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Some refinements for the numerical radius of Hilbert space operators

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Abstract. Our aim in this paper is to give several new numerical radius inequalities for products of operators in a Hilbert space. These inequalities improve on some earlier related inequalities.

1 Introduction

In this paper, $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. By $\mathcal{B}(\mathcal{H})$, we denote the algebra of all bounded linear operators acting on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, A^* denotes the adjoint of A and $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of A.

Recall that for $A \in \mathcal{B}(\mathcal{H})$, the usual operator norm of A is defined as

$$\|A\| := \sup_{\|x\|=1} \|Ax\|.$$

The numerical range of A, denoted by $W(A)$, is defined as

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H},\ \|x\| = 1\}.$$

The numerical radius is defined as

$$\omega(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H},\ \|x\| = 1\}.$$

For more material about the numerical radius and other information on the basic theory of numerical range, we refer the reader to [9].

It is well known that $\omega(\cdot)$ is a norm on $\mathcal{B}(\mathcal{H})$ and for all $A \in \mathcal{B}(\mathcal{H})$ the following inequality holds [9, Theorem 1.3]

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1.1}$$

That is, $\omega(\cdot)$ defines an equivalent norm to $\|\cdot\|$ on $\mathcal{B}(\mathcal{H})$. The first inequality becomes equality if $A^2 = 0$ and the second one turns into equality if A is normal. Also, the numerical radius satisfies the following power inequality

$$\omega(A^n) \leq \omega^n(A) \quad \text{for all } n \in \mathbb{N}.$$

Several numerical radius inequalities improving the inequalities in (1.1) have been recently given in [1, 3, 6, 8, 11].

In [13], Kittaneh improved the second inequality in (1.1) in the following form

$$\omega(A) \leq \frac{1}{2}\||A| + |A^*|\| \leq \|A\|.$$

Another refinement of the second inequality in (1.1) has been established in [14]. This refinement asserts that if $A \in \mathcal{B}(\mathcal{H})$, then

$$\omega^2(A) \leq \frac{1}{2}\| |A|^2 + |A^*|^2 \|. \tag{1.2}$$

After that, a generalization of (1.2) was given in [6] as follows

$$\omega^{2r}(A) \leq \frac{1}{2}\| |A|^{2r} + |A^*|^{2r} \| \text{ for all } r \geq 1. \tag{1.3}$$

In 2021, Bhunia and Paul in [3] proved that

$$\omega(A) \leq \frac{1}{\sqrt{2}}\omega\left(|A| + i\,|A^*|\right). \tag{1.4}$$

In [5], Dragomir generalized (1.3) for product of two operators, that if $A, B \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$, then

$$\omega^r(B^*A) \leq \frac{1}{2}\| |A|^{2r} + |B|^{2r} \|. \tag{1.5}$$

In this work, we will continue working in this direction and we will prove several new refinements of numerical radius inequalities for products of operators. Further, some applications of our inequalities are also provided.

2 Prerequisites

In this section, we present some well-known lemmas. The first lemma is the Buzano extension of Schwarz's inequality.

Lemma 2.1. *[4] If a, b, e are vectors in $\mathcal{H}$ and $\|e\| = 1$, then*

$$|\langle a, e\rangle \langle e, b\rangle| \leq \frac{1}{2}\left(\|a\| \|b\| + |\langle a, b\rangle|\right).$$

Lemma 2.2. *[16] Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

(i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for $r \geq 1$.
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for $0 < r \leq 1$.

Lemma 2.3. *[12] Let $A \in \mathcal{B}(\mathcal{H})$, and let f and g be non-negative continuous functions on $[0, \infty)$ such that $f(t)\, g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\langle Ax, y\rangle|^2 \leq \left\langle f^2\left(|A|\right) x, x\right\rangle \left\langle g^2\left(|A^*|\right) y, y\right\rangle$$

for all $x, y \in \mathcal{H}$.

In particular, if $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ with $0 \leq \alpha \leq 1$, then we have the mixed Cauchy-Schwarz inequality

$$|\langle Ax, y\rangle|^2 \leq \left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle.$$

Lemma 2.4. *[2] Let $x, y \in \mathcal{H}$. Then for any $\alpha \geq 0$,*

$$|\langle x, y\rangle|^2 \leq \frac{1}{1+\alpha}\|x\| \|y\| |\langle x, y\rangle| + \frac{\alpha}{1+\alpha}\|x\|^2 \|y\|^2 \leq \|x\|^2 \|y\|^2.$$

Lemma 2.5. *[10] Let a, b be two positive real numbers and $0 \leq \alpha \leq 1$. Then*

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)\, b.$$

In particular, for $\alpha = \frac{1}{2}$, we obtain

$$\sqrt{ab} \leq \frac{a+b}{2},$$

which is the arithmetic–geometric mean inequality.

Lemma 2.6. *[17] Let a_i $(i = 1, 2, ..., n)$ be positive real numbers and $r \geq 1$. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1} \sum_{i=1}^{n} a_i^r.$$

Lemma 2.7. *[15] Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then*

$$\|A + iB\| \leq \|A + B\|.$$

Lemma 2.8. *[15] Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators. Then*

$$\omega^2 (A + iB) \leq \left\|A^2 + B^2\right\|.$$

Lemma 2.9. *[15] Let $A, B \in \mathcal{B}(\mathcal{H})$ be normal operators. Then*

$$\omega (A + B) \leq \sqrt{2}\omega (|A| + i |B|).$$

3 Main results

In this section, we present our results. Our first result can be stated as follows.

