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ON THE NORM EQUALITIES OF ELEMENTARY OPERATORS

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Abstract. Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. Given $A, B \in \mathcal{B}(\mathcal{H})$, the basic elementary operator $M_{A,B}$ on $\mathcal{B}(\mathcal{H})$ is defined by $M_{A,B}(X) = AXB$ for all $X \in \mathcal{B}(\mathcal{H})$.

In this paper, we establish some necessary and sufficient conditions on the operators $A_k, B_k \in \mathcal{B}(\mathcal{H})$ $(1 \leq k \leq 3)$ under which the norm $\left\|\sum_{k=1}^{3} M_{A_k,B_k}\right\|$ equals its optimal value $\sum_{k=1}^{3} \|A_k\|\|B_k\|$.

1 Introduction and preliminaries

Throughout this paper, let $\mathcal{H}$ denote a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. We denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$. The identity operator on $\mathcal{H}$ is represented by I.

Let $T \in \mathcal{B}(\mathcal{H})$, we designate by $\|T\|$ and T^* the operator norm and the adjoint of T, respectively. The numerical range $W(T)$, the maximal numerical range $W_0(T)$ and the normalized maximal numerical range $W_N(T)$ of T are respectively given by:

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\};$$
$$W_0(T) = \{\lim_{n\to+\infty} \langle Tx_n, x_n\rangle : x_n \in \mathcal{H}, \|x_n\| = 1, \lim_{n\to+\infty} \|Tx_n\| = \|T\|\}$$

and

$$W_N(T) = \begin{cases} W_0\left(\frac{T}{\|T\|}\right) & \text{if } T \neq 0, \\ \{0\} & \text{if } T = 0. \end{cases}$$

It is well-known that these sets are non-empty and convex. Furthermore, both the maximal numerical range $W_0(T)$ and the normalized maximal numerical range $W_N(T)$ of an operator $T \in \mathcal{B}(\mathcal{H})$ are closed and included in the closure of the numerical range $W(T)$. Note that, in general, the numerical range $W(T)$ is not closed. In addition, it satisfies the following properties:

(1) $W(T^*) = \{\overline{\lambda} : \lambda \in W(T)\}$;

(2) $W(aT + bI) = aW(T) + b$ for every $a, b \in \mathbb{C}$;

(3) $W(U^*TU) = W(T)$ for any unitary operator $U \in \mathcal{B}(\mathcal{H})$.

Here $\overline{\lambda}$ stands for the complex conjugate of a scalar $\lambda \in \mathbb{C}$. Notice also that the maximal numerical range $W_0(T)$ of T satisfies properties (1) and (3) above (see [13]). However, it does not verify the translation property by scalar, i.e., $W_0(T + b) \neq W_0(T) + b$ for $b \neq 0$ (see [13, 20]). Further, we have $W_0(T + \lambda_1) \cap W_0(T + \lambda_2) = \emptyset$ for $\lambda_1 \neq \lambda_2$ (see [20, Lemma

4]). For more information regarding these sets and their applications, we refer the reader to [1, 11, 12, 13, 18, 20] and references therein.

For two n-tuples $\mathbf{A} = (A_1, \cdots, A_n)$ and $\mathbf{B} = (B_1, \cdots, B_n)$ of elements of $\mathcal{B}(\mathcal{H})$, the elementary operator $R_{\mathbf{A},\mathbf{B}}$ induced by $\mathbf{A}$ and $\mathbf{B}$ is the linear operator on $\mathcal{B}(\mathcal{H})$ given by:

$$R_{\mathbf{A},\mathbf{B}}(X) = \sum_{k=1}^{n} A_k X B_k \ \ (X \in \mathcal{B}(\mathcal{H})).$$

Obviously, $R_{\mathbf{A},\mathbf{B}}$ is bounded and $\|R_{\mathbf{A},\mathbf{B}}\| \leq \sum_{k=1}^{n} \|A_k\|\|B_k\|$. One of the most well-known examples of elementary operators is the basic elementary operator $M_{A,B}$ associated with the operators $A, B \in \mathcal{B}(\mathcal{H})$. It is defined as follows:

$$M_{A,B}(X) = AXB \ \ (X \in \mathcal{B}(\mathcal{H})).$$

Another familiar example of elementary operators is the generalized derivation $\delta_{A,B}$ given by:

$$\delta_{A,B} = M_{A,I} + M_{I,-B}.$$

Elementary operators have attracted the attention of many researchers in operator theory due to their numerous applications in different branches of mathematics, such as Banach algebras, quantum information theory, etc. For more details about the elementary operators, we refer the reader to [2, 3, 4, 5, 6, 7, 8, 9, 14, 15, 16, 17, 19, 20, 21, 22, 23] and references therein.

