Palestine Journal of Mathematics

Vol.15(3), (2026) , 1188–1204 © Palestine Polytechnic University-PPU 2026

A COVID-19 Stochastic Epidemic Model's Mathematical Analysis

A. Kadri, A. Boudaoui, Fawaz K. Alalhareth and N. Laksaci

Communicated by Thabet Abdeljawad

MSC 2020 Classifications: 37A50, 60G10, 37H30.

Keywords and phrases: COVID-19 epidemic model, disease extinction, stationary distribution (**SD**), numerical simulation.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper

Corresponding Author: A. Boudaoui

Abstract. In this article, we extend a discrete COVID-19 epidemic model from a deterministic to a stochastic framework by formulating it using a stochastic differential equation (SDE). The model classifies the population into four subgroups: susceptible, infected, exposed, and recovered. First, we formulate the model and we establish the existence and uniqueness of a global positive solution. We have found conditions that are enough for extinction and the existence of a unique stationary distribution. In conclusion, we use a numerical simulation to show the outcomes of the analytical work.

1 Introduction

SARS-CoV-2 is the infectious illness that causes COVID-19 [5]. In December 2019, Wuhan, China, reported the first COVID-19 case [6]. All eight of the world's continents, Africa, Europe, Central America, the Caribbean, Oceania, Asia, and North and South America, had been affected by the pandemic by January [7]. The World Health Organization announced the new COVID-19 virus as an international health emergency on March 11, 2020. Due to the illness's quick progression and the new virus's apparent high contagiousness, a worldwide pandemic has been declared. While some people infected with SARS-CoV-2 show no symptoms or very mild symptoms, COVID-19 is deadly for others. The COVID-19 death toll worldwide reached nearly 2,000,000 in just one year of the disease's spread[8]. Around 4,000,000 deaths associated with COVID-19 had been recorded globally as of July 2021 [9].

Mathematical models are indispensable for in-depth investigation into how deadly diseases spread among populations and for formulating effective methods to control or prevent their transmission. These models enable scientists to replicate the patterns of disease spread, assess the effectiveness of control measures such as vaccination and isolation, and predict the progression of outbreaks under different conditions. By integrating data from biological, demographic, and behavioral sources, mathematical modeling provides a structured approach to understanding the complex processes underlying infectious disease transmission and supports evidence-based decision-making in public health planning. Many models have lately been developed by researchers to describe and study the transmission of infectious diseases [10, 11, 22, 15, 23, 18]. When COVID-19 was first identified as a global issue, numerous studies were conducted using mathematical models to predict its future trajectory [19, 20, 14, 17, 25].

In [25] Thanin et al. discussed modeling the propagation of COVID-19 using a discrete $\mathcal{SEIR}$ framework, with the aim of understanding disease transmission dynamics and identifying

strategies to control its spread. The model analyzed is as follows:

$$\begin{aligned}
\mathcal{S}(1+k)-\mathcal{S}(n) &= \Xi-\wp_1\mathcal{E}(k)\mathcal{S}(k)-\wp_2\mathcal{I}(k)\mathcal{S}(k)+\zeta\mathcal{R}(k)-\mho\mathcal{S}(k)\\
\mathcal{E}(1+k)-\mathcal{E}(k) &= \wp_1\mathcal{E}(k)\mathcal{S}(k)+\wp_2\mathcal{I}(k)\mathcal{S}(k)-(\mho+\xi)\mathcal{E}(k)\\
\mathcal{I}(1+k)-\mathcal{I}(k) &= \xi\mathcal{E}(k)-(\mho++\varsigma)\mathcal{I}(k)\\
\mathcal{R}(1+k)-\mathcal{R}(k) &= \varsigma\mathcal{I}(k)-(\mho+\zeta)\mathcal{R}(k),
\end{aligned} \tag{1.1}$$

where: $\mathcal{S}$ susceptible , $\mathcal{E}$ exposed, $\mathcal{I}$ infected and recovered $\mathcal{R}$. The parameters are shown in table 1.

Numerous deterministic epidemic models have been explored in earlier research. However, SDE models are generally considered more effective for capturing the complexity of biological phenomena, as they incorporate randomness and thus offer a more realistic representation. Unlike deterministic models that yield a single outcome, repeated simulations of stochastic models produce a range of possible scenarios, offering richer and more practical insights into disease dynamics.

Infectious disease spread is inherently stochastic, necessitating the inclusion of environmental noise in models to accurately reflect fluctuations in environmental conditions. The random, uncorrelated oscillations known as "white noise" are used in epidemic modeling to assist account for the uncertainty associated with disease transmission. Deterministic models frequently include it to reflect uncertainties in human behavior and transmission dynamics. This method offers a useful way to include stochasticity, which makes epidemic predictions more realistic. To better understand the stochastic influences driving disease dynamics, several scholars have recently included white noise in their models [12, 13, 16].

This paper is structured as follows: In Section 2, a stochastic epidemic model for COVID-19 is proposed. In Section 3, we demonstrate that a unique positive solution exists, and the **SD** and the extinction of disease are investigated. In Section 4, we simulate the analytical results by using the stochastic 4th order Runge-Kutta (**RK4**) method. In the last section, concluding observations are presented.

2 Model formulation

Motivated by the above and based on the model (1.1), in this section, we develop a stochastic model to understand COVID-19's behavior. The total population $\mathcal{N}(t)$: is constant and is partitioned into the four subpopulations described above, where $\mathcal{R}(t)+\mathcal{I}(t)+\mathcal{E}(t)+\mathcal{S}(t)=\mathcal{N}(t)$, where t is the current time. To be biologically relevant, we assume that no model variables or parameters have negative values. Assuming that the white noise intensity scales with $\mathcal{S},\mathcal{E},\mathcal{I}$ and $\mathcal{R}$. Thus, we obtain

$$\begin{aligned}
d\mathcal{S}(t) &= \Big[\zeta\mathcal{R}(t)+\Xi-\mho\mathcal{S}(t)-\frac{\wp_1\mathcal{E}(t)\mathcal{S}(t)}{\mathcal{N}}-\frac{\wp_2\mathcal{I}(t)\mathcal{S}(t)}{\mathcal{N}}\Big]dt+\varrho_1\mathcal{S}(t)dB_1(t)\\
d\mathcal{E}(t) &= \Big[\frac{\wp_1\mathcal{E}(t)\mathcal{S}(t)}{\mathcal{N}}+\frac{\wp_2\mathcal{I}(t)\mathcal{S}(t)}{\mathcal{N}}-(\mho+\xi)\mathcal{E}(t)\Big]dt+\varrho_2\mathcal{E}(t)dB_2(t)\\
d\mathcal{I}(t) &= \Big[\xi\mathcal{E}(t)-(\mho_0+\mho+\varsigma)\mathcal{I}(t)\Big]dt+\varrho_3\mathcal{I}(t)dB_3(t)\\
d\mathcal{R}(t) &= \Big[-(\mho+\zeta)\mathcal{R}(t)+\varsigma\mathcal{I}(t)\Big]dt+\varrho_4\mathcal{R}(t)dB_4(t),
\end{aligned} \tag{2.1}$$

where $B_\ell(t)\ \ (\ell=1,2,3,4)$, are the independent Brownian motions, $\varrho_\ell\ \ (\ell=1,2,3,4)$, are intensities of the standard Gaussian white noises. In addition, the Brownian motion satisfies: $B_\ell(0)=0$, for $\ell=1,2,3,4$. The following table displays the biological significance of the model's parameters:

Table 1. parameter description.

Notation	Description
Ξ	the birth rate.
ξ	progression rate from $\mathcal{E}$ to $\mathcal{I}$.
$\wp_1$	the passage ratio is from $\mathcal{S}$ to $\mathcal{E}$ as a result of coming into contact with $\mathcal{E}$.
$\wp_2$	rate of passage from $\mathcal{S}$ to $\mathcal{E}$ as a result of coming into contact with $\mathcal{I}$.
$\mho$	natural mortality rate.
$\mho_0$	mortality rate by infection.
ς	recovery rate of infectious individuals.
ζ	rate of infective people recover and return to susceptible class.

If $\varrho_\ell = 0$ for each $\ell = 1, 2, 3, 4$, we obtain the deterministic case of Model (2.1)

$$\begin{aligned}
\mathcal{S}'(t) &= -\mho\mathcal{S}(t) + \Xi + \zeta\mathcal{R}(t) - \wp_1\frac{\mathcal{E}(t)\mathcal{S}(t)}{\mathcal{N}} - \wp_2\frac{\mathcal{I}(t)\mathcal{S}(t)}{\mathcal{N}} \\
\mathcal{E}'(t) &= \wp_1\frac{\mathcal{E}(t)\mathcal{S}(t)}{\mathcal{N}} + \wp_2\frac{\mathcal{I}(t)\mathcal{S}(t)}{\mathcal{N}} - (\mho + \xi)\mathcal{E}(t) \\
\mathcal{I}'(t) &= \xi\mathcal{E}(t) - (\mho + \mho_0 + \varsigma)\mathcal{I}(t) \\
\mathcal{R}'(t) &= -(\mho + \zeta)\mathcal{R}(t) + \varsigma\mathcal{I}(t).
\end{aligned} \tag{2.2}$$

We consider a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, equipped with a filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ that meets the usual conditions, as the foundational setting of this work.

3 Main results

3.1 Global positive solutions: existence and uniqueness results

To explore the dynamics of model (2.1), we start by establishing the existence of a global positive solution.

The solution of model (2.1) may exhibit finite-time explosion, as the coefficients fail to satisfy the linear growth condition, despite meeting the locally Lipschitz condition. To address such cases, Khasminskii and Mao introduced a Lyapunov function approach-a powerful criterion for proving the non-explosion of solutions in the absence of the linear growth condition-commonly known as the Khasminskii-Mao theorem [27, 26].