Theorem 3.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega^r (B^* A) \leq \frac{1}{4} \left\||A|^{2r} + |B|^{2r}\right\| + \frac{1}{2}\omega (|B|^r |A|^r),$$

for all $r \geq 2$.

Proof. Let $x \in \mathcal{H}$ be a unit vector. Then

$$\begin{aligned}
|\langle B^* Ax, x\rangle|^r &= |\langle Ax, Bx\rangle|^r \\
&\leq \|Ax\|^r \|Bx\|^r \\
&= \left\langle |A|^2 x, x\right\rangle^{\frac{r}{2}} \left\langle |B|^2 x, x\right\rangle^{\frac{r}{2}} \\
&\leq \langle |A|^r x, x\rangle \langle |B|^r x, x\rangle \text{ (by Lemma 2.2 (i))} \\
&\leq \langle |A|^r x, x\rangle \langle x, |B|^r x\rangle \\
&\leq \frac{1}{2} \left(\| |A|^r x\| \| |B|^r x\| + \langle |A|^r x, |B|^r x\rangle\right) \text{ (by Lemma 2.1)} \\
&= \frac{1}{2} \left(\sqrt{\langle |A|^r x, |A|^r x\rangle}\sqrt{\langle |B|^r x, |B|^r x\rangle} + |\langle |B|^r |A|^r x, x\rangle|\right) \\
&= \frac{1}{2} \left(\sqrt{\left\langle |A|^{2r} x, x\right\rangle}\sqrt{\left\langle |B|^{2r} x, x\right\rangle} + |\langle |B|^r |A|^r x, x\rangle|\right) \\
&\leq \frac{1}{4} \left(\left\langle |A|^{2r} x, x\right\rangle + \left\langle |B|^{2r} x, x\right\rangle\right) + \frac{1}{2} |\langle |B|^r |A|^r x, x\rangle| \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{4} \left\langle \left(|A|^{2r} + |B|^{2r}\right) x, x\right\rangle + \frac{1}{2} |\langle |B|^r |A|^r x, x\rangle| \\
&\leq \frac{1}{4} \left\||A|^{2r} + |B|^{2r}\right\| + \frac{1}{2}\omega (|B|^r |A|^r).
\end{aligned}$$

Therefore,

$$|\langle B^*Ax,x\rangle|^r \leq \frac{1}{4}\left\||A|^{2r}+|B|^{2r}\right\|+\frac{1}{2}\omega\left(|B|^r|A|^r\right).$$

By taking the supremum in the above inequality over $x\in\mathcal{H}$ with $\|x\|=1$, we get

$$\omega^r\left(B^*A\right)\leq\frac{1}{4}\left\||A|^{2r}+|B|^{2r}\right\|+\frac{1}{2}\omega\left(|B|^r|A|^r\right),$$

as required. □

Remark 3.2. We note that the inequalities obtained in Theorem 3.1 refine the inequality (1.5). Indeed, using (1.5) we observe that

$$\begin{aligned}\omega^r\left(B^*A\right) &\leq \frac{1}{4}\left\||A|^{2r}+|B|^{2r}\right\|+\frac{1}{2}\omega\left(|B|^r|A|^r\right)\\ &\leq \frac{1}{4}\left\||A|^{2r}+|B|^{2r}\right\|+\frac{1}{4}\left\||A|^{2r}+|B|^{2r}\right\|\\ &= \frac{1}{2}\left\||A|^{2r}+|B|^{2r}\right\|.\end{aligned}$$

Theorem 3.3. *Let $A,B\in\mathcal{B}(\mathcal{H})$. Then*

$$\omega^r\left(B^*A\right)\leq\frac{\sqrt{2}}{4}\left\||A|^{2r}+i|B|^{2r}\right\|+\frac{1}{2}\omega\left(|B|^r|A|^r\right),$$

for all $r\geq 2$.