One of the famous problems relating to elementary operators is the norm problem for elementary operators which consists of finding a formula that describes the norm of an elementary operator $R_{\mathbf{A},\mathbf{B}}$ in terms of its coefficients A_i and B_i, $i = 1, 2, \cdots, n$. In [23] Timoney presented the first formula for the norm of the elementary operator $R_{\mathbf{A},\mathbf{B}}$ in the Hilbert space setting, but it is rather complicated. Notice that this problem has been investigated by several authors in different situations, see for instance [4, 5, 8, 9, 14, 16, 17, 19, 20, 22] and references therein.

In [10], Fialkow proved that $\|M_{A,B}\| = \|A\|\|B\|$ for every $A, B \in \mathcal{B}(\mathcal{H})$. Therefore, a natural question that can be raised is:

Question: Do we have $\|R_{\mathbf{A},\mathbf{B}}\| = \sum_{k=1}^{n} \|A_k\|\|B_k\|$?

Unfortunately, the answer to this question is negative in general. Indeed, Stampfli [20] showed the following equivalence:

Theorem 1.1 ([20, Theorem 7]). *Let $A, B \in \mathcal{B}(\mathcal{H}) \setminus \{0\}$. Then the following assertions are equivalent:*

(i) $\|\delta_{A,B}\| = \|A\| + \|B\|$;

(ii) $W_N(A) \cap W_N(-B) \neq \emptyset$.

Later on, Boumazgour [8] generalized Theorem 1.1 as follows:

Theorem 1.2 ([8, Theorem 3.1]). *Let $A, B, C, D \in \mathcal{B}(\mathcal{H}) \setminus \{0\}$. Then the following assertions are equivalent:*

(i) $\|M_{A,B} + M_{C,D}\| = \|A\|\|B\| + \|C\|\|D\|$;

(ii) $W_N(C^*A) \cap W_N(DB^*) \neq \emptyset$, $\|C^*A\| = \|C\|\|A\|$ *and* $\|DB^*\| = \|D\|\|B\|$.

Following these results, our purpose in this paper is to establish necessary and sufficient conditions on the operators $A_k, B_k \in \mathcal{B}(\mathcal{H})$ $(1 \leq k \leq 3)$ under which the norm $\left\|\sum_{k=1}^{3} M_{A_k,B_k}\right\|$ equals its optimal value $\sum_{k=1}^{3} \|A_k\|\|B_k\|$.

For $\lambda \in \mathbb{C}$, let $\Re(\lambda)$ denote the real part of λ.

2 Main results

In this section, we give our main results. We begin with the following theorem.

Theorem 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H})\backslash\{0\}$. If $AB^* = B^*A = \|A\|\|B\|I$, then the following assertions are equivalent:*

(i) $\|I + M_{A,B} + M_{B,A}\| = 1 + 2\|A\|\|B\|$;

(ii) $W_N(A) \cap W_N(B^*) \neq \emptyset$.

Proof. (i) $\Rightarrow$ (ii) : Using the triangular inequality and the assumption in Part (i), we obtain that $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$. Hence, it follows from Theorem 1.2 that

$$W_N(A) \cap W_N(B^*) \neq \emptyset.$$

(ii) $\Rightarrow$ (i) : We suppose that $W_N(A) \cap W_N(B^*) \neq \emptyset$. So, according to Theorem 1.1, we get that

$$\|\delta_{A,-B^*}\| = \|A\| + \|B\|.$$

Hence, there exist two sequences $\{X_n\} \subseteq \mathcal{B}(\mathcal{H})$ and $\{x_n\} \subseteq \mathcal{H}$ such that $\|X_n\| = \|x_n\| = 1$ for each n and

$$\lim_{n\to+\infty} \|AX_nx_n + X_nB^*x_n\| = \|A\| + \|B\|. \tag{2.1}$$

For each $n \in \mathbb{N}$, we have

$$\|AX_nx_n + X_nB^*x_n\|^2 = \|AX_nx_n\|^2 + \|X_nB^*x_n\|^2 + 2\Re(\langle AX_nx_n, X_nB^*x_n\rangle)\cdot \tag{2.2}$$

