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Complexity Profiles and Computational Classification of Defectless Polynomials over Henselian Valued Fields

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Abstract. Let (K, v) be a henselian valued field and let $F \in K[x]$ be a defectless polynomial. Motivated by the theory of inductive valuations, optimal MacLane chains, and the valuation-theoretic classification of defectless polynomials through strong types and Okutsu equivalence, we develop a quantitative framework for the analysis and classification of defectless polynomial extensions. Associated with an optimal MacLane chain

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r,$$

we introduce the computational complexity invariant

$$\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F),$$

where $\mathrm{Depth}(F)$ denotes the MacLane depth,

$$E(F) = \sum_{i=0}^{r} e_i, \qquad R(F) = \sum_{i=0}^{r} f_i, \qquad S(F) = \sum_{i=1}^{r} \lambda_i,$$

represent the cumulative ramification, residual degree and Newton polygon contributions arising from the associated valuation-theoretic data. Here (e_i) and (f_i) denote the ramification and residual degree sequences attached to the MacLane chain, while (λ_i) denotes the sequence of principal Newton polygon slopes. To preserve the individual contribution of these arithmetic components, we introduce the complexity profile

$$\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big),$$

which provides a multidimensional numerical representation of the valuation-theoretic structure of F. Complexity profiles naturally induce a metric structure through the computational distance

$$D(F, G) = \|\mathbf{C}(F) - \mathbf{C}(G)\|_2 ,$$

leading to the notions of complexity classes, profile matrices, complexity spectra and arithmetic diversity for families of defectless polynomials. We establish several structural properties of the proposed framework, including positivity, monotonicity, normalization and metric compatibility. Effective algorithms for constructing complexity profiles are developed and implemented computationally. The resulting numerical framework transforms valuation-theoretic information into quantitative data suitable for geometric analysis, statistical investigation and computational classification. Extensive computational experiments are performed for representative families of defectless and Eisenstein-type polynomials. The numerical results illustrate the influence of MacLane depth, ramification growth, residual degree behavior and Newton polygon geometry on the resulting complexity profiles. They further demonstrate that complexity profiles, distance

matrices and complexity spectra provide effective tools for the comparison, visualization and classification of valuation-theoretic structures. The proposed framework offers a new quantitative perspective on defectless polynomials over henselian valued fields and establishes a bridge between valuation theory, computational algebra, arithmetic geometry and data-driven approaches to the study of valued field extensions.

1 Introduction

The theory of valuations on polynomial rings has become one of the fundamental tools in modern algebra, arithmetic geometry, and number theory. Its origins may be traced back to the pioneering works of MacLane [13, 14], who introduced key polynomials and developed a systematic approach to the construction of valuations on rational function fields. These foundational ideas have profoundly influenced subsequent developments in valuation theory and have led to deep connections between ramification theory, local field arithmetic, polynomial factorization, and algebraic extensions of valued fields. At the same time, significant progress in related areas of algebra and number theory has continued to enrich the broader mathematical framework in which valuation-theoretic methods operate. Recent contributions published in the *Palestine Journal of Mathematics* include investigations of commutative rings with Artinian spectrum [2], integrally closed ring extensions [3], quartic number fields and their integral bases [20], generalizations of prime radicals in noncommutative rings [5], and structural properties of zero-dimensional rings and their spectra [12]. These works illustrate the continuing interaction between algebraic structures, field extensions, and arithmetic properties that motivate the development of new theoretical and computational approaches.