Proof. Let $x\in\mathcal{H}$ be a unit vector. Then

$$\begin{aligned}|\langle B^*Ax,x\rangle|^r &= |\langle Ax,Bx\rangle|^r\\ &\leq \|Ax\|^r\|Bx\|^r\\ &= \left\langle|A|^2x,x\right\rangle^{\frac{r}{2}}\left\langle|B|^2x,x\right\rangle^{\frac{r}{2}}\\ &\leq \langle|A|^rx,x\rangle\langle|B|^rx,x\rangle \text{ (by Lemma 2.2 (i))}\\ &= \langle|A|^rx,x\rangle\langle x,|B|^rx\rangle\\ &\leq \frac{1}{2}\left(\||A|^rx\|\,\||B|^rx\|+\langle|A|^rx,|B|^rx\rangle\right) \text{ (by Lemma 2.1)}\\ &= \frac{1}{2}\left(\sqrt{\langle|A|^rx,|A|^rx\rangle}\sqrt{\langle|B|^rx,|B|^rx\rangle}+|\langle|B|^r|A|^rx,x\rangle|\right)\\ &= \frac{1}{2}\left(\sqrt{\left\langle|A|^{2r}x,x\right\rangle}\sqrt{\left\langle|B|^{2r}x,x\right\rangle}+|\langle|B|^r|A|^rx,x\rangle|\right)\\ &\leq \frac{1}{4}\left(\left\langle|A|^{2r}x,x\right\rangle+\left\langle|B|^{2r}x,x\right\rangle\right)+\frac{1}{2}|\langle|B|^r|A|^rx,x\rangle|\\ &\qquad\text{(by the arithmetic-geometric mean inequality)}\\ &\leq \frac{\sqrt{2}}{4}\left|\left\langle\left(|A|^{2r}+i|B|^{2r}\right)x,x\right\rangle\right|+\frac{1}{2}|\langle|B|^r|A|^rx,x\rangle|.\end{aligned}$$

Here we have used the fact that $|a+b|\leq\sqrt{2}\,|a+ib|$ for $a,b\in\mathbb{R}$. Thus,

$$\begin{aligned}|\langle B^*Ax,x\rangle|^r &\leq \frac{\sqrt{2}}{4}\left|\left\langle\left(|A|^{2r}+i|B|^{2r}\right)x,x\right\rangle\right|+\frac{1}{2}|\langle|B|^r|A|^rx,x\rangle|\\ &\leq \frac{\sqrt{2}}{4}\left\||A|^{2r}+i|B|^{2r}\right\|+\frac{1}{2}\omega\left(|B|^r|A|^r\right).\end{aligned}$$

Taking the supremum over $x\in\mathcal{H}$ with $\|x\|=1$ in the above inequality, we get

$$\omega^r\left(B^*A\right)\leq\frac{\sqrt{2}}{4}\left\||A|^{2r}+i|B|^{2r}\right\|+\frac{1}{2}\omega\left(|B|^r|A|^r\right).$$

□

Corollary 3.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$. If $|B|\,|A| = 0$, then*

$$\omega\left(B^*A\right) \leq \frac{\sqrt{2}}{4}\left\||A|^2 + i\,|B|^2\right\| \leq \frac{1}{2}\left\||A|^2 + |B|^2\right\|.$$

Proof. If $|B|\,|A| = 0$, then by using Lemma 2.7 we have

$$\begin{aligned}\omega\left(B^*A\right) &\leq \frac{\sqrt{2}}{4}\left\||A|^2 + i\,|B|^2\right\| \\ &\leq \frac{\sqrt{2}}{4}\left\||A|^2 + |B|^2\right\| \\ &\leq \frac{1}{2}\left\||A|^2 + |B|^2\right\|.\end{aligned}$$

□

Theorem 3.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega^r\left(B^*A\right) \leq \frac{1}{4}\omega^2\left(|A|^r + i\,|B|^r\right) + \frac{1}{8}\left\||A|^{2r} + |B|^{2r}\right\| + \frac{1}{4}\omega\left(|B|^r\,|A|^r\right),$$

for all $r \geq 2$.

Proof. Let $x \in \mathcal{H}$ be a unit vector. Then

$$\begin{aligned}&\left|\langle B^*Ax, x\rangle\right|^r \\ =\ &\left|\langle Ax, Bx\rangle\right|^r \\ \leq\ &\|Ax\|^r\,\|Bx\|^r \\ =\ &\left\langle |A|^2 x, x\right\rangle^{\frac{r}{2}}\left\langle |B|^2 x, x\right\rangle^{\frac{r}{2}} \\ \leq\ &\left\langle |A|^r x, x\right\rangle\left\langle |B|^r x, x\right\rangle \text{ (by Lemma 2.2 (i))} \\ \leq\ &\frac{1}{4}\left(\left\langle |A|^r x, x\right\rangle + \left\langle |B|^r x, x\right\rangle\right)^2 \\ \leq\ &\frac{1}{4}\left(\left\langle |A|^r x, x\right\rangle^2 + \left\langle |B|^r x, x\right\rangle^2 + 2\left\langle |A|^r x, x\right\rangle\left\langle |B|^r x, x\right\rangle\right) \\ =\ &\frac{1}{4}\left|\left\langle |A|^r x, x\right\rangle + i\left\langle |B|^r x, x\right\rangle\right|^2 + \frac{1}{2}\left\langle |A|^r x, x\right\rangle\left\langle x, |B|^r x\right\rangle \\ \leq\ &\frac{1}{4}\left|\left\langle \left(|A|^r + i\,|B|^r\right)x, x\right\rangle\right|^2 + \frac{1}{4}\left(\||A|^r x\|\,\||B|^r x\| + \left\langle |A|^r x, |B|^r x\right\rangle\right) \\ \leq\ &\frac{1}{4}\left|\left\langle \left(|A|^r + i\,|B|^r\right)x, x\right\rangle\right|^2 + \frac{1}{8}\left(\||A|^r x\|^2 + \||B|^r x\|^2\right) \\ &+\frac{1}{4}\left|\left\langle |B|^r\,|A|^r x, x\right\rangle\right| \\ =\ &\frac{1}{4}\left|\left\langle \left(|A|^r + i\,|B|^r\right)x, x\right\rangle\right|^2 + \frac{1}{8}\left\langle\left(|A|^{2r} + |B|^{2r}\right)x, x\right\rangle + \frac{1}{4}\left|\left\langle |B|^r\,|A|^r x, x\right\rangle\right| \\ \leq\ &\frac{1}{4}\omega^2\left(|A|^r + i\,|B|^r\right) + \frac{1}{8}\left\||A|^{2r} + |B|^{2r}\right\| + \frac{1}{4}\omega\left(|B|^r\,|A|^r\right).\end{aligned}$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired inequality. □

