023). https://doi.org/10.1007/s41478-023-00581-8

[12] M. Rossafi, F. D. Nhari, C. Park, S. Kabbaj, Continuous g-frames with C*-valued bounds and their properties, Complex Anal. Operator Theory 16, 44 (2022). https://doi.org/10.1007/s11785-022-01229-4

[13] M. Rossafi, F. D. Nhari, P. Sam Johnson, Controlled continuous g-frames and their duals in Hilbert spaces, Palest. J. Math. 15 (1) (2026), 21-31.

[14] N. K. Sahu, Controlled g-frames in Hilbert C*-modules, Math. Anal. Contemp. Appl. 3 (2021), 65-82.

[15] X. Zhao, P. Li, Weaving Frames in Hilbert C*-Modules, J. Math. Vol. 2021, Article ID 2228397, 13 pages. https://doi.org/10.1155/2021/2228397

## Author information

Abdelilah Karara, Laboratory Analysis, Geometry and Applications, Higher School of Education and Training, University of Ibn Tofail, P. O. Box 242, Kenitra 14000, Morocco.
E-mail: abdelilah.karara@uit.ac.ma

Khadija Mabrouk, Laboratory Analysis, Geometry and Applications, University of Ibn Tofail, Kenitra, Morocco.
E-mail: khadija.mabrouk@uit.ac.ma

Mohamed Rossafi, Laboratory Analysis, Geometry and Applications, Higher School of Education and Training, University of Ibn Tofail, P. O. Box 242, Kenitra 14000, Morocco.
E-mail: rossafimohamed@gmail.com; mohamed.rossafi@uit.ac.ma

Received: 2025-01-30
Accepted: 2026-07-04