Since $\|AX_nx_n\| \leq \|A\|\|X_nx_n\| \leq \|A\|$, $\|X_nB^*x_n\| \leq \|B^*x_n\| \leq \|B\|$ and $|\langle AX_nx_n, X_nB^*x_n\rangle| \leq \|A\|\|B\|$, then by combining (2.1) and (2.2), we deduce that

$$\lim_{n\to+\infty} \|X_nx_n\| = 1; \tag{2.3}$$

$$\lim_{n\to+\infty} \|AX_nx_n\| = \|A\|; \tag{2.4}$$

$$\lim_{n\to+\infty} \|X_nB^*x_n\| = \lim_{n\to+\infty} \|B^*x_n\| = \|B\|; \tag{2.5}$$

$$\lim_{n\to+\infty} \Re(\langle AX_nx_n, X_nB^*x_n\rangle) = \|A\|\|B\|. \tag{2.6}$$

Set $y_n = \frac{B^*x_n}{\|B\|}$. We claim that

$$\lim_{n\to+\infty} \|AX_nBy_n + BX_nAy_n + X_ny_n\| = 1 + 2\|A\|\|B\|. \tag{2.7}$$

A simple calculation shows that

$$\begin{aligned}\|AX_nBy_n + BX_nAy_n + X_ny_n\|^2 = {} & \|AX_nBy_n\|^2 + \|BX_nAy_n\|^2 + \|X_ny_n\|^2 \\ & + 2\Re(\langle AX_nBy_n, BX_nAy_n\rangle) + 2\Re(\langle AX_nBy_n, X_ny_n\rangle) \\ & + 2\Re(\langle BX_nAy_n, X_ny_n\rangle)\end{aligned}$$

for each $n \in \mathbb{N}$. Therefore, to prove (2.7), it suffices to show that

$$\lim_{n\to+\infty} \|AX_nBy_n\| = \|A\|\|B\|; \tag{2.8}$$

$$\lim_{n\to+\infty} \|BX_nAy_n\| = \|A\|\|B\|; \tag{2.9}$$

$$\lim_{n\to+\infty} \|X_ny_n\| = 1; \tag{2.10}$$

$$\lim_{n\to+\infty} \Re(\langle AX_nBy_n, BX_nAy_n\rangle) = \|A\|^2\|B\|^2; \tag{2.11}$$

$$\lim_{n\to+\infty} \Re(\langle AX_nBy_n, X_ny_n\rangle) = \|A\|\|B\|; \tag{2.12}$$

$$\lim_{n\to+\infty} \Re(\langle BX_nAy_n, X_ny_n\rangle) = \|A\|\|B\|. \tag{2.13}$$

Let us first observe that, according to (2.5), there exists a sequence $\{\varepsilon_n\}$ of elements of $\mathcal{H}$ which converges to 0 such that

$$BB^*x_n = \|B\|^2 x_n + \varepsilon_n \ \ (n \in \mathbb{N}). \tag{2.14}$$

Let us now show (2.11). By using (2.14), we have for each $n \in \mathbb{N}$:

$$\begin{aligned}
\Re(\langle AX_nBy_n, BX_nAy_n\rangle) &= \Re(\langle B^*AX_nBy_n, X_nAy_n\rangle)\\
&= \|A\|\|B\|\Re(\langle X_nBy_n, X_nAy_n\rangle)\\
&= \|A\|\Big(\|B\|^2\Re(\langle X_nx_n, X_nAy_n\rangle) + \Re(\langle X_n\varepsilon_n, X_nAy_n\rangle)\Big)\\
&= \|A\|\Big(\|B\|^2\Re\left(\left\langle X_nx_n, X_nA\frac{B^*x_n}{\|B\|}\right\rangle\right) + \Re(\langle X_n\varepsilon_n, X_nAy_n\rangle)\Big)\\
&= \|A\|\Big(\|B\|^2\|A\|\|X_nx_n\|^2 + \Re(\langle X_n\varepsilon_n, X_nAy_n\rangle)\Big).
\end{aligned}$$

Letting $n \to \infty$ and using (2.3), we conclude (2.11).