Remark 3.6. By using Lemma 2.8 and the inequality (1.5), it can be observed that

$$\begin{aligned}\omega^r\left(B^*A\right) &\leq \frac{1}{4}\omega^2\left(|A|^r + i\,|B|^r\right) + \frac{1}{8}\left\||A|^{2r} + |B|^{2r}\right\| + \frac{1}{4}\omega\left(|B|^r\,|A|^r\right) \\ &\leq \frac{1}{4}\left\||A|^{2r} + |B|^{2r}\right\| + \frac{1}{8}\left\||A|^{2r} + |B|^{2r}\right\| + \frac{1}{8}\left\||A|^{2r} + |B|^{2r}\right\| \\ &= \frac{1}{2}\left\||A|^{2r} + |B|^{2r}\right\|.\end{aligned}$$

Theorem 3.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$. Then*

$$\begin{aligned}\omega^2(B^*A) &\leq \frac{1}{4}\left\|\alpha|A|^4 + (1-\alpha)|A^*|^4 + \beta|B|^4 + (1-\beta)|B^*|^4\right\| \\ &\quad + \frac{1}{8}\left\||A|^4 + |B|^4\right\| + \frac{1}{4}\omega\left(|B|^2|A|^2\right).\end{aligned}$$

Proof. Let $x \in \mathcal{H}$ be a unit vector. Then

$$\begin{aligned}&|\langle B^*Ax, x\rangle|^2 \\ =\ &|\langle Ax, Bx\rangle|^2 \\ \leq\ &\|Ax\|^2\|Bx\|^2 \\ =\ &\left\langle |A|^2x, x\right\rangle\left\langle |B|^2x, x\right\rangle \\ \leq\ &\frac{1}{4}\left(\left\langle |A|^2x, x\right\rangle + \left\langle |B|^2x, x\right\rangle\right)^2 \\ =\ &\frac{1}{4}\left(\left\langle |A|^2x, x\right\rangle^2 + \left\langle |B|^2x, x\right\rangle^2 + 2\left\langle |A|^2x, x\right\rangle\left\langle |B|^2x, x\right\rangle\right) \\ =\ &\frac{1}{4}\left(\left\langle |A|^2x, x\right\rangle^2 + \left\langle |B|^2x, x\right\rangle^2 + 2\left\langle |A|^2x, x\right\rangle\left\langle x, |B|^2x\right\rangle\right) \\ \leq\ &\frac{1}{4}\left(\left\langle |A|^{4\alpha}x, x\right\rangle\left\langle |A^*|^{4(1-\alpha)}x, x\right\rangle + \left\langle |B|^{4\alpha}x, x\right\rangle\left\langle |B^*|^{4(1-\alpha)}x, x\right\rangle\right) \\ &+\frac{1}{4}\left(\left\||A|^2x\right\|\left\||B|^2x\right\| + \left\langle |A|^2x, |B|^2x\right\rangle\right) \text{ (by Lemma 2.3)} \\ \leq\ &\frac{1}{4}\left(\left\langle |A|^4x, x\right\rangle^{\alpha}\left\langle |A^*|^4x, x\right\rangle^{1-\alpha} + \left\langle |B|^4x, x\right\rangle^{\beta}\left\langle |B^*|^4x, x\right\rangle^{1-\beta}\right) \\ &+\frac{1}{8}\left\langle\left(|A|^4 + |B|^4\right)x, x\right\rangle + \frac{1}{4}\left|\left\langle |B|^2|A|^2x, x\right\rangle\right| \text{ (by Lemma 2.2 (ii))} \\ \leq\ &\frac{1}{4}\left(\left\langle \alpha|A|^4x, x\right\rangle + \left\langle (1-\alpha)|A^*|^4x, x\right\rangle + \left\langle \beta|B|^4x, x\right\rangle + (1-\beta)\left\langle |B^*|^4x, x\right\rangle\right) \\ &+\frac{1}{8}\left\langle\left(|A|^4 + |B|^4\right)x, x\right\rangle + \frac{1}{4}\left|\left\langle |B|^2|A|^2x, x\right\rangle\right| \text{ (by Lemma 2.5)} \\ =\ &\frac{1}{4}\left\langle\left\{\alpha|A|^4 + (1-\alpha)|A^*|^4 + \beta|B|^4 + (1-\beta)|B^*|^4\right\}x, x\right\rangle \\ &+\frac{1}{8}\left\langle\left(|A|^4 + |B|^4\right)x, x\right\rangle + \frac{1}{4}\left|\left\langle |B|^2|A|^2x, x\right\rangle\right| \\ \leq\ &\frac{1}{4}\left\|\alpha|A|^4 + (1-\alpha)|A^*|^4 + \beta|B|^4 + (1-\beta)|B^*|^4\right\| \\ &+\frac{1}{8}\left\||A|^4 + |B|^4\right\| + \frac{1}{4}\omega\left(|B|^2|A|^2\right).\end{aligned}$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\begin{aligned}\omega^2(B^*A) &\leq \frac{1}{4}\left\|\alpha|A|^4 + (1-\alpha)|A^*|^4 + \beta|B|^4 + (1-\beta)|B^*|^4\right\| \\ &\quad + \frac{1}{8}\left\||A|^4 + |B|^4\right\| + \frac{1}{4}\omega\left(|B|^2|A|^2\right).\end{aligned}$$