Theorem 3.1. *Given any initial condition $\mathfrak{S}(0) \in \mathbb{R}^4_+$, there is a unique solution $\mathfrak{S}(t) = (\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{R}(t))$ to System (2.1) on $t \geq 0$. Moreover, the solution is positive $\forall\ t \geq 0$ almost surely $(a.s.)$.*

Proof. Because the coefficients of the system are locally Lipschitz continuous for every starting value $\mathfrak{S}(0)$ in $\mathbb{R}^4_+$, there exists a unique solution $\mathfrak{S}(t)$ on $t \in [0, \varpi_E)$, with ϖ_E representing the explosion time. The globality of the solution is shown by demonstrating that $\varpi_E = \infty$ a.s. Assuming we have a sufficiently big nonnegative integer m such that $\mathcal{S}(0), \mathcal{E}(0), \mathcal{I}(0)$ and $\mathcal{R}(0)$ all fall inside the interval $[\frac{1}{m_0}, m_0]$. For every integer $m \geq m_0$, define the stopping time as:

$$\varpi_m = \begin{cases} \inf \underbrace{\left\{ t \in [0, \varpi_E) : \begin{array}{l} \min\Big\{\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{R}(t)\Big\} \leq \dfrac{1}{m} \text{ Or} \\ \max\Big\{\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{R}(t)\Big\} \geq m \end{array} \right\}}_{\Delta_m}, \\ \infty \qquad \text{if} \qquad \Delta_m = \emptyset, \end{cases}$$

In view of ϖ_m definition, we conclude that ϖ_m is increasing as $m \to \infty$. Set $\varpi_\infty = \lim_{m\to\infty} \varpi_m$, with $\varpi_E \geq \varpi_\infty$ almost surely. Thus, if we present that $\varpi_\infty = \infty$ a.s., then

$\varpi_E = \infty$ and $\mathfrak{S}(t) \in \mathbb{R}^4_+$ $a.s.$ $\forall\, t \geq 0$. Now we need to prove the assertion $\varpi_\infty = \infty$, the proof goes by contradiction. Hence, we suppose that $\varpi_\infty < \infty$, hence, there exist constants $\tau > 0$ and $(0,1) \ni \varepsilon$ satisfying $\varepsilon \leq \mathbb{P}\{\varpi_\infty \leq \tau\}$. In other words, we can find an integer $p \geq m_0$, such that

$$\forall m \geq p \qquad \mathbb{P}\{\varpi_m \leq \tau\} \geq \varepsilon. \tag{3.1}$$

We now introduce a $\mathcal{C}^2$-function, $W : \mathbb{R}^4_+ \to \mathbb{R}_+$:

$$W(\mathfrak{S}) = \mathcal{R} + \mathcal{I} + \mathcal{E} + \mathcal{S} - 4 - (\ln \mathcal{R} + \ln \mathcal{I} + \ln \mathcal{E} + \ln \mathcal{S}).$$

Using the function $F(\kappa) = \kappa - (1 + \ln \kappa) \geq 0,\ \forall \kappa > 0$, then the function W is non-negative.Itô's formula is applied to yield

$$\begin{aligned}
dW(\mathfrak{S}) =& \frac{1}{2\mathcal{S}^2}(d\mathcal{S})^2 + \left(-\frac{1}{\mathcal{S}} + 1\right)d\mathcal{S} + \left(-\frac{1}{\mathcal{E}} + 1\right)d\mathcal{E} + \frac{1}{2\mathcal{E}^2}(d\mathcal{E})^2 \\
&+ \frac{1}{2\mathcal{I}^2}(d\mathcal{I})^2 + \left(-\frac{1}{\mathcal{I}} + 1\right)d\mathcal{I} + \left(-\frac{1}{\mathcal{R}} + 1\right)d\mathcal{R} + \frac{1}{2\mathcal{R}^2}(d\mathcal{R})^2 \\
=& \left(1 - \frac{1}{\mathcal{S}}\right)\left([\Xi - \wp_1 \frac{\mathcal{E}(t)\mathcal{S}(t)}{\mathcal{N}} - \wp_2 \frac{\mathcal{I}(t)\mathcal{S}(t)}{\mathcal{N}} + \zeta\mathcal{R}(t) - \mho\mathcal{S}(t)]dt + \varrho_1\mathcal{S}(t)dB_1(t)\right) \\
&+ \frac{1}{2\mathcal{S}^2}\left([\Xi - \wp_1 \frac{\mathcal{E}(t)\mathcal{S}(t)}{\mathcal{N}} - \wp_2 \frac{\mathcal{I}(t)\mathcal{S}(t)}{\mathcal{N}} + \zeta\mathcal{R}(t) - \mho\mathcal{S}(t)]dt + \varrho_1\mathcal{S}(t)dB_1(t)\right)^2 \\
&+ \left(1 - \frac{1}{\mathcal{E}}\right)\left([\wp_1 \frac{\mathcal{S}(t)\mathcal{E}(t)}{\mathcal{N}} + \wp_2 \frac{\mathcal{S}(t)\mathcal{I}(t)}{\mathcal{N}} - (\mho + \xi)\mathcal{E}(t)]dt + \varrho_2\mathcal{E}(t)dB_2(t)\right) \\
&+ \frac{1}{2\mathcal{E}^2}\left([\wp_1 \frac{\mathcal{E}(t)\mathcal{S}(t)}{\mathcal{N}} + \wp_2 \frac{\mathcal{I}(t)\mathcal{S}(t)}{\mathcal{N}} - (\mho + \xi)\mathcal{E}(t)]dt + \varrho_2\mathcal{E}(t)dB_2(t)\right)^2 \\
&+ \left(1 - \frac{1}{\mathcal{I}}\right)\left([\xi\mathcal{E}(t) - (\mho_0 + \mho + \varsigma)\mathcal{I}(t)]dt + \varrho_3\mathcal{I}(t)dB_3(t)\right) \\
&+ \frac{1}{2\mathcal{I}^2}\left([\xi\mathcal{E}(t) - (\varsigma + \mho + \mho_0)\mathcal{I}(t)]dt + \varrho_3\mathcal{I}(t)dB_3(t)\right)^2 \\
&+ \left(1 - \frac{1}{\mathcal{R}}\right)\left([\varsigma\mathcal{I}(t) - (\mho + \zeta)\mathcal{R}(t)]dt + \varrho_4\mathcal{R}(t)dB_3(t)\right) \\
&+ \frac{1}{2\mathcal{R}^2}\left([\varsigma\mathcal{I}(t) - (\mho + \zeta)\mathcal{R}(t)]dt + \varrho_4\mathcal{R}(t)dB_4(t)\right)^2 \\
=& LW(\mathfrak{S})dt + \varrho_1(-1 + \mathcal{S}(t))dB_1(t) + \varrho_2(-1 + \mathcal{E}(t))dB_2(t) + \varrho_3(-1 + \mathcal{I}(t))dB_3(t) \\
&+ \varrho_4(-1 + \mathcal{R}(t))dB_4(t).
\end{aligned}$$

Here $LW :\ \mathbb{R}^4_+ \to \mathbb{R}_+$ obtained according to the following definition

$$\begin{aligned}
LW(\mathfrak{S}) =& \Xi - \mho(\mathcal{E} + \mathcal{S} + \mathcal{R} + \mathcal{I}) - \mho_0\mathcal{I} - \frac{\Xi}{\mathcal{S}} + \wp_1\frac{\mathcal{E}}{\mathcal{N}} + \wp_2\frac{\mathcal{I}}{\mathcal{N}} - \zeta\frac{\mathcal{R}}{\mathcal{S}} - \wp_1\frac{\mathcal{S}}{\mathcal{N}} - \wp_2\frac{\mathcal{S}\mathcal{I}}{\mathcal{E}\mathcal{N}} \\
&- \xi\frac{\mathcal{E}}{\mathcal{I}} - \varsigma\frac{\mathcal{I}}{\mathcal{R}} + 4\mho + \zeta + \varsigma + \mho_0 + \xi + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_3^2 + \varrho_4^2}{2} \\
\leq& \Xi + \wp_1 + \wp_2 + \xi + 4\mho + \zeta + \mho_0 + \varsigma + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_3^2 + \varrho_4^2}{2} :== M.
\end{aligned}$$

Here, M is a constant that does not depend on $\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{R}$ or t. Therefore,

$$dW(\mathfrak{S}) \leq Mdt + \varrho_1(-1+\mathcal{S})dB_1(t) + \varrho_2(-1+\mathcal{E})dB_2(t) + \varrho_3(-1+\mathcal{I})dB_3(t) + \varrho_4(-1+\mathcal{I})dB_4(t).$$

The integration from 0 to $\varpi_m \wedge \tau = \min\{\varpi_m, \tau\}$ yields

$$\begin{aligned}
\int_0^{\varpi_m\wedge\tau} dW(\mathfrak{S}) \leq& \int_0^{\varpi_m\wedge\tau} Mdr + \int_0^{\varpi_m\wedge\tau} \varrho_1(\mathcal{S}(r) - 1)dB_1(r) + \int_0^{\varpi_m\wedge\tau} \varrho_2(\mathcal{E}(r) - 1)dB_2(r) \\
&+ \int_0^{\varpi_m\wedge\tau} \varrho_3(\mathcal{I}(r) - 1)dB_3(r) + \int_0^{\varpi_m\wedge\tau} \varrho_4(\mathcal{R}(r) - 1)dB_4(r).
\end{aligned}$$

Taking expectation, we obtain

$$\mathbb{E}\Big[W\big(\mathfrak{S}(\varpi_m \wedge \tau)\big)\Big] \leq W\big(\mathfrak{S}(0)\big) + \mathbb{E}\Big[\int_0^{\varpi_m \wedge \tau} M dr\Big],$$

that is

$$\mathbb{E}\Big[W\big(\mathfrak{S}(\varpi_m \wedge \tau)\big)\Big] \leq W\big(\mathfrak{S}(0)\big) + MT.$$

Let $\Omega_m = \{\varpi_m \leq \tau\}\ \forall\ m \geq k_1$, and by (3.1) then $\mathbb{P}(\Omega_m) \geq \varepsilon$. For every ω from Ω_m, there exists one or more than one $\mathcal{S}(\varpi_m, \omega), \mathcal{E}(\varpi_m, \omega), \mathcal{I}(\varpi_m, \omega)$ and $\mathcal{R}(\varpi_m \wedge \tau)$, that is equal to m or $\frac{1}{m}$. As a result,

$$W\big(\mathfrak{S}(\varpi_m, \omega)\big) \geq F(m) \wedge F(\frac{1}{m}),$$

therefore,

$$\begin{aligned} W\big(\mathfrak{S}(0)\big) + MT &\geq \mathbb{E}\Big[\mathbf{1}_{\Omega_m}(\omega) W\big(\mathfrak{S}(\varpi_m \wedge \tau)\big)\Big] \\ &= \mathbb{E}\Big[\mathbf{1}_{\Omega_m}(\omega) W\big(\mathfrak{S}(\varpi_m, \omega)\big)\Big] \\ &\geq \mathbb{E}\Big[\mathbf{1}_{\Omega_m}(\omega) F(m) \wedge F(\frac{1}{m})\Big] \\ &\geq \mathbb{E}\Big[\mathbf{1}_{\Omega_m}(\omega)\Big] F(m) \wedge F(\frac{1}{m}) \\ &\geq \varepsilon F(m) \wedge F(\frac{1}{m}). \end{aligned}$$

Here, $\mathbf{1}_{\Omega_m}(\omega)$ denotes the indicator function of Ω_m. By letting $m \to \infty$, we obtain

$$\infty > MT + W\big(\mathfrak{S}(0)\big) = \infty,$$

that is impossible. Then, $\varpi_\infty = \infty\ a.s.$ □

3.2 Extinction of the disease

In this subsection, we investigate the conditions for the extinction of the disease, which depend on intensities of white noise and the system parameters. In order to keep things simple, we will use the notation:

$$\langle \Theta(t) \rangle = \frac{1}{t} \int_0^t \Theta(s) ds.$$

Lemma 3.2. *[12] Let $\mathfrak{X} = \{\mathfrak{X}_t\}_{t \geq 0}$, be a continuous local martingale with real values such that $\mathfrak{X}(0) = 0$, then*

$$\lim_{t \to \infty} \langle \mathfrak{X}, \mathfrak{X} \rangle_t = \infty\ \ a.s.\ \Rightarrow 0 = \lim_{t \to \infty} \frac{\mathfrak{X}_t}{\langle \mathfrak{X}, \mathfrak{X} \rangle_t}\ a.s.$$

and

$$\lim_{t \to \infty} \sup t^{-1} \langle \mathfrak{X}, \mathfrak{X} \rangle_t < \infty\ \ a.s.\ \Rightarrow 0 = \lim_{t \to \infty} t^{-1} \mathfrak{X}_t.$$