The equalities in (2.8) and (2.9) follow directly from (2.11) and the following inequality

$$\Re(\langle AX_nBy_n, BX_nAy_n\rangle) \le \|AX_nBy_n\|\|BX_nAy_n\| \le \|A\|^2\|B\|^2 \ \ (n \in \mathbb{N}).$$

Clearly, the equality (2.10) is an immediate consequence of (2.8) and (2.12). It therefore suffices to prove the equality (2.12). Applying (2.14) again, we obtain that

$$\begin{aligned}
\Re(\langle AX_nBy_n, X_ny_n\rangle) &= \frac{\Re(\langle AX_nBB^*x_n, X_nB^*x_n\rangle)}{\|B\|^2}\\
&= \Re(\langle AX_nx_n, X_nB^*x_n\rangle) + \frac{\Re(\langle AX_n\varepsilon_n, X_nB^*x_n\rangle)}{\|B\|^2},
\end{aligned}$$

for every $n \in \mathbb{N}$. Letting $n \to \infty$ and using (2.6) we deduce the equality in (2.12). It remains to prove (2.13). Note first that by virtue of the equality in (2.9) and the hypothesis, we infer that $\lim_{n\to+\infty}\|BX_nx_n\| = \|B\|$. Consequently, there exists a sequence $\{z_n\} \subseteq \mathcal{H}$ such that $\lim_{n\to+\infty} z_n = 0$ and

$$B^*BX_nx_n = \|B\|^2X_nx_n + z_n \ \ (n \in \mathbb{N}). \tag{2.15}$$

We now turn to the proof of (2.13). For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\Re(\langle BX_nAy_n, X_ny_n\rangle) &= \Re\left(\left\langle \frac{\|A\|\|B\|BX_nx_n}{\|B\|}, \frac{X_nB^*x_n}{\|B\|}\right\rangle\right)\\
&= \Re\left(\left\langle \frac{AB^*BX_nx_n}{\|B\|^2}, X_nB^*x_n\right\rangle\right)\\
&= \Re\left(\langle AX_nx_n, X_nB^*x_n\rangle\right) + \Re\left(\left\langle \frac{Az_n}{\|B\|^2}, X_nB^*x_n\right\rangle\right) \ \text{(by (2.15))}.
\end{aligned}$$

Letting $n \to \infty$ and using the equality in (2.6), we get the desired result. This completes the proof. □

Our second main result is stated as follows.

Theorem 2.2. *Let $A, B, C, D, E \in \mathcal{B}(\mathcal{H}) \setminus \{0\}$ satisfy the following conditions:*

(C_1) $DB^* = B^*D = \|D\|\|B\|I$;

(C_2) $CE^* = E^*C = \|C\|\|E\|I$.

Then, the following assertions are equivalent:

(i) $\|M_{A,B} + M_{C,D} + M_{E,D^*}\| = \|A\|\|B\| + \|C\|\|D\| + \|E\|\|D\|$;

(ii) $W_N(C^*A + E^*A) \cap W_N(I + D^*B^*) \neq \emptyset$, $\|C^*A + E^*A\| = \|C\|\|A\| + \|E\|\|A\|$ *and* $\|I + D^*B^*\| = 1 + \|D\|\|B\|$.

In order to prove Theorem 2.2, we need the following auxiliary lemmas.

Lemma 2.3. *Let $A, B, C, D, E \in \mathcal{B}(\mathcal{H})$ satisfy the conditions (C_1) and C_2 of Theorem 2.2. Assume that exist two unit sequences $\{X_n\} \subseteq \mathcal{B}(\mathcal{H})$ and $\{x_n\} \subseteq \mathcal{H}$ such that*

$$\lim_{n\to+\infty} \|AX_nBx_n + CX_nDx_n + EX_nD^*x_n\| = \|A\|\|B\| + \|C\|\|D\| + \|E\|\|D\|. \tag{2.16}$$

Then the following equalities hold:

$$\lim_{n\to+\infty} \|AX_nBx_n\| = \|A\|\|B\|; \tag{2.17}$$

$$\lim_{n\to+\infty} \|CX_nDx_n\| = \|C\|\|D\|; \tag{2.18}$$

$$\lim_{n\to+\infty} \|EX_nD^*x_n\| = \|E\|\|D\|; \tag{2.19}$$

$$\lim_{n\to+\infty} \Re(\langle AX_nBx_n, CX_nDx_n\rangle) = \|A\|\|B\|\|C\|\|D\|; \tag{2.20}$$

$$\lim_{n\to+\infty} \Re(\langle AX_nBx_n, EX_nD^*x_n\rangle) = \|A\|\|B\|\|E\|\|D\|; \tag{2.21}$$

$$\lim_{n\to+\infty} \Re(\langle CX_nDx_n, EX_nD^*x_n\rangle) = \|C\|\|E\|\|D\|^2; \tag{2.22}$$

$$\lim_{n\to+\infty} \|Bx_n\| = \|B\|. \tag{2.23}$$

Proof. A simple calculation shows that

$$\begin{aligned}\|AX_nBx_n + CX_nDx_n + EX_nD^*x_n\|^2 &= \|AX_nBx_n\|^2 + \|CX_nDx_n\|^2 + \|EX_nD^*\|^2 \\ &+ 2\Re(\langle AX_nBx_n, CX_nDx_n\rangle) + 2\Re(\langle AX_nBx_n, EX_nD^*x_n\rangle) \\ &+ 2\Re(\langle CX_nDx_n, EX_nD^*x_n\rangle).\end{aligned} \tag{2.24}$$