□

The following result presents an upper bound for $\omega^{2r}(B^*A)$ that interpolates between the two bounds $\frac{1}{2}\left\||A|^{2r} + |B|^{2r}\right\|\omega(B^*A)$ and $\frac{1}{2}\left\||A|^{2r} + i|B|^{2r}\right\|$, as α goes from 0 to ∞.

Theorem 3.8. *Let $A, B \in \mathcal{B}(\mathcal{H})$, and let $\alpha \geq 0, r \geq 1$. Then*

$$\omega^{2r}(B^*A) \leq \frac{1}{2(1+\alpha)} \left\| |A|^{2r} + |B|^{2r} \right\| \omega(B^*A) + \frac{\alpha}{2(1+\alpha)} \left\| |A|^{2r} + i|B|^{2r} \right\|.$$

Proof. Let $x \in \mathcal{H}$ be a unit vector. By using Lemma 2.4, we have

$$\begin{aligned}
|\langle B^*Ax, x\rangle|^{2r} &= |\langle Ax, Bx\rangle|^{2r} \\
&\leq \left(\frac{1}{1+\alpha} \|Ax\| \|Bx\| |\langle Ax, Bx\rangle| + \frac{\alpha}{1+\alpha} \|Ax\|^2 \|Bx\|^2 \right)^r \\
&\leq \frac{1}{1+\alpha} \|Ax\|^r \|Bx\|^r |\langle Ax, Bx\rangle|^r + \frac{\alpha}{1+\alpha} \|Ax\|^{2r} \|Bx\|^{2r} \\
&\leq \frac{1}{2(1+\alpha)} \left(\|Ax\|^{2r} + \|Bx\|^{2r} \right) |\langle Ax, Bx\rangle|^r \\
&\quad + \frac{\alpha}{2(1+\alpha)} \left(\|Ax\|^{4r} + \|Bx\|^{4r} \right) \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&\leq \frac{1}{2(1+\alpha)} \left(\left\langle |A|^2 x, x \right\rangle^r + \left\langle |B|^2 x, x \right\rangle^r \right) |\langle B^*Ax, x\rangle|^r \\
&\quad + \frac{\alpha}{2(1+\alpha)} \left(\left\langle |A|^2 x, x \right\rangle^{2r} + \left\langle |B|^2 x, x \right\rangle^{2r} \right) \\
&\leq \frac{1}{2(1+\alpha)} \left(\left\langle |A|^{2r} x, x \right\rangle + \left\langle |B|^{2r} x, x \right\rangle \right) |\langle B^*Ax, x\rangle|^r \\
&\quad + \frac{\alpha}{2(1+\alpha)} \left(\left\langle |A|^{2r} x, x \right\rangle^2 + \left\langle |B|^{2r} x, x \right\rangle^2 \right) \\
&\quad \text{(by Lemma 2.2 (i))} \\
&= \frac{1}{2(1+\alpha)} \left(\left\langle |A|^{2r} x, x \right\rangle + \left\langle |B|^{2r} x, x \right\rangle \right) |\langle B^*Ax, x\rangle|^r \\
&\quad + \frac{\alpha}{2(1+\alpha)} \left(\left| \left\langle |A|^{2r} x, x \right\rangle + i \left\langle |B|^{2r} x, x \right\rangle \right|^2 \right) \\
&= \frac{1}{2(1+\alpha)} \left\langle \left(|A|^{2r} + |B|^{2r} \right) x, x \right\rangle |\langle B^*Ax, x\rangle|^r \\
&\quad + \frac{\alpha}{2(1+\alpha)} \left| \left\langle \left(|A|^{2r} + i|B|^{2r} \right) x, x \right\rangle \right|^2 \\
&\leq \frac{1}{2(1+\alpha)} \left\| |A|^{2r} + |B|^{2r} \right\| \omega^r(B^*A) \\
&\quad + \frac{\alpha}{2(1+\alpha)} \left\| |A|^{2r} + i|B|^{2r} \right\|.
\end{aligned}$$

□

Corollary 3.9. *Let $A, B \in \mathcal{B}(\mathcal{H})$, and let $\alpha \geq 0$. Then*

$$\begin{aligned}
\omega^2(B^*A) &\leq \frac{1}{2(1+\alpha)} \omega \left(|A|^2 + |B|^2 \right) \omega(B^*A) + \frac{\alpha}{2(1+\alpha)} \omega^2 \left(|A|^2 + i|B|^2 \right) \\
&\leq \frac{1}{2} \left\| |A|^4 + |B|^4 \right\|.
\end{aligned}$$