Lemma 3.3. *Consider that $\mathfrak{S}(t)$, is a solution to Model (2.1) for any initial value, then*

$$\lim_{t \to \infty} t^{-1} \mathcal{S}(t) = 0,\ \ \lim_{t \to \infty} t^{-1} \mathcal{E}(t) = 0,\ \ \lim_{t \to \infty} t^{-1} \mathcal{I}(t) = 0,\ \ \lim_{t \to \infty} t^{-1} \mathcal{R}(t) = 0.\quad a.s. \tag{3.2}$$

Proof. System (2.1) allows us to write

$$\begin{aligned} d\mathcal{N}(t) &= -\mho \mathcal{N}(t) + \Xi - \mho_0 \mathcal{I}(t) + \varrho_1 \mathcal{S}(t) dB_1(t) + \varrho_2 \mathcal{E}(t) dB_2(t) \\ &\quad + \varrho_3 \mathcal{I}(t) dB_3(t) + \varrho_4 \mathcal{R}(t) dB_4(t) \\ &\leq -\mho \mathcal{N}(t) + \Xi + \varrho_1 \mathcal{S}(t) dB_1(t) + \varrho_2 \mathcal{E}(t) dB_2(t) + \varrho_3 \mathcal{I}(t) dB_3(t) + \varrho_4 \mathcal{R}(t) dB_4(t). \end{aligned}$$

By integration we have

$$\begin{aligned}\mathcal{N}(t) \leq & (\mathcal{N}(0) - \frac{\Xi}{\mho})e^{-\mho t} + \frac{\Xi}{\mho} + \varrho_1 \int_0^t \mathcal{S}(r)e^{-\mho(t-r)}dB_1(r) + \varrho_2 \int_0^t \mathcal{E}(r)e^{-\mho(t-r)}dB_2(r) \\ & + \varrho_3 \int_0^t \mathcal{I}(r)e^{-\mho(t-r)}dB_3(r) + \varrho_4 \int_0^t \mathcal{R}(r)e^{-\mho(t-r)}dB_4(r) \\ \leq & \frac{\Xi}{\mho}(1 - e^{-\mho t}) + \mathcal{N}(0) + X(t) - \Big(\mathcal{N}(0)(1 - e^{-\mho t})\Big) \quad \forall t \geq 0,\end{aligned}$$

with

$$\begin{aligned}X(t) = & \varrho_1 \int_0^t \mathcal{S}(r)e^{-\mho(t-r)}dB_1(r) + \varrho_2 \int_0^t \mathcal{E}(r)e^{-\mho(t-r)}dB_2(r) \\ & + \varrho_3 \int_0^t \mathcal{I}(r)e^{-\mho(-r+t)}dB_3(r) + \varrho_4 \int_0^t \mathcal{R}(r)e^{-\mho(t-r)}dB_4(r),\end{aligned}$$

is a continuous local martingale process, and $X(0) = 0$.
Therefore,

$$\lim_{t\to\infty}\left[\mathcal{N}(0) + \frac{\Xi}{\mho}(1 - e^{-\mho t}) + X(t) - \Big(\mathcal{N}(0)(1 - e^{-\mho t})\Big)\right] < \infty \text{ a.s.}$$

Hence,

$$\limsup_{t\to\infty} \mathcal{N}(t) < \infty.\ a.s. \tag{3.3}$$

It can be concluded that (3.2) holds. □

Theorem 3.4. *Suppose that $\mathfrak{S}(t)$, is a solution of model (2.1) with $\mathfrak{S}(0) \in \mathbb{R}^4_+$. The following conditions are imposed:*

$$a)\ \frac{\mho\wp_1 + \Xi\wp_2}{\mho(\mho + \xi + \frac{\varrho_2^2}{2})} < 1, \textit{and} \qquad b)\ \frac{\xi\Xi}{\mho(\mho + \mho_0 + \varsigma + \frac{\varrho_3^2}{2})} < 1.$$

Then, the following properties hold

$$\lim_{t\to\infty} \sup \frac{\ln \mathcal{E}(t)}{t} < 0, \textit{and} \qquad \lim_{t\to\infty} \sup \frac{\ln \mathcal{I}(t)}{t} < 0 \qquad a.s.$$

namely, $\mathcal{E}(t), \mathcal{I}(t) \longrightarrow 0$ exponentially a.s. Equivalently, the disease will vanish with probability one. Moreover,

$$\lim_{t\to\infty} \langle \mathcal{S}(t) \rangle = \frac{\Xi}{\mho}, \quad \lim_{t\to\infty} \langle \mathcal{R}(t) \rangle = 0.$$

Proof. System (2.1) is integrated over $[0; t]$ and division by t gives:

$$\begin{aligned}\frac{-\mathcal{S}(0) + \mathcal{S}(t)}{t} &= \frac{\varrho_1}{t}\int_0^t \mathcal{S}(r)dB_1(r) + \Xi - \wp_1\langle\frac{\mathcal{E}(t)\mathcal{S}(t)}{\mathcal{N}}\rangle - \wp_2\langle\frac{\mathcal{I}(t)\mathcal{S}(t)}{\mathcal{N}}\rangle \\ &\quad - \mho\langle\mathcal{S}(t)\rangle + \varsigma\langle\mathcal{R}(t)\rangle \\ \frac{-\mathcal{E}(0) + \mathcal{E}(t)}{t} &= \frac{\varrho_2}{t}\int_0^t \mathcal{E}(r)dB_2(r) + \wp_1\langle\frac{\mathcal{E}(t)\mathcal{S}(t)}{\mathcal{N}}\rangle + \wp_2\langle\frac{\mathcal{I}(t)\mathcal{S}(t)}{\mathcal{N}}\rangle - (\mho + \xi)\langle\mathcal{E}(t)\rangle \\ \frac{-\mathcal{I}(0) + \mathcal{I}(t)}{t} &= \frac{\varrho_3}{t}\int_0^t \mathcal{I}(r)dB_3(r) - (\mho + \mho_0 + \varsigma)\langle\mathcal{I}(t)\rangle + \xi\langle\mathcal{E}(t)\rangle \\ \frac{-\mathcal{R}(0) + \mathcal{R}(t)}{t} &= \frac{\varrho_4}{t}\int_0^t \mathcal{R}(r)dB_4(r) + \varsigma\langle\mathcal{I}(t)\rangle - (\mho + \zeta)\langle\mathcal{R}(t)\rangle.\end{aligned} \tag{3.4}$$

Adding the above four equations, gives

$$\frac{\mathcal{S}(t)-\mathcal{R}(0)}{t}+\frac{\mathcal{I}(t)-\mathcal{I}(0)}{t}+\frac{\mathcal{R}(t)-\mathcal{S}(0)}{t}+\frac{\mathcal{E}(t)-\mathcal{E}(0)}{t}=\Xi-\mho_0\langle\mathcal{I}(t)\rangle$$
$$-\mho(\langle\mathcal{S}(t)\rangle+\langle\mathcal{R}(t)\rangle+\langle\mathcal{I}(t)\rangle+\langle\mathcal{E}(t)\rangle)+\frac{\varrho_1}{t}\int_0^t\mathcal{S}(r)dB_1(r)+\frac{\varrho_2}{t}\int_0^t\mathcal{E}(r)dB_2(r)$$
$$+\frac{\varrho_3}{t}\int_0^t\mathcal{I}(r)dB_3(r)+\frac{\varrho_4}{t}\int_0^t\mathcal{R}(r)dB_4(r).$$

We compute that

$$\langle\mathcal{S}(t)\rangle=-\langle\mathcal{R}(t)\rangle+\frac{\Xi}{\mho}-\langle\mathcal{E}(t)\rangle+\Phi(t)-\frac{(\mho+\mho_0)}{\mho}\langle\mathcal{I}(t)\rangle, \tag{3.5}$$

and

$$\langle\mathcal{E}(t)\rangle=\frac{\Xi}{\mho}-\langle\mathcal{S}(t)\rangle-\langle\mathcal{R}(t)\rangle+\Phi(t)-\frac{(\mho_0+\mho)}{\mho}\langle\mathcal{I}(t)\rangle, \tag{3.6}$$

where Φ function is defined by

$$\Phi(t)=\frac{1}{\mho}\Big[\frac{\mathcal{R}(0)-\mathcal{R}(t)}{t}+\frac{\mathcal{I}(0)-\mathcal{E}(t)}{t}+\frac{\mathcal{E}(0)-\mathcal{I}(t)}{t}+\frac{\mathcal{S}(0)-\mathcal{S}(t)}{t}+\frac{\varrho_1}{t}\int_0^t\mathcal{S}(r)dB_1(r)$$
$$+\frac{\varrho_2}{t}\int_0^t\mathcal{E}(r)dB_2(r)+\frac{\varrho_3}{t}\int_0^t\mathcal{I}(r)dB_3(r)+\frac{\eta_4}{t}\int_0^t\mathcal{R}(r)dB_4(r)\Big].$$

Setting:

$$M_1=\varrho_1\int_0^t\mathcal{S}(r)dB_1(r),\qquad M_2=\varrho_2\int_0^t\mathcal{E}(r)dB_2(r),\qquad M_3=\varrho_3\int_0^t\mathcal{I}(r)dB_3(r),$$
$$M_4=\varrho_4\int_0^t\mathcal{R}(r)dB_4(r),$$

are continuous local martingales whose quadratic variations are

$$\langle M_1,M_1\rangle=\varrho_1^2\int_0^t\mathcal{S}^2(r)dr\le\Big(\varrho_1^2\sup_{t\ge0}\mathcal{S}^2(t)\Big)t,$$
$$\langle M_2,M_2\rangle=\varrho_2^2\int_0^t\mathcal{E}^2(r)dr\le\Big(\varrho_2^2\sup_{t\ge0}\mathcal{E}^2(t)\Big)t,$$
$$\langle M_3,M_3\rangle=\varrho_3^2\int_0^t\mathcal{I}^2(r)dr\le\Big(\varrho_3^2\sup_{t\ge0}\mathcal{I}^2(t)\Big)t,$$
$$\langle M_4,M_4\rangle=\varrho_4^2\int_0^t\mathcal{R}^2(r)dr\le\Big(\varrho_4^2\sup_{t\ge0}\mathcal{R}^2(t)\Big)t.$$

By Lemma (3.2), we obtain

$$\lim_{t\to\infty}\frac{M_j(t)}{t}=0\ a.s,\quad(j=1,2,3,4).$$

Obviously, $\lim_{t\to\infty}\Phi(t)=0$ a.s.