Hence, by combining (2.16) and (2.24), we easily derive the desired equalities. □

Lemma 2.4. *Under the same assumptions and notations of Lemma 2.3, we have the following equalities:*

$$\lim_{n\to+\infty} \|C^*AX_ny_n\| = \|C\|\|A\|; \tag{2.25}$$

$$\lim_{n\to+\infty} \|E^*AX_ny_n\| = \|E\|\|A\|; \tag{2.26}$$

$$\lim_{n\to+\infty} \|X_ny_n\| = 1; \tag{2.27}$$

$$\lim_{n\to+\infty} \|X_nD^*B^*y_n\| = \|D\|\|B\|; \tag{2.28}$$

$$\lim_{n\to+\infty} \Re(\langle C^*AX_ny_n, E^*AX_ny_n\rangle) = \|C\|\|E\|\|A\|^2; \tag{2.29}$$

$$\lim_{n\to+\infty} \Re(\langle C^*AX_ny_n, X_ny_n\rangle) = \|C\|\|A\|, \tag{2.30}$$

where $y_n = \frac{Bx_n}{\|B\|}$ for all $n \in \mathbb{N}$.

Proof. First of all, observe that from Equality (2.23), we can find a sequence $\{\epsilon_n\} \subseteq \mathcal{H}$, with $\lim_{n\to+\infty} \|\epsilon_n\| = 0$ such that

$$B^*Bx_n = \|B\|^2x_n + \epsilon_n \ \ (n \in \mathbb{N}). \tag{2.31}$$

We start by showing the equality in (2.29). For each n, we have

$$\begin{aligned}\Re(\langle C^*AX_ny_n, E^*AX_ny_n\rangle) &= \Re(\langle AX_ny_n, CE^*AX_ny_n\rangle) \\ &= \|C\|\|E\|\Re\left(\left\langle AX_n\frac{Bx_n}{\|B\|}, AX_n\frac{Bx_n}{\|B\|}\right\rangle\right) \text{ [by Hypothesis } (C_2)] \\ &= \frac{\|C\|\|E\|\|AXBx_n\|^2}{\|B\|^2}.\end{aligned}$$

Using the equality in (2.17) and letting $n \to \infty$, we obtain the required equality.

The equalities in (2.25) and (2.26) follow directly from (2.29).

Clearly, the equality in (2.27) is an immediate consequence of (2.30). Let us now show the equality in (2.30). For each n, we have

$$\begin{aligned}\Re(\langle C^*AX_ny_n, X_ny_n\rangle) &= \frac{\Re(\langle C^*AX_nBx_n, X_nBx_n\rangle)}{\|B\|^2}\\ &= \frac{\Re(\langle C^*AX_nBx_n, X_nDB^*Bx_n\rangle)}{\|B\|^3\|D\|} \text{ [by Hypothesis } (C_1)]\\ &= \frac{\Re(\langle C^*AX_nBx_n, X_nDx_n\rangle)}{\|B\|\|D\|} + \frac{\Re(\langle C^*AX_nBx_n, X_nD\epsilon_n\rangle)}{\|B\|^3\|D\|} \text{ [by (2.31)]}\\ &= \frac{\Re(\langle AX_nBx_n, CX_nDx_n\rangle)}{\|B\|\|D\|} + \frac{\Re(\langle C^*AX_nBx_n, X_nD\epsilon_n\rangle)}{\|B\|^3\|D\|}.\end{aligned}$$

Using (2.20) and letting $n \to \infty$, we infer the desired equality.