Proof. Using the inequality (1.5), it follows that

$$\begin{aligned}\omega^2(B^*A) &\leq \frac{1}{2(1+\alpha)}\omega\left(|A|^2+|B|^2\right)\omega(B^*A)+\frac{\alpha}{2(1+\alpha)}\omega^2\left(|A|^2+i|B|^2\right)\\ &\leq \frac{1}{4(1+\alpha)}\omega^2\left(|A|^2+|B|^2\right)+\frac{\alpha}{2(1+\alpha)}\omega^2\left(|A|^2+i|B|^2\right)\\ &\leq \frac{1}{2(1+\alpha)}\omega^2\left(|A|^2+i|B|^2\right)+\frac{\alpha}{2(1+\alpha)}\omega^2\left(|A|^2+i|B|^2\right)\end{aligned}$$

(by Lemma 2.9)

$$\begin{aligned}&= \frac{1}{2}\omega^2\left(|A|^2+i|B|^2\right)\\ &\leq \frac{1}{2}\left\||A|^4+|B|^4\right\| \text{ (by Lemma 2.8).}\end{aligned}$$

□

In [3] the authors proved the following theorem.

Theorem 3.10. *Let $A_i, X_i, B_i \in \mathcal{B}(\mathcal{H})$ $(i=1,2,...,n)$ and let f and g be non-negative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t)=t$ $(t\in[0,\infty))$. Then for all $r\geq 1$, we have*

$$\omega^r\left(\sum_{i=1}^n A_i^*X_iB_i\right)\leq\frac{n^{r-1}}{\sqrt{2}}\omega\left(\sum_{i=1}^n\left(\left[A_i^*g^2(|X_i^*|)A_i\right]^r+i\left[B_i^*f^2(|X_i|)B_i\right]^r\right)\right).$$

In the following theorem we introduce a refinement of the above theorem.

Theorem 3.11. *Let $A_i, X_i, B_i \in \mathcal{B}(\mathcal{H})$ $(i=1,2,...,n)$ and let f and g be non-negative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t)=t$ $(t\in[0,\infty))$. Then for all $r\geq 1$, we have*

$$\begin{aligned}\omega^r\left(\sum_{i=1}^n A_i^*X_iB_i\right) &\leq \frac{n^{r-1}}{\sqrt{2}}\omega\left(\sum_{i=1}^n\left(\left[A_i^*g^2(|X_i^*|)A_i\right]^r+i\left[B_i^*f^2(|X_i|)B_i\right]^r\right)\right)\\ &\quad -\frac{n^{r-1}}{2}\inf_{\|x\|=1}\eta(x),\end{aligned}$$

where

$$\eta(x)=\sum_{i=1}^n\left(\left\langle\left[A_i^*g^2(|X_i^*|)A_i\right]^rx,x\right\rangle^{\frac{1}{2}}-\left\langle\left[B_i^*f^2(|X_i|)B_i\right]^rx,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Proof. Let $x\in\mathcal{H}$ be a unit vector. Then

$$\begin{aligned}&\left|\left\langle\left(\sum_{i=1}^n A_i^*X_iB_i\right)x,x\right\rangle\right|^r\\ =&\left|\sum_{i=1}^n\langle(A_i^*X_iB_i)x,x\rangle\right|^r\\ \leq&\sum_{i=1}^n|\langle A_i^*X_iB_ix,x\rangle|^r\\ =&\sum_{i=1}^n|\langle X_iB_ix,A_ix\rangle|^r\\ \leq&\left(\sum_{i=1}^n\left\langle f^2(|X_i|)B_ix,B_ix\right\rangle^{\frac{1}{2}}\left\langle g^2(|X_i^*|)A_ix,A_ix\right\rangle^{\frac{1}{2}}\right)^r\end{aligned}$$

(by Lemma 2.3)

$$\leq n^{r-1}\sum_{i=1}^{n}\left\langle f^2\left(|X_i|\right)B_ix,B_ix\right\rangle^{\frac{r}{2}}\left\langle g^2\left(|X_i^*|\right)A_ix,A_ix\right\rangle^{\frac{r}{2}}$$

(by Lemma 2.6)

$$\leq n^{r-1}\sum_{i=1}^{n}\left\langle \left(B_i^*f^2\left(|X_i|\right)B_i\right)^r x,x\right\rangle^{\frac{1}{2}}\left\langle \left(A_i^*g^2\left(|X_i^*|\right)A_i\right)^r x,x\right\rangle^{\frac{1}{2}}$$