Let (K, v) be a valued field and let $K(x)$ denote the rational function field in one indeterminate. One of the central problems in valuation theory is the description of all valuations on $K(x)$ extending the valuation v of K. Significant progress toward the solution of this problem was achieved through the theory of key polynomials and augmented valuations developed by Vaquié [21, 22], Herrera Govantes, Olalla Acosta and Spivakovsky [9], and later refined by Herrera Govantes, Mahboub, Olalla Acosta and Spivakovsky [10]. These contributions established a powerful framework for the explicit study of algebraic extensions and their valuation-theoretic invariants. The importance of these ideas becomes particularly apparent in the setting of henselian valued fields. If (K, v) is henselian and $F(x) \in K[x]$ is a monic irreducible polynomial, then the arithmetic structure of the finite extension $L = K(\theta)$, where $F(\theta) = 0$, may be described through valuation-theoretic data attached to F. In particular, MacLane chains, Newton polygons, residual polynomial operators, ramification indices, residual degrees, and Okutsu invariants provide a rich collection of arithmetic information encoding the behavior of the extension. These tools have proved extremely effective for studying factorization problems and local arithmetic phenomena. The modern computational theory of valuations has undergone substantial development through the works of Guàrdia, Montes and Nart [7, 6], who introduced powerful techniques based on Newton polygons and Okutsu invariants. Their methods led to efficient algorithms for polynomial factorization, ideal arithmetic, and the construction of integral bases in number fields. Further developments concerning residual ideals, types, and valuation-theoretic structures were obtained in [4, 8, 16]. These investigations established strong links between valuation theory and computational algebra and demonstrated the effectiveness of valuation-theoretic methods in explicit arithmetic computations. A central role within this framework is played by defectless extensions. Recall that every finite extension L/K satisfies the classical identity $[L : K] = e(L/K)f(L/K)d(L/K)$, where $e(L/K)$ denotes the ramification index, $f(L/K)$ denotes the residual degree, and $d(L/K)$ denotes the defect. The condition $d(L/K) = 1$ characterizes defectless extensions and occupies a distinguished position in valuation theory. Defectless polynomials exhibit particularly rich arithmetic properties and admit a description in terms of MacLane chains, key polynomials, and residual polynomial operators. Important contributions concerning defectless extensions, valuation defects, and factorization phenomena over henselian valued fields may be found in the works of Vaquié [22], Aghigh and Khanduja [1], Popescu and Popescu [19], and Jakhar, Khanduja and Sangwan [11]. A major breakthrough in the subject was achieved by Moraes de Oliveira and Nart [15], who established a precise correspondence

between defectless prime polynomials, strong types, and inductive valuations over henselian fields. Their results provide a complete valuation-theoretic description of defectless polynomials and clarify the role played by Okutsu equivalence classes, MacLane chains, and residual polynomial operators. Closely related computational aspects concerning residual polynomial operators were subsequently investigated in [15], while a comprehensive treatment of key polynomials over valued fields was developed by Nart [17].

Let $\mu_0 \to \mu_1 \to \cdots \to \mu_r$ be an optimal MacLane chain associated with a defectless polynomial F. Attached to this chain are sequences of ramification indices $e_0, e_1, \ldots, e_r$, residual degrees $f_0, f_1, \ldots, f_r$, and principal Newton polygon slopes $\lambda_1, \lambda_2, \ldots, \lambda_r$. These numerical quantities reflect the arithmetic complexity of the extension generated by a root of F. Although the structural properties of these invariants have been extensively studied during the last decades, comparatively little attention has been devoted to their quantitative interaction. Indeed, most existing works focus on classification problems, factorization algorithms, residual structures, key polynomials, strong types, and arithmetic properties of valued field extensions. To the best of our knowledge, no systematic attempt has been made to construct numerical descriptors capable of measuring simultaneously the complexity encoded in MacLane chains, ramification data, residual degree growth, and Newton polygon geometry. The primary objective of the present paper is to initiate such a quantitative approach. More precisely, we introduce a computational complexity invariant $\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F)$, where $\mathrm{Depth}(F) = r$ denotes the MacLane depth of F, $E(F) = \sum_{i=0}^{r} e_i$, $R(F) = \sum_{i=0}^{r} f_i$, and $S(F) = \sum_{i=1}^{r} \lambda_i$. This invariant combines information arising from ramification theory, residual extensions, and Newton polygon geometry into a single numerical quantity. Associated with $\mathrm{Comp}(F)$, we define the complexity profile $\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big)$, which preserves separately the contribution of each arithmetic component. This leads naturally to the notions of complexity classes, complexity spectra, and complexity distances, providing new tools for the comparison and classification of defectless polynomials.