Itô's formula is applied to the 2^{nd} equation of Model (2.1), resulting in

$$d\ln\mathcal{E}(t)=\Big[\wp_1\frac{\mathcal{S}(t)}{\mathcal{N}}+\wp_2\frac{\mathcal{S}(t)\mathcal{I}(t)}{\mathcal{E}(t)\mathcal{N}}-(\mho+\xi+\frac{\varrho_2^2}{2})\Big]dt+\varrho_2dB_2(t),\ \text{for}\ \mathcal{E}\neq0. \tag{3.7}$$

Integrating (3.7) over $[0,t]$, and division by t implies that

$$\frac{\ln\mathcal{E}(t)}{t}=\wp_1\langle\frac{\mathcal{S}(t)}{\mathcal{N}}\rangle+\wp_2\langle\frac{\mathcal{I}(t)\mathcal{S}(t)}{\mathcal{E}(t)\mathcal{N}}\rangle-(\mho+\xi+\frac{\varrho_2^2}{2})+\frac{\varrho_2}{t}\int_0^t dB_2(r)+\frac{\ln\mathcal{E}(0)}{t}$$
$$\le-(\mho+\xi+\frac{\varrho_2^2}{2})+\wp_1+\wp_2\langle\mathcal{S}(t)\rangle+\frac{\varrho_2}{t}\int_0^t dB_2(r)+\frac{\ln\mathcal{E}(0)}{t}. \tag{3.8}$$

Substituting (3.5) in (3.8) yields

$$\frac{\ln \mathcal{E}(t)}{t} \leq \wp_1 + \frac{\wp_2 \Xi}{\mho} - \wp_2\langle \mathcal{E}(t)\rangle - \wp_2\langle \mathcal{R}(t)\rangle - \frac{\wp_2(\mho_0+\mho)}{\mho}\langle \mathcal{I}(t)\rangle + \wp_2\Phi(t)$$
$$- (\mho + \xi + \frac{\varrho_2^2}{2}) + \frac{\varrho_2}{t}\int_0^{\iota} dB_2(r) + \frac{\ln \mathcal{E}(0)}{t}.$$

Then

$$\frac{\ln \mathcal{E}(t)}{t} \leq (\mho + \xi + \frac{\varrho_2^2}{2})\Big(\frac{\mho\wp_1 + \wp_2\Xi}{\mho(\mho + \xi + \frac{\varrho_2^2}{2})} - 1\Big) - \wp_2\langle \mathcal{E}(t)\rangle - \wp_2\langle \mathcal{R}(t)\rangle$$
$$- \frac{\wp_2(\mho_0+\mho)}{\mho}\langle \mathcal{I}(t)\rangle + \wp_2\Phi(t) + \frac{\varrho_2}{t}\int_0^t dB_2(r) + \frac{\ln \mathcal{E}(0)}{t}$$
$$\leq (\mho + \xi + \frac{\varrho_2^2}{2})\Big(\frac{\mho\wp_1 + \wp_2\Xi}{\mho(\mho + \xi + \frac{\varrho_2^2}{2})} - 1\Big) + \wp_2\Phi(t) + \frac{\varrho_2}{t}\int_0^t dB_2(r) + \frac{\ln \mathcal{E}(0)}{t}.$$

When Condition (a) holds, then

$$\lim_{t\to\infty} \sup \frac{\ln \mathcal{E}(t)}{t} \leq (\mho + \xi + \frac{\varrho_2^2}{2})\Big(\frac{\mho\wp_1 + \wp_2\Xi}{\mho(\mho + \xi + \frac{\varrho_2^2}{2})} - 1\Big) < 0,$$

thus, $\lim_{t\to\infty} \mathcal{E}(t) = 0$ and $\lim_{t\to\infty}\langle \mathcal{E}(t)\rangle = 0 \quad$ a.s.
Similarly, Itô's formula is applied to the 3^{rd} equation of Model (2.1), followed by integration over $[0;t]$ and division by t , yielding

$$\frac{\ln \mathcal{I}(t)}{t} = \xi\langle\frac{\mathcal{E}(t)}{\mathcal{I}(t)}\rangle - (\mho + \mho_0 + \varsigma + \frac{\varrho_3^2}{2}) + \frac{\varrho_3}{t}\int_0^t dB_3(r) + \frac{\ln \mathcal{I}(0)}{t}$$
$$\leq \xi\langle \mathcal{E}(t)\rangle - (\mho + \mho_0 + \varsigma + \frac{\varrho_3^2}{2}) + \frac{\varrho_3}{t}\int_0^t dB_3(r) + \frac{\ln \mathcal{I}(0)}{t}. \tag{3.9}$$

Substituting (3.6) in (3.9) yields

$$\frac{\ln \mathcal{I}(t)}{t} \leq (\mho + \mho_0 + \varsigma + \frac{\varrho_3^2}{2})\Big(\frac{\xi\Xi}{\mho(\mho + \mho_0 + \varsigma + \frac{\varrho_3^2}{2})} - 1\Big) - \xi\Big(\langle \mathcal{R}(t)\rangle + \langle \mathcal{S}(t)\rangle + \frac{\mho + \mho_0}{\mho}\langle \mathcal{I}(t)\rangle\Big)$$
$$+\xi\Phi(t) + \frac{\varrho_3}{t}\int_0^t dB_3(r) + \frac{\ln \mathcal{I}(0)}{t}$$
$$\leq (\mho + \mho_0 + \varsigma + \frac{\varrho_3^2}{2})\Big(\frac{\xi\Xi}{\mho(\mho + \mho_0 + \varsigma + \frac{\varrho_3^2}{2})} - 1\Big) + \xi\Phi(t) + \frac{\varrho_3}{t}\int_0^t dB_3(r) + \frac{\ln \mathcal{I}(0)}{t}.$$

Using hypothesis (b) and taking the limit superior, imply that

$$\lim_{t\to\infty} \sup \frac{\ln \mathcal{I}(t)}{t} \leq (\mho + \mho_0 + \varsigma + \frac{\varrho_3^2}{2})\Big(\frac{\xi\Xi}{\mho(\mho + \mho_0 + \varsigma + \frac{\varrho_3^2}{2})} - 1\Big) < 1 \quad a.s.$$

Hence, $\lim_{t\to\infty} \mathcal{I}(t) = 0 \quad a.s.$ and $\lim_{t\to\infty}\langle \mathcal{I}(t)\rangle = 0 \quad a.s.$
From the last equation in (3.4), we find

$$\langle \mathcal{R}(t)\rangle = \frac{1}{(\mho + \zeta)}\left[\varsigma\langle \mathcal{I}(t)\rangle + \frac{\mathcal{R}(0) - \mathcal{R}(t)}{t} + \frac{\varrho_4}{t}\int_0^t \mathcal{R}(r)dB_4(r)\right].$$

Hence, $\lim_{t\to\infty}\langle \mathcal{R}(t)\rangle = 0 \ a.s.$
Taking the limit of equation (3.5), and from the above discussion, we obtain

$$\lim_{t\to\infty}\langle \mathcal{S}(t)\rangle = \frac{\Xi}{\mho} \text{ a.s.}$$

□

3.3 Stationary distribution and ergodicity

In epidemiology, determining the conditions under which a disease persists within a population is of critical importance. In this subsection, we demonstrate that, according to the theory of Has'minskii [1], there exists an ergodic **SD** of Model (2.1), which, in the biology sense, reveals that the infection will not die out.

Let $Y(t)$ be a homogeneous Markov process in $\mathbb{R}_+^N$, defined by

$$dY(t) = \sum_{r=1}^{N} \sigma_r(Y)dB_r(t) + b(Y)dt.$$

The following defines the diffusion matrix.

$$A(x) = \Big[\mathfrak{a}_{ij}(x)\Big]; \quad \mathfrak{a}_{ij}(x) = \sum_{r=1}^{N} \sigma_r^i(x)\sigma_r^j(y).$$

Lemma 3.5. *The **SD** of the Markov process $X(t)$ is unique if there exists a bounded open domain $U \subset \mathbb{R}_+^N$ with a regular border and the following characteristics:*

- *a) The lowest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero in the domain U and some neighborhood thereof.*
- *b) If $x \in \mathbb{R}_+^N \setminus U$, the mean time ϖ for a route from x to reach the set U is finite, and $\sup_{x\in m} \mathbb{E}^x \varpi$ for each compact subset $m \subset \mathbb{R}^N$. Additionally, if $h(.)$is an integrable function with respect to the measure μ, then the statement is true. Then,*

$$1 = \mathbb{P}\left\{ \lim_{\tau\to\infty} \tau^{-1} \int_0^\tau h(X_x(t))dt = \int_{\mathbb{R}_+^N} h(x)\mu(dx) \right\} \qquad \forall x \in \mathbb{R}_+^N.$$

Remark 3.6. [24]

- Condition (a) can be validated by proving there is a constant $\mathbf{m} \geq 0$ such that

$$\sum_{i,j=1}^{N} \mathfrak{a}_{ij}\gamma_i\gamma_j \geq \mathbf{m}|\gamma|^2 \; x \in U, \; \gamma \in \mathbb{R}_+^N.$$

- To verify (b), it is enough to show the existence of a nonnegative $\mathcal{C}^2$-function $\mathbb{V}$ and a neighborhood U such that, $L\mathbb{V} < 0$ for every $\mathbb{R}_+^N \setminus U$.