(by Lemma 2.2 (i))

$$= \frac{n^{r-1}}{2}\left[\sum_{i=1}^{n}\left(\left\langle \left(A_i^*g^2\left(|X_i^*|\right)A_i\right)^r x,x\right\rangle+\left\langle \left(B_i^*f^2\left(|X_i|\right)B_i\right)^r x,x\right\rangle\right)\right]$$
$$-\frac{n^{r-1}}{2}\sum_{i=1}^{n}\left(\left\langle \left(B_i^*f^2\left(|X_i|\right)B_i\right)^r x,x\right\rangle^{\frac{1}{2}}-\left\langle \left(A_i^*g^2\left(|X_i^*|\right)A_i\right)^r x,x\right\rangle^{\frac{1}{2}}\right)^2$$

$$\left(\text{since } \sqrt{a}\sqrt{b}\right. = \left.\frac{a+b}{2}-\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2},\ \text{where } a,b\geq 0\right)$$

$$\leq \frac{n^{r-1}}{\sqrt{2}}\left[\sum_{i=1}^{n}\left|\left\langle \left(B_i^*f^2\left(|X_i|\right)B_i\right)^r x,x\right\rangle+i\left\langle \left(A_i^*g^2\left(|X_i^*|\right)A_i\right)^r x,x\right\rangle\right|\right]$$
$$-\frac{n^{r-1}}{2}\sum_{i=1}^{n}\left(\left\langle \left(B_i^*f^2\left(|X_i|\right)B_i\right)^r x,x\right\rangle^{\frac{1}{2}}-\left\langle \left(A_i^*g^2\left(|X_i^*|\right)A_i\right)^r x,x\right\rangle^{\frac{1}{2}}\right)^2$$

$$= \frac{n^{r-1}}{\sqrt{2}}\left[\sum_{i=1}^{n}\left\langle \left|\left(A_i^*f^2\left(|X_i|\right)A_i\right)^r+i\left(B_i^*g^2\left(|X_i^*|\right)B_i\right)^r\right|x,x\right\rangle\right]$$
$$-\frac{n^{r-1}}{2}\sum_{i=1}^{n}\left(\left\langle \left(B_i^*f^2\left(|X_i|\right)B_i\right)^r x,x\right\rangle^{\frac{1}{2}}-\left\langle \left(A_i^*g^2\left(|X_i^*|\right)A_i\right)^r x,x\right\rangle^{\frac{1}{2}}\right)^2$$

$$\left(\text{as } |a+b|\right. \leq \left.\sqrt{2}\,|a+ib|,\ \forall a,b\in\mathbb{R}\right)$$

$$\leq \frac{n^{r-1}}{\sqrt{2}}\omega\left(\sum_{i=1}^{n}\left(\left[A_i^*g^2\left(|X_i^*|\right)A_i\right]^r+i\left[B_i^*f^2\left(|X_i|\right)B_i\right]^r\right)\right)$$
$$-\frac{n^{r-1}}{2}\inf_{\|x\|=1}\eta\left(x\right),$$

where

$$\eta\left(x\right)=\sum_{i=1}^{n}\left(\left\langle \left[A_i^*g^2\left(|X_i^*|\right)A_i\right]^r x,x\right\rangle^{\frac{1}{2}}-\left\langle \left[B_i^*f^2\left(|X_i|\right)B_i\right]^r x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

It is clear that $\eta\left(x\right)\geq 0$ for all $x\in\mathcal{H}$ with $\|x\|=1$. Then, $\inf\limits_{\|x\|=1}\eta\left(x\right)\geq 0$.

Now, by taking the supremum over $x\in\mathcal{H}$ with $\|x\|=1$, we get

$$\omega^r\left(\sum_{i=1}^{n}A_i^*X_iB_i\right) \leq \frac{n^{r-1}}{\sqrt{2}}\omega\left(\sum_{i=1}^{n}\left(\left[A_i^*g^2\left(|X_i^*|\right)A_i\right]^r+i\left[B_i^*f^2\left(|X_i|\right)B_i\right]^r\right)\right)$$
$$-\frac{n^{r-1}}{2}\inf_{\|x\|=1}\eta\left(x\right).$$

This completes the proof. □

Corollary 3.12. *Let $A_i,X_i,B_i\in B\left(\mathcal{H}\right)$ $(i=1,2,...,n)$ and let f and g be non-negative continuous functions on $[0,\infty)$ satisfying the relation $f\left(t\right)g\left(t\right)=t$ $(t\in[0,\infty))$. Then*

$$\omega^r\left(\sum_{i=1}^{n}X_i\right)\leq\frac{n^{r-1}}{\sqrt{2}}\omega\left(\sum_{i=1}^{n}\left|g^{2r}\left(|X_i^*|\right)+if^{2r}\left(|X_i|\right)\right|\right)-\frac{n^{r-1}}{2}\inf_{\|x\|=1}\eta\left(x\right),$$

where

$$\eta(x)=\sum_{i=1}^{n}\left(\left\langle g^{2r}\left(|X_i^*|\right)x,x\right\rangle^{\frac{1}{2}}-\left\langle f^{2r}\left(|X_i|\right)x,x\right\rangle^{\frac{1}{2}}\right)^2,$$

for all $r\geq 1$.

Proof. Taking $A_i=B_i=I$ $(i=1,2,...,n)$ in Theorem 3.11 we deduce the desired result. □

Next, taking $n=1$, $r=1$ and $f(t)=g(t)=t^{\frac{1}{2}}$ in Corollary 3.12 we get the following inequality.