It remains to establish the equality in (2.28). It follows from (2.19) that

$$\lim_{n\to+\infty} \|X_nD^*x_n\| = \|D\|.$$

Thus, by combining this equality with (2.31), we obtain the equality in (2.28). This completes the proof. □

Lemma 2.5. *Under the same conditions and notations of Lemma 2.3, the following equalities hold:*

$$\lim_{n\to+\infty} \Re(\langle C^*AX_ny_n, X_nD^*B^*y_n\rangle) = \|C\|\|A\|\|D\|\|B\|; \tag{2.32}$$

$$\lim_{n\to+\infty} \Re(\langle E^*AX_ny_n, X_nD^*B^*y_n\rangle) = \|E\|\|A\|\|D\|\|B\|; \tag{2.33}$$

$$\lim_{n\to+\infty} \Re(\langle E^*AX_ny_n, X_ny_n\rangle) = \|E\|\|A\|; \tag{2.34}$$

$$\lim_{n\to+\infty} \Re(\langle X_ny_n, X_nD^*B^*y_n\rangle) = \|D\|\|B\|. \tag{2.35}$$

Proof. Note first that there exist two sequences $\{\epsilon'_n\}$ and $\{\epsilon''_n\}$ of elements of $\mathcal{H}$ both converging to 0 such that for each n, we have

$$CC^*EX_nD^*x_n = \|C\|^2EX_nD^*x_n + \epsilon'_n; \tag{2.36}$$

$$C^*CX_nDx_n = \|C\|^2X_nDx_n + \epsilon''_n. \tag{2.37}$$

Indeed, using (2.22) and the equality

$$\Re(\langle CX_nDx_n, EX_nD^*x_n\rangle) = \Re(\langle X_nDx_n, C^*EX_nD^*x_n\rangle) \ \ (n \in \mathbb{N}),$$

we get that

$$\lim_{n\to+\infty} \|C^*EX_nD^*x_n\| = \|C\|\|E\|\|D\|. \tag{2.38}$$

Hence, by combining (2.19) and (2.38), we can infer (2.36). The equality (2.37) follows from (2.18) by using a similar reasoning.

We now show Equality (2.32). For each n, we have

$$\begin{aligned}
\Re(\langle C^*AX_ny_n, X_nD^*B^*y_n\rangle) &= \frac{\Re(\langle C^*AX_nBx_n, X_nD^*B^*Bx_n\rangle)}{\|B\|^2}\\
&= \Re(\langle C^*AX_nBx_n, X_nD^*x_n\rangle)\\
&\qquad + \frac{\Re(\langle C^*AX_nBx_n, X_nD^*\epsilon_n\rangle)}{\|B\|^2} \quad \text{[by (2.31)]}\\
&= \frac{\Re(\langle AX_nBx_n, CC^*EX_nD^*x_n\rangle)}{\|E\|\|C\|}\\
&\qquad + \frac{\Re(\langle C^*AX_nBx_n, X_nD^*\epsilon_n\rangle)}{\|B\|^2} \quad \text{[by } (C_2)\text{]}\\
&= \frac{\|C\|\Re(\langle AX_nBx_n, EX_nD^*x_n\rangle)}{\|E\|} + \frac{\Re(\langle AX_nBx_n, \epsilon_n'\rangle)}{\|E\|\|C\|}\\
&\qquad + \frac{\Re(\langle C^*AX_nBx_n, X_nD^*\epsilon_n\rangle)}{\|B\|^2} \quad \text{[by (2.36)]}.
\end{aligned}$$

Hence, we achieve the required equality by letting $n \to \infty$ and using (2.21).

The equality (2.33) comes directly from (2.21) and (2.31). It remains to show the equalities in (2.34) and (2.35). To do this, we need the following two equalities. The first one is as follows:

$$\begin{aligned}
\Re(\langle E^*AX_ny_n, X_ny_n\rangle) &= \frac{\Re(\langle AX_nBx_n, EX_nBx_n\rangle)}{\|B\|^2}\\
&= \frac{\Re(\langle AX_nBx_n, EX_nDB^*Bx_n\rangle)}{\|D\|\|B\|^3} \quad \text{[by } (C_1)\text{]}\\
&= \frac{\Re(\langle AX_nBx_n, EX_nDx_n\rangle)}{\|D\|\|B\|} + \frac{\Re(\langle AX_nBx_n, EX_nD\epsilon_n\rangle)}{\|D\|\|B\|^3} \quad \text{[by (2.31)]}\\
&= \frac{\Re(\langle AX_nBx_n, EC^*CX_nDx_n\rangle)}{\|D\|\|B\|\|C\|^2} - \frac{\Re(\langle AX_nBx_n, E\epsilon_n''\rangle)}{\|D\|\|B\|\|C\|^2}\\
&\qquad + \frac{\Re(\langle AX_nBx_n, EX_nD\epsilon_n\rangle)}{\|D\|\|B\|^3} \quad \text{[by (2.37)]}\\
&= \frac{\|E\|\Re(\langle AX_nBx_n, CX_nDx_n\rangle)}{\|D\|\|B\|\|C\|} - \frac{\Re(\langle AX_nBx_n, E\epsilon_n''\rangle)}{\|D\|\|B\|\|C\|^2}\\
&\qquad + \frac{\Re(\langle AX_nBx_n, EX_nD\epsilon_n\rangle)}{\|D\|\|B\|^3} \quad \text{[by } (C_2)\text{]},
\end{aligned}$$