Theorem 3.7. *Let us consider the system (2.1) with $\mathfrak{S}(0)$ in $\mathbb{R}_+^4$. Assume that*

$$R_0^{\mathcal{S}} := \frac{\mho\wp_2\xi}{(\mho + \frac{\varrho_1^2}{2})(\mho + \xi + \frac{\varrho_2^2}{2})(\mho_0 + \mho + \varsigma + \frac{\varrho_3^2}{2})} > 1.$$

*Hence, system (2.1) admits a unique **SD** $\mu(.)$ and is ergodic.*

Proof. System (2.1) has the following diffusion matrix:

$$\begin{pmatrix} \varrho_1^2\mathcal{S}^2 & 0 & 0 & 0 \\ 0 & \varrho_2^2\mathcal{E}^2 & 0 & 0 \\ 0 & 0 & \varrho_3^2\mathcal{I}^2 & 0 \\ 0 & 0 & 0 & \varrho_4^2\mathcal{R}^2 \end{pmatrix},$$

and we have

$$\sum_{i,j=1}^{4} a_{ij}(\mathfrak{S})\gamma_i\gamma_j = \varrho_1^2\mathcal{S}^2\gamma_1^2 + \varrho_2^2\mathcal{E}^2\gamma_2^2 + \varrho_3^2\mathcal{I}^2\gamma_3^2 + \varrho_4^2\mathcal{R}^2\gamma_4^2$$

$$\geq \mathbf{m}|\gamma|^2, \;\; \forall\, \mathfrak{S} \in U, \; \gamma \in \mathbb{R}_+^4,$$

here, $\mathbf{m} = \min\{\varrho_1^2\mathcal{S}^2, \varrho_2^2\mathcal{E}^2, \varrho_3^2\mathcal{I}^2, \varrho_4^2\mathcal{R}^2\}$. Hence, Condition (a) is verified.
We proceed to analyze Condition (b), we will construct a non-negative $\mathcal{C}^2$-function $\mathbb{V}$ that maps for any $\mathfrak{S} = (\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{R}) \in \mathbb{R}^4_+$. We define

$$W_1(\mathfrak{S}) = -w_1 \ln \mathcal{S} + \mathcal{S} - w_2 \ln \mathcal{E} + \mathcal{E} - w_3 \ln \mathcal{I} + \mathcal{I} + \mathcal{R}.$$

Using Itô formula, we derive

$$\begin{aligned}
L(\mathcal{R} + \mathcal{S} + \mathcal{E} + \mathcal{I}) &= \Xi - \mho(\mathcal{R} + \mathcal{I} + \mathcal{E} + \mathcal{S}) - \mho_0\mathcal{I},\\
L(-\ln \mathcal{S}) &= -\frac{\Xi}{\mathcal{S}} + \frac{\wp_1\mathcal{E}}{\mathcal{N}} + \frac{\wp_2\mathcal{I}}{\mathcal{N}} - \frac{\zeta\mathcal{R}}{\mathcal{S}} + \frac{\varrho_1^2}{2} + \mho,\\
L(-\ln \mathcal{E}) &= -\frac{\wp_2\mathcal{S}\mathcal{I}}{\mathcal{N}\mathcal{E}} - \frac{\wp_1\mathcal{S}}{\mathcal{N}} + \mho + \xi + \frac{\varrho_2^2}{2},\\
L(-\ln \mathcal{I}) &= -\frac{\xi\mathcal{E}}{\mathcal{I}} + \mho + \mho_0 + \varsigma + \frac{\varrho_3^2}{2}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
LW_1 =& \Xi - \mho(\mathcal{S} + \mathcal{E} + \mathcal{I} + \mathcal{R}) - \mho_0\mathcal{I} - \frac{w_1\Xi}{\mathcal{S}} + \frac{w_1\wp_1\mathcal{E}}{\mathcal{N}} + \frac{w_1\wp_2\mathcal{I}}{\mathcal{N}} - \frac{w_1\zeta\mathcal{R}}{\mathcal{S}} + w_1(\mho + \frac{\varrho_1^2}{2})\\
&- \frac{w_2\wp_1\mathcal{S}}{\mathcal{N}} - \frac{w_2\wp_2\mathcal{S}\mathcal{I}}{\mathcal{N}\mathcal{E}} + w_2(\mho + \xi + \frac{\varrho_2^2}{2}) - \frac{w_3\xi\mathcal{E}}{\mathcal{I}} + w_3(\mho + \mho_0 + \varsigma + \frac{\varrho_3^2}{2}).
\end{aligned}$$

Using the fact

$$\sqrt[4]{sruv} \le \frac{1}{4}(s + r + u + v), \qquad s, r, u, v > 0,$$

we obtain

$$\begin{aligned}
LW_1 \le& -4\sqrt[4]{\mho(\mathcal{S} + \mathcal{E} + \mathcal{I} + \mathcal{R})\frac{w_1\Xi}{\mathcal{S}}\frac{w_2\wp_2 SI}{EN}\frac{w_3\xi\mathcal{E}}{\mathcal{I}}} + \Xi - \mho_0\mathcal{I} + \frac{w_1\wp_1\mathcal{E}}{\mathcal{N}} + \frac{w_2\wp_2\mathcal{I}}{\mathcal{N}} - \frac{w_1\zeta\mathcal{R}}{\mathcal{S}}\\
&+ w_1(\mho + \frac{\varrho_1^2}{2}) - \frac{w_2\wp_1\mathcal{S}}{\mathcal{N}} + w_2(\mho + \xi + \frac{\varrho_2^2}{2}) + w_3(\mho_0 + \mho + \varsigma + \frac{\varrho_3^2}{2})\\
=& -4\sqrt[4]{\mho w_1\Xi w_2\wp_2 w_3\xi} + \Xi - \mho_0\mathcal{I} + \frac{w_1\wp_1\mathcal{E}}{\mathcal{N}} + \frac{w_2\wp_2\mathcal{I}}{\mathcal{N}} - \frac{w_1\zeta\mathcal{R}}{\mathcal{S}} + w_1(\mho + \frac{\varrho_1^2}{2}) - \frac{w_2\wp_1\mathcal{S}}{\mathcal{N}}\\
&+ w_2(\mho + \xi + \frac{\varrho_2^2}{2}) + w_3(\mho_0 + \mho + \varsigma + \frac{\varrho_3^2}{2}).
\end{aligned}$$

Let us put

$$\Xi = w_1(\mho + \frac{\varrho_1^2}{2}) = w_2(\mho + \xi + \frac{\varrho_2^2}{2}) = w_3(\mho_0 + \mho + \varsigma + \frac{\varrho_3^2}{2}),$$

and from the above equation we have

$$w_1 = \frac{\Xi}{(\mho + \frac{\varrho_1^2}{2})}, \quad w_2 = \frac{\Xi}{(\mho + \xi + \frac{\varrho_2^2}{2})}, \quad w_3 = \frac{\Xi}{(\mho_0 + \mho + \varsigma + \frac{\varrho_3^2}{2})}.$$

As a result,

$$\begin{aligned}
LW_1 \le& -4\sqrt[4]{\frac{\mho\wp_2\Xi^4\xi}{(\mho + \frac{\varrho_1^2}{2})(\mho + \xi + \frac{\varrho_2^2}{2})(\mho_0 + \mho + \varsigma + \frac{\varrho_3^2}{2})}} + 4\Xi - \mho_0\mathcal{I} + \frac{w_1\wp_1\mathcal{E} + w_2\wp_2\mathcal{I}}{\mathcal{N}}\\
&- \frac{w_1\zeta\mathcal{R}}{\mathcal{S}} - \frac{w_2\wp_1\mathcal{S}}{\mathcal{N}}\\
\le& -4\Xi\Big(\sqrt[4]{\frac{\mho\wp_2\xi}{(\mho + \frac{\varrho_1^2}{2})(\mho + \xi + \frac{\varrho_2^2}{2})(\mho_0 + \mho + \varsigma + \frac{\varrho_3^2}{2})}} - 1\Big) + \frac{w_1\wp_1\mathcal{E} + w_2\wp_2\mathcal{I}}{\mathcal{N}} - \frac{w_2\wp_1\mathcal{S}}{\mathcal{N}}\\
=& -4\Xi\big[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}} - 1\big] + \frac{w_1\wp_1\mathcal{E} + w_2\wp_2\mathcal{I}}{\mathcal{N}} - \frac{w_2\wp_1\mathcal{S}}{\mathcal{N}}.
\end{aligned}$$

We define

$$W_2(\mathfrak{S}) = w_4\big(-w_1\ln\mathcal{S}+\mathcal{E}-w_2\ln\mathcal{E}+\mathcal{S}-w_3\ln\mathcal{I}+\mathcal{I}+\mathcal{R}\big)+\mathcal{R}+\mathcal{E}+\mathcal{I}+\mathcal{S}-\ln\mathcal{I}-\ln\mathcal{R}-\ln\mathcal{S}.$$

It is simple to verify that

$$\liminf_{p\to+\infty,\mathfrak{S}\in\mathbb{R}^4_+\setminus U_p} W_2(\mathfrak{S}) = +\infty,$$

where $U_p =]\frac{1}{p}, p[^4$.
On the other hand

$$\frac{\partial W_2(\mathfrak{S})}{\partial\mathcal{S}} = w_4+1-\frac{w_4w_1+1}{\mathcal{S}}, \qquad \frac{\partial W_2(\mathfrak{S})}{\partial\mathcal{E}} = w_4+1-\frac{w_4w_2}{\mathcal{E}},$$

$$\frac{\partial W_2(\mathfrak{S})}{\partial\mathcal{I}} = w_4+1-\frac{w_4w_3+1}{\mathcal{I}}, \qquad \frac{\partial W_2(\mathfrak{S})}{\partial\mathcal{R}} = w_4+1-\frac{1}{\mathcal{R}},$$

the function W_2 has a unique stagnation point,

$$(\mathcal{S}_*,\mathcal{E}_*,\mathcal{I}_*,\mathcal{R}_*) = \Big(\frac{w_4w_1+1}{1+w_4}, \frac{w_4w_2}{w_4+1}, \frac{w_4w_3+1}{1+w_4}, \frac{1}{w_4+1}\Big).$$

Furthermore, the Hessian matrix of function W_2 at $(\mathcal{S}_*,\mathcal{E}_*,\mathcal{I}_*,\mathcal{R}_*)$ is written as

$$\begin{pmatrix} \frac{w_4w_1+1}{\mathcal{S}_*^2} & 0 & 0 & 0\\ 0 & \frac{w_4w_2}{\mathcal{E}_*^2} & 0 & 0\\ 0 & 0 & \frac{w_4w_3+1}{\mathcal{I}_*^2} & 0\\ 0 & 0 & 0 & \frac{1}{\mathcal{R}_*^2}\end{pmatrix},$$

The Hessian matrix exhibits positive definiteness and W_2 is a continuous function. Thus, W_2 attains a unique minimum value. $(\mathcal{S}_*,\mathcal{E}_*,\mathcal{I}_*,\mathcal{R}_*)\in\mathbb{R}^4_+$.
Let

$$V(\mathfrak{S}) = W_2(\mathfrak{S}) - W_2(\mathcal{S}_*,\mathcal{E}_*,\mathcal{I}_*,\mathcal{R}_*),$$

applying Itô formula we obtain

$$\begin{aligned} L\mathbb{V}(\mathfrak{S}) &= LW_2(\mathfrak{S})\\ &\le w_4\Big[-4\Xi\big[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\big]+\frac{w_1\wp_1\mathcal{E}+w_2\wp_2\mathcal{I}}{\mathcal{N}}-\frac{w_2\wp_1\mathcal{S}}{\mathcal{N}}\Big]+\Xi-\mho(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{R})\\ &-\frac{\Xi}{\mathcal{S}}+\frac{\wp_1\mathcal{E}}{\mathcal{N}}+\frac{\wp_2\mathcal{I}}{\mathcal{N}}-\frac{\zeta\mathcal{R}}{\mathcal{S}}-\frac{\varsigma\mathcal{I}}{\mathcal{R}}+2\mho+\zeta+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2}-\frac{\xi\mathcal{E}}{\mathcal{I}}+\mho+\mho_0+\varsigma+\frac{\varrho_3^2}{2}\\ &= -4w_4\Xi\big[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\big]+\frac{w_4w_1\wp_1\mathcal{E}+w_4w_2\wp_2\mathcal{I}}{\mathcal{N}}-\frac{w_4w_2\wp_1\mathcal{S}}{\mathcal{N}}+\Xi-\mho(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{R})\\ &-\frac{\Xi}{\mathcal{S}}+\frac{\wp_1\mathcal{E}}{\mathcal{N}}+\frac{\wp_2\mathcal{I}}{\mathcal{N}}-\frac{\zeta\mathcal{R}}{\mathcal{S}}-\frac{\varsigma\mathcal{I}}{\mathcal{R}}+2\mho+\zeta+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2}-\frac{\xi\mathcal{E}}{\mathcal{I}}+\mho+\mho_0+\varsigma+\frac{\varrho_3^2}{2}. \end{aligned}$$

Let

$$U = \Big\{\varepsilon_1\le\mathcal{S}\le\frac{1}{\varepsilon_2}, \varepsilon_3\le\mathcal{E}\le\frac{1}{\varepsilon_4}, \varepsilon_5\le\mathcal{I}\le\frac{1}{\varepsilon_6}, \varepsilon_7\le\mathcal{R}\le\frac{1}{\varepsilon_8}\Big\},$$

where $0<\varepsilon_\ell<1$ for $\ell=1,...,8$, are very small constants. We select positive constants ε_ℓ and w_4 to satisfy the following conditions.