Corollary 3.13. *Let $A\in\mathcal{B}(\mathcal{H})$. Then*

$$\omega(A)\leq\frac{1}{\sqrt{2}}\omega\left(|A^*|+i|A|\right)-\frac{1}{2}\inf_{\|x\|=1}\eta(x),$$

where

$$\eta(x)=\left(\langle|A^*|x,x\rangle^{\frac{1}{2}}-\langle|A|x,x\rangle^{\frac{1}{2}}\right)^2.$$

Remark 3.14. Since $\eta(x)\geq 0$ for every unit vector x, we have $\inf_{\|x\|=1}\eta(x)\geq 0$. Therefore, Corollary 3.13 provides a sharper bound than

$$\omega(A)\leq\frac{1}{\sqrt{2}}\omega\left(|A^*|+i|A|\right),$$

which was proved in [3, Corollary 2.15]. Also, it is easy to observe that

$$\omega^2(A)\leq\frac{1}{2}\omega^2\left(|A^*|+i|A|\right)\leq\frac{1}{2}\left\||A^*|^2+|A|^2\right\|.$$

Hence, Corollary 3.13 refines (1.2).

For $X_i=I$ $(i=1,2,...,n)$, $r\geq 1$ and $f(t)=g(t)=t^{\frac{1}{2}}$ $(t\in[0,\infty))$ in Theorem 3.11 we get the following inequality.

Corollary 3.15. *Let $A_i,B_i\in B(\mathcal{H})$ $(i=1,2,...,n)$ and $r\geq 1$. Then*

$$\begin{aligned}\omega^r\left(\sum_{i=1}^{n}A_i^*B_i\right)\leq&\frac{n^{r-1}}{\sqrt{2}}\omega\left(\sum_{i=1}^{n}\left(|A_i|^{2r}+i|B_i|^{2r}\right)\right)\\&-\frac{n^{r-1}}{2}\inf_{\|x\|=1}\eta(x),\end{aligned}$$

where

$$\eta(x)=\sum_{i=1}^{n}\left(\left\langle|A_i|^{2r}x,x\right\rangle^{\frac{1}{2}}-\left\langle|B_i|^{2r}x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Remark 3.16. For $n=1$ in Corollary 3.15, we get

$$\omega^r(A^*B)\leq\frac{1}{\sqrt{2}}\omega\left(|A|^{2r}+i|B|^{2r}\right)-\frac{1}{2}\inf_{\|x\|=1}\eta(x),$$

where

$$\eta(x)=\left(\left\langle|A|^{2r}x,x\right\rangle^{\frac{1}{2}}-\left\langle|B|^{2r}x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Data availability No data were used in this study.

Declaration

Conflict of interest The authors declare no conflict of interest.

References

[1] A. Abu-Omar, F. Kittaneh, Upper and lower bounds for the numerical radius with an application to involution operators, Rocky Mountain J. Math. 45 (2015), no. 4, 1055-1065.

[2] M. Al-Dolata, I. Jaradat, A refinement of the Cauchy-Schwarz inequality accompanied by new numerical radius upper bounds, Filomat 37:3 (2023), 971–977. https://doi.org/10.2298/FIL2303971A

[3] P. Bhunia, K. Paul, New upper bounds for the numerical radius of Hilbert space operators, Bull. Sci. Math. 167 (2021), Paper No. 102959, 11 pp

[4] M. L. Buzano, Generalizzatione della disuguaglianza di Cauchy-Schwarz, (Italian) Rend. Sem. Mat. Univ. e Politec. Torino 31 (1974), 405–409.

[5] S. S. Dragomir, Power inequalities for the numerical radius of a product of two operators in Hilbert spaces, Sarajevo J. Math. 5(18) (2009), no. 2, 269-278.

[6] M. El-Haddad, Mohammad, F. Kittaneh, Numerical radius inequalities for Hilbert space operators, II. Studia Math. 182 (2007), no. 2, 133-140.

[7] M. Guesba, On some numerical radius inequalities for normal operators in Hilbert spaces, Journal of Interdisciplinary Mathematics, 25(2) (2022), 463-470.

[8] M. Guesba, Some generalizations of A-numerical radius inequalities for semi-Hilbert space operators, Boll. Unione Mat. Ital., 14 (2021), no. 4, pp. 681–692.

[9] K. E. Gustafson, D. K. M. Rao. Numerical Range. Springer-Verlag, New York, 1997.

[10] G. H. Hardy, J. E. Littlewood, G.Pólya, Inequalities, 2nd ed, Cambridge University Press, Cambridge, 1988.

[11] A. Hosseini and M. Hassani, Inequalities involving norm and numerical radius of Hilbert space operators, Palestine J. Math, 13(4)(2024), 657–663.

[12] F. Kittaneh, Notes on some inequalities for Hilbert space operators, Publ. RIMS Kyoto Univ. 24(1988) 283–293.

[13] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, Studia Math. 158 (2003), no. 1, 11-17.

[14] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Studia Math. 168 (2005), no. 1, 73-80.

[15] H. Moradi and M. Sababheh, New estimates for the numerical radius, Filomat 35 (2020), 4957–4962.

[16] B. Simon, Trace Ideals and Their Applications, Cambridge University Press, 1979.

[17] M.P. Vasić, D.J. Kećkić, Some inequalities for complex numbers, Math. Balk. 1 (1971) 282–286.

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