the second equality is:

$$\begin{aligned}
\Re(\langle X_ny_n, X_nD^*B^*y_n\rangle) &= \frac{\Re(\langle X_nBx_n, X_nD^*B^*Bx_n\rangle)}{\|B\|^2}\\
&= \Re(\langle X_nBx_n, X_nD^*x_n\rangle) + \frac{\Re(\langle X_nBx_n, X_nD^*\epsilon_n\rangle)}{\|B\|^2} \quad \text{[by (2.31)]}\\
&= \frac{\Re(\langle X_nDB^*Bx_n, C^*EX_nD^*x_n\rangle)}{\|B\|\|D\|\|C\|\|E\|}\\
&\qquad + \frac{\Re(\langle X_nBx_n, X_nD^*\epsilon_n\rangle)}{\|B\|^2} \quad \text{[by } (C_1) \text{ and } (C_2)\text{]}\\
&= \frac{\|B\|\Re(\langle CX_nDx_n, EX_nD^*x_n\rangle)}{\|D\|\|C\|\|E\|} + \frac{\Re(\langle X_nD\epsilon_n, C^*EX_nD^*x_n\rangle)}{\|B\|\|D\|\|C\|\|E\|}\\
&\qquad + \frac{\Re(\langle X_nBx_n, X_nD^*\epsilon_n\rangle)}{\|B\|^2} \quad \text{[by (2.31)]}.
\end{aligned}$$

So, by letting $n \to \infty$ and using (2.20) and (2.22), we obtain the required equalities. This completes the proof. □

We are now in a position to give the proof of Theorem 2.2.

Proof of Theorem 2.2.

(i) $\Rightarrow$ (ii) : As per Theorem 1.1, to establish (ii), it suffices to show that

$$\begin{aligned}\|\delta_{C^*A+E^*A;-(I+D^*B^*)}\| &= \|C^*A + E^*A\| + \|I + D^*B^*\| \\ &= \|C\|\|A\| + \|E\|\|A\| + 1 + \|D\|\|B\|.\end{aligned} \tag{2.39}$$

To do so, observe first that from assumption (i) there exist two unit sequences $\{X_n\} \subseteq \mathcal{B}(\mathcal{H})$ and $\{x_n\} \subseteq \mathcal{H}$ such that

$$\lim_{n\to+\infty} \|AX_nBx_n + CX_nDx_n + EX_nD^*x_n\| = \|A\|\|B\| + \|C\|\|D\| + \|E\|\|D\|.$$

Therefore, in order to obtain Equality (2.39), we only need to prove that

$$\begin{aligned}\lim_{n\to+\infty} \|(C^*A + E^*A)X_ny_n + X_n(I + D^*B^*)y_n\| = \|C\|\|A\|+\|E\|\|A\| \\ + 1 + \|D\|\|B\|,\end{aligned} \tag{2.40}$$

where $y_n = \frac{Bx_n}{\|B\|}$ for all $n \in \mathbb{N}$. By a straightforward calculation, we have

$$\begin{aligned}&\|(C^*A + E^*A)X_ny_n + X_n(I + D^*B^*)y_n\|^2 \\ &= \|C^*AX_ny_n\|^2 + \|E^*AX_ny_n\|^2 + \|X_ny_n\|^2 + \|X_nD^*B^*y_n\|^2 \\ &+ 2\Re(\langle C^*AX_ny_n, E^*AX_ny_n\rangle) + 2\Re(\langle C^*AX_ny_n, X_ny_n\rangle) + 2\Re(\langle C^*AX_ny_n, X_nD^*B^*y_n\rangle) \\ &+ 2\Re(\langle E^*AX_ny_n, X_ny_n\rangle) + 2\Re(\langle E^*AX_ny_n, X_nD^*B^*y_n\rangle) \\ &+ 2\Re(\langle X_ny_n, X_nD^*B^*y_n\rangle),\end{aligned}$$

for every n. Therefore, by applying Lemmas 2.4 and 2.5, we obtain the equality in (2.40). Consequently, the equality in (2.39) is reached using the triangular inequality and (2.40).