- $-\frac{\Xi}{\varepsilon_1}+\mathbf{F}\le-1.$
- $(w_4w_1+1)\wp_1\varepsilon_1+\mathbf{G}-4w_4\Xi\big[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\big]\le-1.$
- $-\frac{\xi}{\varepsilon_3}+\mathbf{F}\le-1.$
- $-\frac{\varsigma}{\varepsilon_5}+\mathbf{F}\le-1.$

- $-\dfrac{\mho}{\varepsilon_2}+\mathbf{F}\leq -1.$
- $-\dfrac{\mho}{\varepsilon_4}+\mathbf{F}\leq -1.$
- $-\dfrac{\mho}{\varepsilon_6}+\mathbf{F}\leq -1.$
- $-\dfrac{\mho}{\varepsilon_8}+\mathbf{F}\leq -1.$

where

$$\mathbf{F}=w_4w_1\wp_1+w_4w_2\wp_2+\Xi+\wp_1+\wp_2+3\mho+\zeta+\mho_0+\varsigma+\frac{\varrho_3^2}{2}+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2},$$

$$\mathbf{G}=w_4w_2\wp_2\Xi+\wp_2+3\mho+\zeta+\mho_0+\varsigma+\frac{\varrho_3^2}{2}+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2}.$$

We divide $\mathbb{R}^4_+\setminus U$ into eight regions, so that $\mathbb{R}^4_+\setminus U=U_1\cup U_2\cup ...\cup U_8$, where,

$$\begin{aligned}
U_1&=\Big\{\mathfrak{S}\in\mathbb{R}^4_+,\varepsilon_1>\mathcal{S}>0\Big\},\quad U_2=\Big\{\mathfrak{S}\in\mathbb{R}^4_+,0<\mathcal{E}<\varepsilon_3,\mathcal{S}>\varepsilon_1\Big\},\\
U_3&=\Big\{\mathfrak{S}\in\mathbb{R}^4_+,0<\mathcal{I}<\varepsilon_5,\mathcal{E}>\varepsilon_3\Big\},\quad U_4=\Big\{\mathfrak{S}\in\mathbb{R}^4_+,0<\mathcal{R}<\varepsilon_7,\mathcal{I}>\varepsilon_5\Big\},\\
U_5&=\Big\{\mathfrak{S}\in\mathbb{R}^4_+,\mathcal{S}>\frac{1}{\varepsilon_2}\Big\},\quad U_6=\Big\{\mathfrak{S}\in\mathbb{R}^4_+,\mathcal{E}>\frac{1}{\varepsilon_4}\Big\},\\
U_7&=\Big\{\mathfrak{S}\in\mathbb{R}^4_+,\mathcal{I}>\frac{1}{\varepsilon_6}\Big\},\quad U_8=\Big\{\mathfrak{S}\in\mathbb{R}^4_+,\mathcal{R}>\frac{1}{\varepsilon_8}\Big\}.
\end{aligned}$$

Next, we show that $L\mathbb{V}\leq -1$ on $\mathbb{R}^4_+\setminus U$, to this end, we consider eight cases:
Case 1. Let $\mathfrak{S}\in U_1$, then we have

$$\begin{aligned}
L\mathbb{V}(\mathfrak{S})\leq&-4w_4\Xi\big[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\big]+\frac{w_4w_1\wp_1\mathcal{E}+w_4w_2\wp_2\mathcal{I}}{\mathcal{N}}-\frac{w_4w_2\wp_1\mathcal{S}}{\mathcal{N}}+\Xi-\mho(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{R})\\
&-\frac{\Xi}{\mathcal{S}}+\frac{\wp_1\mathcal{E}}{\mathcal{N}}+\frac{\wp_2\mathcal{I}}{\mathcal{N}}-\frac{\zeta\mathcal{R}}{\mathcal{S}}-\frac{\varsigma\mathcal{I}}{\mathcal{R}}+2\mho+\zeta+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2}-\frac{\xi\mathcal{E}}{\mathcal{I}}+\mho+\mho_0+\varsigma+\frac{\varrho_3^2}{2}\\
\leq&\mathbf{F}-\frac{\Xi}{\mathcal{S}}\leq\mathbf{F}-\frac{\Xi}{\varepsilon_1}\leq -1.
\end{aligned}$$

Case 2. If $\mathfrak{S}\in U_2$, then we have

$$\begin{aligned}
L\mathbb{V}(\mathfrak{S})\leq&-4w_4\Xi\big[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\big]+\frac{w_4w_1\wp_1\mathcal{E}+w_4w_2\wp_2\mathcal{I}}{\mathcal{N}}-\frac{w_4w_2\wp_1\mathcal{S}}{\mathcal{N}}+\Xi-\mho(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{R})\\
&-\frac{\Xi}{\mathcal{S}}+\frac{\wp_1\mathcal{E}}{\mathcal{N}}+\frac{\wp_2\mathcal{I}}{\mathcal{N}}-\frac{\zeta\mathcal{R}}{\mathcal{S}}-\frac{\varsigma\mathcal{I}}{\mathcal{R}}+2\mho+\zeta+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2}-\frac{\xi\mathcal{E}}{\mathcal{I}}+\mho+\mho_0+\varsigma+\frac{\varrho_3^2}{2}\\
\leq&(w_4w_1+1)\wp_1\frac{\mathcal{E}}{\mathcal{N}}+\mathbf{G}-4w_4\Xi\big[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\big]\\
\leq&(w_4w_1+1)\wp_1\frac{\mathcal{E}}{\mathcal{S}}+\mathbf{G}-4w_4\Xi\big[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\big]\\
\leq&(w_4w_1+1)\wp_1\frac{\varepsilon_3}{\varepsilon_1}+\mathbf{G}-4w_4\Xi\big[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\big].
\end{aligned}$$

By choosing, $\varepsilon_3=\varepsilon_1^2$, we have,

$$L\mathbb{V}(\mathfrak{S})\leq(w_4w_1+1)\wp_1\varepsilon_1+\mathbf{G}-4w_4\Xi\big[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\big]\leq -1.$$

Case 3. If $\mathfrak{S} \in U_3$, then we have

$$\begin{aligned} L\mathbb{V}(\mathfrak{S}) \leq & -4w_4\Xi\left[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\right]+\frac{w_4w_1\wp_1\mathcal{E}+w_4w_2\wp_2\mathcal{I}}{\mathcal{N}}-\frac{w_4w_2\wp_1\mathcal{S}}{\mathcal{N}}+\Xi-\mho(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{R}) \\ & -\frac{\Xi}{\mathcal{S}}+\frac{\wp_1\mathcal{E}}{\mathcal{N}}+\frac{\wp_2\mathcal{I}}{\mathcal{N}}-\frac{\zeta\mathcal{R}}{\mathcal{S}}-\frac{\varsigma\mathcal{I}}{\mathcal{R}}+2\mho+\zeta+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2}-\frac{\xi\mathcal{E}}{\mathcal{I}}+\mho+\mho_0+\varsigma+\frac{\varrho_3^2}{2} \\ \leq & \mathbf{F}-\xi\frac{\mathcal{E}}{\mathcal{I}}\leq \mathbf{F}-\xi\frac{\varepsilon_3}{\varepsilon_5}. \end{aligned}$$

Here, we choose, $\varepsilon_5=\varepsilon_3^2$, thus

$$L\mathbb{V}(\mathfrak{S}) \leq \mathbf{F}-\frac{\xi}{\varepsilon_3}\leq -1.$$

Case 4. If $\mathfrak{S} \in U_4$ then we have

$$\begin{aligned} L\mathbb{V}(\mathfrak{S}) \leq & -4w_4\Xi\left[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\right]+\frac{w_4w_1\wp_1\mathcal{E}+w_4w_2\wp_2\mathcal{I}}{\mathcal{N}}-\frac{w_4w_2\wp_1\mathcal{S}}{\mathcal{N}}+\Xi-\mho(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{R}) \\ & -\frac{\Xi}{\mathcal{S}}+\frac{\wp_1\mathcal{E}}{\mathcal{N}}+\frac{\wp_2\mathcal{I}}{\mathcal{N}}-\frac{\zeta\mathcal{R}}{\mathcal{S}}-\frac{\varsigma\mathcal{I}}{\mathcal{R}}+2\mho+\zeta+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2}-\frac{\xi\mathcal{E}}{\mathcal{I}}+\mho+\mho_0+\varsigma+\frac{\varrho_3^2}{2} \\ \leq & \mathbf{F}-\varsigma\frac{\mathcal{I}}{\mathcal{R}}\leq \mathbf{F}-\varsigma\frac{\varepsilon_5}{\varepsilon_7}. \end{aligned}$$

Letting $\varepsilon_7=\varepsilon_5^2$, yields

$$L\mathbb{V}(\mathfrak{S}) \leq \mathbf{F}-\frac{\varsigma}{\varepsilon_5}\leq -1.$$

Case 5. If $\mathfrak{S} \in U_5$, then we have

$$\begin{aligned} L\mathbb{V}(\mathfrak{S}) \leq & -4w_4\Xi\left[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\right]+\frac{w_4w_1\wp_1\mathcal{E}+w_4w_2\wp_2\mathcal{I}}{\mathcal{N}}-\frac{w_4w_2\wp_1\mathcal{S}}{\mathcal{N}}+\Xi-\mho(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{R}) \\ & -\frac{\Xi}{\mathcal{S}}+\frac{\wp_1\mathcal{E}}{\mathcal{N}}+\frac{\wp_2\mathcal{I}}{\mathcal{N}}-\frac{\zeta\mathcal{R}}{\mathcal{S}}-\frac{\varsigma\mathcal{I}}{\mathcal{R}}+2\mho+\zeta+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2}-\frac{\xi\mathcal{E}}{\mathcal{I}}+\mho+\mho_0+\varsigma+\frac{\varrho_3^2}{2} \\ \leq & \mathbf{F}-\mho\mathcal{S}\leq \mathbf{F}-\frac{\mho}{\varepsilon_2}\leq -1. \end{aligned}$$

Case 6. If $\mathfrak{S} \in U_6$, then we have

$$\begin{aligned} L\mathbb{V}(\mathfrak{S}) \leq & -4w_4\Xi\left[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\right]+\frac{w_4w_1\wp_1\mathcal{E}+w_4w_2\wp_2\mathcal{I}}{\mathcal{N}}-\frac{w_4w_2\wp_1\mathcal{S}}{\mathcal{N}}+\Xi-\mho(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{R}) \\ & -\frac{\Xi}{\mathcal{S}}+\frac{\wp_1\mathcal{E}}{\mathcal{N}}+\frac{\wp_2\mathcal{I}}{\mathcal{N}}-\frac{\zeta\mathcal{R}}{\mathcal{S}}-\frac{\varsigma\mathcal{I}}{\mathcal{R}}+2\mho+\zeta+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2}-\frac{\xi\mathcal{E}}{\mathcal{I}}+\mho+\mho_0+\varsigma+\frac{\varrho_3^2}{2} \\ \leq & \mathbf{F}-\mho\mathcal{E}\leq \mathbf{F}-\frac{\mho}{\varepsilon_4}\leq -1. \end{aligned}$$

Case 7. If $\mathfrak{S} \in U_7$, then we have

$$\begin{aligned} L\mathbb{V}(\mathfrak{S}) \leq & -4w_4\Xi\left[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\right]+\frac{w_4w_1\wp_1\mathcal{E}+w_4w_2\wp_2\mathcal{I}}{\mathcal{N}}-\frac{w_4w_2\wp_1\mathcal{S}}{\mathcal{N}}+\Xi-\mho(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{R}) \\ & -\frac{\Xi}{\mathcal{S}}+\frac{\wp_1\mathcal{E}}{\mathcal{N}}+\frac{\wp_2\mathcal{I}}{\mathcal{N}}-\frac{\zeta\mathcal{R}}{\mathcal{S}}-\frac{\varsigma\mathcal{I}}{\mathcal{R}}+2\mho+\zeta+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2}-\frac{\xi\mathcal{E}}{\mathcal{I}}+\mho+\mho_0+\varsigma+\frac{\varrho_3^2}{2} \\ \leq & \mathbf{F}-\mho\mathcal{I}\leq \mathbf{F}-\frac{\mho}{\varepsilon_6}\leq -1. \end{aligned}$$

Case 8. If $\mathfrak{S} \in U_8$, then we have

$$\begin{aligned} L\mathbb{V}(\mathfrak{S}) \leq & -4w_4\Xi\left[(\mathcal{R}_0^{\mathcal{S}})^{\frac{1}{4}}-1\right]+\frac{w_4w_1\wp_1\mathcal{E}+w_4w_2\wp_2\mathcal{I}}{\mathcal{N}}-\frac{w_4w_2\wp_1\mathcal{S}}{\mathcal{N}}+\Xi-\mho(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{R}) \\ & -\frac{\Xi}{\mathcal{S}}+\frac{\wp_1\mathcal{E}}{\mathcal{N}}+\frac{\wp_2\mathcal{I}}{\mathcal{N}}-\frac{\zeta\mathcal{R}}{\mathcal{S}}-\frac{\varsigma\mathcal{I}}{\mathcal{R}}+2\mho+\zeta+\frac{\varrho_1^2}{2}+\frac{\varrho_4^2}{2}-\frac{\xi\mathcal{E}}{\mathcal{I}}+\mho+\mho_0+\varsigma+\frac{\varrho_3^2}{2} \\ \leq & \mathbf{F}-\mho\mathcal{R}\leq \mathbf{F}-\frac{\mho}{\varepsilon_8}\leq -1. \end{aligned}$$

We may deduce from the above reasons that $L\mathbb{V} \leq -1$ for all $\mathfrak{S} \in \mathbb{R}^4_+ \setminus U$, therefore, Condition (b) of remark 3.6 holds. By Lemma (3.5) the stochastic system (2.1) is ergodic and admits a unique **SD** $\mu(.)$. □

4 Numerical simulations

To help the reader better comprehend our findings, in this section we will perform numerical simulations. To illustrate and validate the theoretical results and display these results, we used MATLAB. Using the stochastic **RK4** method, we derive the discretized form of System (2.1) as follows:

$$\begin{aligned}
\mathcal{S}_{k+1} &= \mathcal{S}_k + \Big[\Xi - \wp_1\frac{\mathcal{S}_k\mathcal{E}_k}{\mathcal{N}_k} - \wp_2\frac{\mathcal{S}_k\mathcal{I}_k}{\mathcal{N}_k} + \zeta\mathcal{R}_k - \mho S_k\Big]dt + \varrho_1\mathcal{S}_k\sqrt{dt}W_{1_k} + \frac{\varrho_1^2}{2}\mathcal{S}_k(W_{1_k}-1)dt,\\
\mathcal{E}_{k+1} &= \mathcal{E}_k + \Big[\wp_1\frac{\mathcal{S}_k\mathcal{E}_k}{\mathcal{N}_k} + \wp_2\frac{\mathcal{S}_k\mathcal{I}_k}{\mathcal{N}_k} - (\mho+\xi)\mathcal{E}_k\Big]dt + \varrho_2\sqrt{dt}\mathcal{E}_kW_{2_k} + \frac{\varrho_2^2}{2}\mathcal{E}_k(W_{2_k}-1)dt,\\
\mathcal{I}_{k+1} &= \mathcal{I}_k + \Big[\xi\mathcal{E}_k - (\mho+\mho_0+\varsigma)\mathcal{I}_k\Big]dt + \varrho_3\sqrt{dt}\mathcal{I}_kW_{3_k} + \frac{\varrho_3^2}{2}\mathcal{I}_k(W_{3_k}-1)dt,\\
\mathcal{R}_{k+1} &= \mathcal{R}_k + \Big[\varsigma\mathcal{I}_k - (\mho+\zeta)\mathcal{R}_k\Big]dt + \varrho_4\sqrt{dt}\mathcal{R}_kW_{4_k} + \frac{\varrho_4^2}{2}\mathcal{R}_k(W_{4_k}-1)dt.
\end{aligned} \tag{4.1}$$

where $W_j \rightsquigarrow \mathcal{N}(0,1)$, $(j = 1,..,4)$ are mutually independent random variables, and dt is time progression. We consider two sets of the parameters values.

Example 4.1. To illustrate extinction of COVID-19, we use the parameters : $\Xi = 11$, $\wp_1 = 0.4$, $\wp_2 = 0.01$, $\xi = 0.1$, $\zeta = 0.6$, $\varsigma = 0.7$, $\mho = 0.6$, $\mho_0 = 0.5$. with $\mathfrak{S}(0) = (50, 40, 30, 10)$. After calculating $\tilde{R_0^{\mathcal{S}}}$, we have

$$\tilde{R_0^{\mathcal{S}}} = \max\left\{\frac{\mho\wp_1 + \Xi\wp_2}{\mho(\mho+\xi+\frac{\varrho_2^2}{2})}, \frac{\xi\Xi}{\mho(\mho+\mho_0+\varsigma+\frac{\varrho_3^2}{2})}\right\} = 0.9643 < 1.$$

The conditions stated in Theorem 3.4 have been met. As can be shown from Figure 1, the disease will thus disappear with probability one.

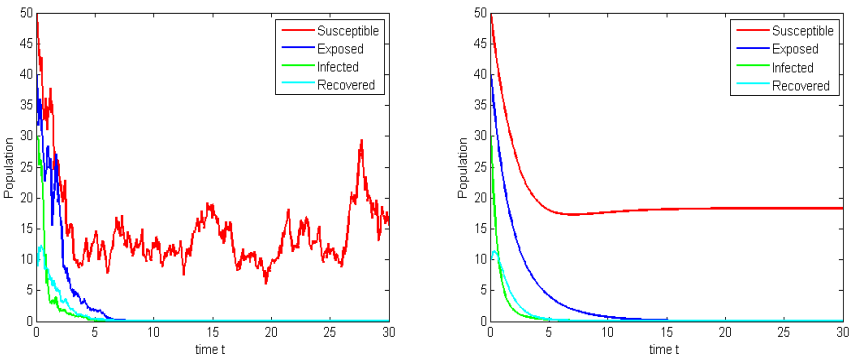


Figure 1. Simulation results for the path of solution based on the parameters in example 4.1 with $dt = 0.06$. (a) the deterministic version means $\varrho_i = 0$ $(i = 1, 2, 3, 4)$, and (b) when $\varrho_1 = 0.3, \varrho_2 = 0.4, \varrho_3 = 0.45, \varrho_4 = 0.325$.

Example 4.2. We consider the following set of parameter values for Model (2.1): $\Xi = 12$, $\wp_1 = 0.1$, $\wp_2 = 0.6$, $\xi = 0.65$, $\zeta = 0.4$, $\varsigma = 0.3$, $\mho = 0.1$, $\mho_0 = 0.01$, with initial condition $\mathfrak{S}(0) = (50, 40, 30, 10)$. Note that

$$R_0^{\mathcal{S}} = \frac{\mho\wp_2\xi}{(\mho+\frac{\varrho_1^2}{2})(\mho+\xi+\frac{\varrho_2^2}{2})(\mho_0+\mho+\varsigma+\frac{\varrho_3^2}{2})} = 1.0435 > 1,$$

according to Theorem 3.7, then system (2.1) has a uniquely determined **SD** . Hence, the disease will continue to spread throughout the population as shown in Figure 2.