(ii) $\Rightarrow$ (i) : Combining Theorem 1.1 together with the assumption in Part (ii), we obtain that

$$\|\delta_{C^*A+E^*A;-(I+D^*B^*)}\| = \|C\|\|A\| + \|E\|\|A\| + 1 + \|D\|\|B\|.$$

From the equality above, we can find two sequences $\{X_n\} \subseteq \mathcal{B}(\mathcal{H})$ and $\{x_n\} \subseteq \mathcal{H}$ with $\|X_n\| = \|x_n\| = 1$ for all n, such that

$$\begin{aligned}\lim_{n\to+\infty} &\|C^*AX_nx_n + E^*AX_nx_n + X_nx_n + X_nD^*B^*x_n\| \\ &= \|C\|\|A\| + \|E\|\|A\| + 1 + \|D\|\|B\|.\end{aligned}$$

This implies the following equalities:

$$\lim_{n\to+\infty} \Re(\langle C^*AX_nx_n, X_nx_n\rangle) = \|C\|\|A\|; \tag{2.41}$$

$$\lim_{n\to+\infty} \Re(\langle E^*AX_nx_n, X_nD^*B^*x_n\rangle) = \|E\|\|A\|\|D\|\|B\|; \tag{2.42}$$

$$\lim_{n\to+\infty} \Re(\langle X_nx_n, X_nD^*B^*x_n\rangle) = \|D\|\|B\|; \tag{2.43}$$

$$\lim_{n\to+\infty} \|B^*x_n\| = \|B\|. \tag{2.44}$$

Set $y_n = \frac{B^*x_n}{\|B\|}$. We want to prove

$$\lim_{n\to+\infty} \|AX_nBy_n + CX_nDy_n + EX_nD^*y_n\| = \|A\|\|B\| + \|C\|\|D\| + \|E\|\|D\|.$$

For this purpose, it suffices to establish the following equalities:

$$\lim_{n\to+\infty} \|AX_nBy_n\| = \|A\|\|B\|; \tag{2.45}$$

$$\lim_{n\to+\infty} \|CX_nDy_n\| = \|C\|\|D\|; \tag{2.46}$$

$$\lim_{n\to+\infty} \|EX_nD^*y_n\| = \|E\|\|D\|; \tag{2.47}$$

$$\lim_{n\to+\infty} \Re(\langle AX_nBy_n, CX_nDy_n\rangle) = \|A\|\|B\|\|C\|\|D\|; \tag{2.48}$$

$$\lim_{n\to+\infty} \Re(\langle AX_nBy_n, EX_nD^*y_n\rangle) = \|A\|\|B\|\|E\|\|D\|; \tag{2.49}$$

$$\lim_{n\to+\infty} \Re(\langle CX_nDy_n, EX_nD^*y_n\rangle) = \|C\|\|E\|\|D\|^2. \tag{2.50}$$

Obviously, the equalities in (2.45), (2.46) and (2.47) are immediate consequences of (2.48), (2.49) and (2.50). Therefore, it is sufficient to show the last three equalities. First, observe from (2.44) that there exists a sequence $\{u_n\}$ of elements of $\mathcal{H}$ such that $\lim_{n\to+\infty} u_n = 0$ and

$$BB^*x_n = \|B\|^2x_n + u_n \quad (n \in \mathbb{N}). \tag{2.51}$$

Let us now begin by showing (2.48). For each n, we have

$$\begin{aligned}\Re(\langle AX_nBy_n, CX_nDy_n\rangle) &= \frac{\Re(\langle AX_nBB^*x_n, CX_nDB^*x_n\rangle)}{\|B\|^2}\\ &= \|B\|\|D\|\Re(\langle C^*AX_nx_n, X_nx_n\rangle)\\ &\quad + \frac{\Re(\langle AX_nu_n, CX_nDB^*x_n\rangle)}{\|B\|^2} \quad [\text{by } (C_1) \text{ and } (2.51)].\end{aligned}$$

Using (2.41) and letting $n \to \infty$, we obtain Equality (2.48). The equality in (2.49) follows directly from (2.42) and (2.41). Combining the assumptions in (C_1) and (C_2) with the equality in (2.43), we directly obtain Equality (2.50). This completes the proof.

□

Data availability.

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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