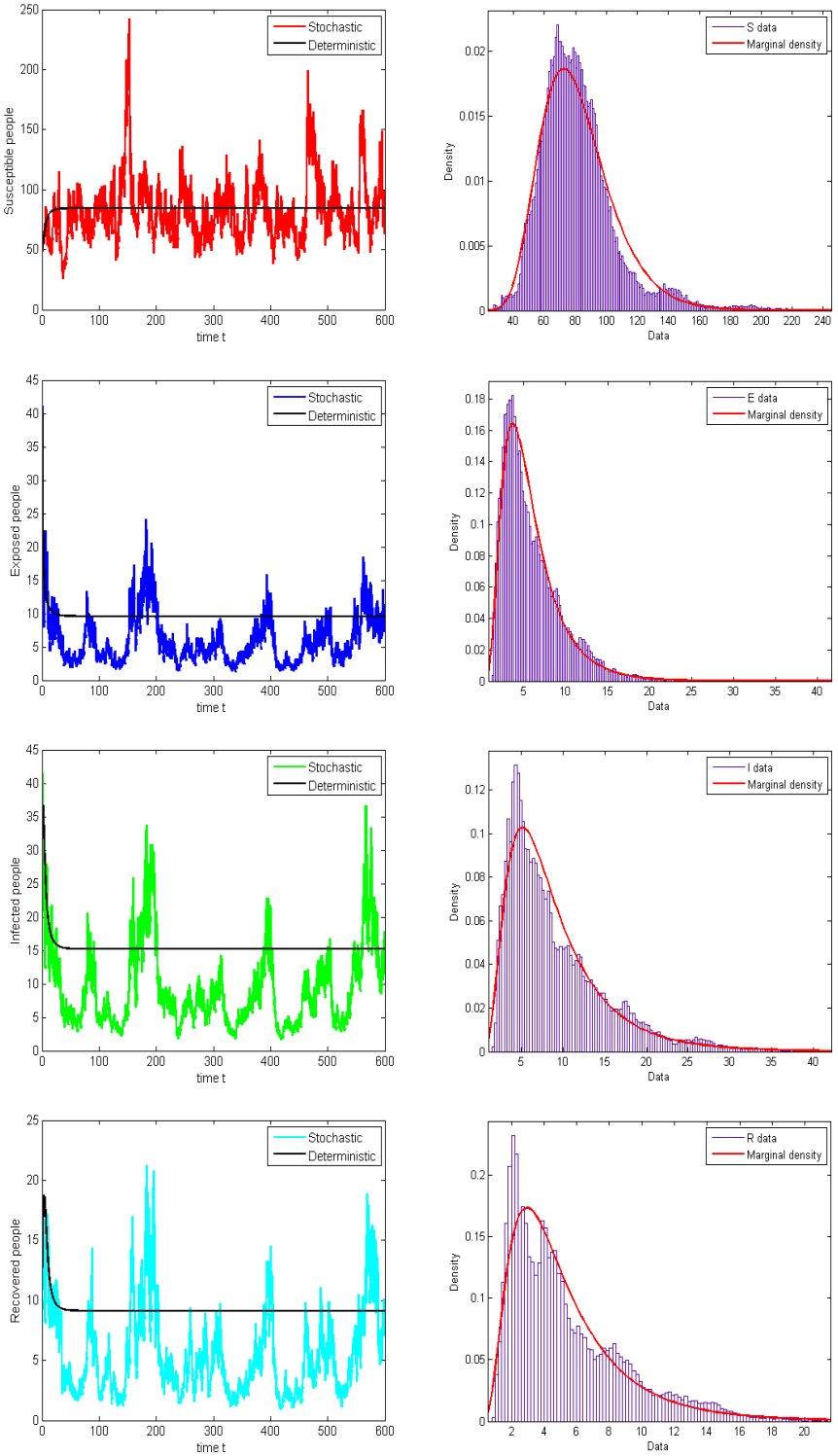


Figure 2. Compartment $\mathcal{S}$, $\mathcal{E}$, $\mathcal{I}$ and $\mathcal{R}$ in the deterministic and stochastic model are simulated in the left hand column using parameter of example 4.2 and the noise intensities $\varrho_1 = 0.15$, $\varrho_2 = 0.25$, $\varrho_3 = 0.2$, $\varrho_4 = 0.225$ with $dt = 0.006$. The density function graphs and frequency histogram for $\mathcal{S}$, $\mathcal{E}$, $\mathcal{I}$, and $\mathcal{R}$ are displayed in the right hand column.

5 Conclusion

This investigation is concerned with the behavior of a stochastic COVID-19 model. We commence by showing that there exists a unique global positive solution through carefully constructed Lyapunov functions. Next, we derive criteria that are sufficient for disease eradication and demonstrate the presence of a unique ergodic **SD** under specific criteria. Integrating theoretical analysis with numerical simulations, our findings reveal that lower levels of white noise permit the model to have a uniquely ergodic **SD** when $R_0^{\mathcal{S}} > 1$. In contrast, higher noise intensities ensure the disappearance of the disease when $\tilde{R}_0^{\mathcal{S}} < 1$. In contrast to deterministic models, the introduction of stochastic white noise has a significant influence on disease dynamics, affecting both persistence and elimination, thereby broadening the range of possible outcomes in epidemic modeling.

References

[1] R.Z. Has'minskii, *Stochastic Stability of Differential Equations*, Sijthoff & Noordhoff, Alphen aan den Rijn, Netherlands, 1980.

[2] C. Zhu, G. Yin, *Asymptotic properties of hybrid diffusion systems*, SIAM J. Control Optim. 46 (2007) 1155–1179.

[3] G. Strang, *Linear Algebra and Its Applications*, Thomson Learning, Inc., 1988.

[4] T.C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, Inc., New York, 1988.

[5] World Health Organization, Naming the coronavirus disease (COVID-19) and the virus that causes it, in Naming the coronavirus disease (COVID-19) and the virus that causes it, https://www.who.int/emergencies/diseases/novel-coronavirus-2019

[6] World Health Organization, Novel coronavirus (2019-nCoV) Situation Report-1, 21 Jan. 2020, in Novel Coronavirus (2019-nCoV) Situation Report-1, 21 Jan 2020, https://www.who.int/emergencies/diseases/novel-coronavirus-2019/situation-reports(2020)

[7] World Health Organization, COVID-19 Weekly Epidemiological Update-17, in COVID19 Weekly Epidemiological Update-17, https://www.who.int/emergencies/diseases/novelcoronavirus-2019, 6 December 2020

[8] World Health Organization, COVID-19 Weekly Epidemiological Update 23, 19 Jan 2021, in COVID-19 Weekly Epidemiological Update 23, 19 Jan 2021, https://www.who.int/emergencies/diseases/novel-coronavirus-2019/situation-reports (2021)

[9] World Health Organization, COVID-19 Weekly Epidemiological Update 47, 6 July 2021, in COVID-19 Weekly Epidemiological Update 47, 6 July 2021, https://www.who.int/emergencies/diseases/novel-coronavirus-2019/situation-reports (2021)

[10] Anwarud Din, Yongjin Li, Qi Liu, *Viral dynamics and control of hepatitis B virus (HBV) using an epidemic model*, Alexandria Engineering Journal, Volume 59, Issue 2, 2020, Pages 667-679, ISSN 1110-0168.

[11] He Huang, Yahong Chen, Zhijun Yan,*Impacts of social distancing on the spread of infectious diseases with asymptomatic infection: A mathematical model*, Applied Mathematics and Computation, Volume 398, 2021, 125983, ISSN 0096-3003.

[12] Anwarud Din, Amir Khan, Dumitru Baleanu, *Stationary distribution and extinction of stochastic coronavirus (COVID-19) epidemic model*, Chaos, Solitons & Fractals, Volume 139, 2020, 110036, ISSN 0960-0779.

[13] Tesfaye, A.W., Satana, T.S., *Stochastic model of the transmission dynamics of COVID-19 pandemic*, Adv Differ Equ 2021, 457 (2021).

[14] Thirthar, A. A., Jawad, Sh., Shah, K., Abdeljawad, Th.,*How does media coverage affect a COVID-19 pandemic model with direct and indirect transmission?*, Journal of Mathematics and Computer Science, 35, no. 2 (2024): 169–181

[15] Thirthar, A. A., Abboubakar, H., Abdesslem L. A., Nisar, K. S., *Dynamical behavior of a fractional-order epidemic model for investigating two fear effect functions*, Results in Control and Optimization, Volume 16, 2024, 100474.

[16] Rukhsar Ikram, Amir Khan, Mostafa Zahri, Anwar Saeed, Mehmet Yavuz, Poom Kumam, *Extinction and stationary distribution of a stochastic COVID-19 epidemic model with time-delay*, Computers in Biology and Medicine, Volume 141, 2022, 105115, ISSN 0010-4825.

[17] Mohammed AS, Khoshnaw SHA, *The impact of vaccination on COVID-19 pandemic in the Kurdistan Region of Iraq: A mathematical modelling*, Palestine Journal of Mathematics, 2023; 12(SI I):87-106.

[18] Achar SJ, Baishya C, *Dynamics of modified fractional illicit Drug Consumption Model*, Palestine Journal of Mathematics, 2022; 11(3):112-126.

[19] Suwardi Annas, Muh. Isbar Pratama, Muh. Rifandi, Wahidah Sanusi, Syafruddin Side, *Stability analysis and numerical simulation of SEIR model for pandemic COVID-19 spread in Indonesia*, Chaos, Solitons & Fractals, Volume 139, 2020, 110072, ISSN 0960-0779.

[20] Abadi Abay Gebremeskel, Hailay Weldegiorgis Berhe, Habtu Alemayehu Atsbaha, *Mathematical modelling and analysis of COVID-19 epidemic and predicting its future situation in Ethiopia*, Results in Physics, Volume 22, 2021, 103853, ISSN 2211-3797.

[21] Anwarud Din, Tahir Khan, Yongjin Li, Hassan Tahir, Asaf Khan, Wajahat Ali Khan, *Mathematical analysis of dengue stochastic epidemic model*, Results in Physics, Volume 20, 2021, 103719, ISSN 2211-3797.

[22] Thirthar, A. A., Jawad, Sh., Panja, P., Mukheimer, A., Abdeljawad, Th., *The role of human shield in prey, cro-raiders and top predator species in southwestern Ethiopia's coffee forests: a modeling study*, Journal of Mathematics and Computer Science, 36, no. 3 (2025): 333–351.

[23] kamel Naji R, Adnan Thirthar A.,*Stability and Bifurcation of an SIS Epidemic Model with Saturated Incidence Rate and Treatment Function*, IJMSI 2020; 15 (2) :129-146.

[24] Yanli Zhou, Weiguo Zhang, Sanling Yuan, *Survival and stationary distribution of a SIR epidemic model with stochastic perturbations*, Applied Mathematics and Computation, Volume 244, 2014, Pages 118-131, ISSN 0096-3003.

[25] Thanin Sitthiwirattham, Anwar Zeb, Saowaluck Chasreechai, Zohreh Eskandari, Mouhcine Tilioua, Salih Djilali, *Analysis of a discrete mathematical COVID-19 model*, Results in Physics, Volume 28, 2021, 104668, ISSN 2211-3797.

[26] X. Mao,*Stochastic Differential Equations and Applications*, Horwood, Chichester, UK, 2nd edition, 2007.

[27] Rafail Khasminskii, *Stochastic Stability of Differential Equations*, Vol. 66. Springer Science & Business Media, 2011.

Author information

A. Kadri, Laboratory of Mathematics Modeling and Applications, University of Adrar, Algeria.
E-mail: kadri.abdeldjalil@univ-adrar.edu.dz

A. Boudaoui, Laboratory of Mathematics Modeling and Applications, University of Adrar, Algeria.
E-mail: ahmedboudaoui@univ-adrar.edu.dz

Fawaz K. Alalhareth, Department of Mathematics, College of Arts & Sciences, Najran University, Kingdom of Saudi Arabia.
E-mail: fkalalhareth@nu.edu.sa

N. Laksaci, Laboratory of Mathematics Modeling and Applications, University of Adrar, Algeria.
E-mail: nor.laksaci@univ-adrar.edu.dz

Received: 2024-12-04
Accepted: 2026-06-26