The contribution of the present work differs substantially from the existing literature on valuations, key polynomials and defectless extensions. The classical works of MacLane [13, 14], Vaquié [21, 22], Guàrdia, Montes and Nart [6, 7], Nart [16, 17], and Moraes de Oliveira and Nart [15] focus primarily on the structural and arithmetic description of valuations on $K(x)$, the construction of inductive valuations, the theory of key polynomials, the characterization of defectless polynomials, and the classification of prime polynomials through strong types and Okutsu equivalence. These works provide deep qualitative and algorithmic descriptions of valuation-theoretic phenomena, but they do not attempt to quantify the global arithmetic complexity encoded by an optimal MacLane chain. The principal objective of the present paper is to introduce a quantitative framework that converts valuation-theoretic information into explicit numerical data. To every defectless polynomial F, we associate numerical invariants extracted from an optimal MacLane chain, namely the MacLane depth $\mathrm{Depth}(F)$, the cumulative ramification contribution $E(F) = \sum_{i=0}^{r} e_i$, the cumulative residual degree contribution $R(F) = \sum_{i=0}^{r} f_i$, and the Newton polygon contribution $S(F) = \sum_{i=1}^{r} \lambda_i$. These quantities are combined into the computational complexity invariant $\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F)$, which provides a global numerical measure of the valuation-theoretic complexity of the polynomial. In order to preserve the contribution of each arithmetic component individually, we further introduce the complexity profile $\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big)$, thereby transforming valuation-theoretic data into a finite-dimensional numerical object suitable for computational analysis. A central contribution of the paper is the development of a complete complexity-based classification framework for defectless polynomials. Complexity profiles naturally induce a metric structure through the distance $D(F, G) = \|\mathbf{C}(F) - \mathbf{C}(G)\|_2$, allowing defectless polynomials to be compared quantitatively rather than solely through qualitative valuation-theoretic criteria. This leads to the introduction of complexity classes, profile matrices, distance matrices, complexity spectra, arithmetic diversity measures, and geometric representations of polynomial families. Consequently, the arithmetic information encoded in MacLane chains can be studied using techniques originating from geometry, numerical analysis, clustering theory and statistical classification. The paper also makes a computational contribution. Explicit algorithms are developed for constructing complexity profiles from valuation-theoretic data and for computing the associated invariants. These procedures are implemented in MATLAB and applied to several families of defectless and Eisenstein-type polynomials. The resulting experiments reveal quantitative relationships between

MacLane depth, ramification growth, residual degree behavior and Newton polygon geometry. The numerical results show that complexity profiles capture significant arithmetic information and provide effective descriptors for distinguishing valuation-theoretic structures that may appear similar from a purely qualitative viewpoint. Finally, the proposed framework establishes a direct connection between valuation theory and modern computational mathematics. By representing defectless polynomials through numerical complexity vectors, one obtains a setting in which geometric visualization, distance-based classification procedures, clustering algorithms and statistical methods become applicable to the study of valued field extensions. To the best of our knowledge, the introduction of complexity profiles, complexity spectra and complexity-based classification methods for defectless polynomials has not previously been investigated in the literature. The present work therefore provides a new quantitative perspective on defectless extensions and opens a direction for further interactions between valuation theory, computational algebra and arithmetic geometry.

The paper is organized as follows. Section 2 reviews the valuation-theoretic background required throughout the paper, including henselian valued fields, valuations on polynomial rings, key polynomials, inductive valuations, Newton polygons, residual polynomial operators, defectless polynomials, and Okutsu equivalence. In Section 3 we introduce the computational complexity invariant together with the associated complexity profiles and establish their basic structural properties. Section 4 is devoted to the development of effective computational procedures for constructing complexity profiles and complexity distances from the data of optimal MacLane chains. Section 5 presents a computational study of several families of defectless and Eisenstein-type polynomials and investigates the behavior of the proposed invariants through numerical experiments and complexity-profile analysis. In Section 6 we develop a classification framework based on complexity equivalence, complexity distances, profile matrices, and complexity spectra, providing quantitative tools for comparing valuation-theoretic structures. Section 7 contains the theoretical analysis of the proposed invariants, where positivity, monotonicity, normalization, metric properties, and arithmetic diversity are investigated in detail. Finally, Section 8 summarizes the main results of the paper and discusses several directions for future research concerning defectless polynomials, inductive valuations, and quantitative approaches to valuation theory.

2 Preliminaries

Throughout this paper, (K, v) denotes a henselian valued field with valuation ring $\mathcal{O}_v = \{a \in K \mid v(a) \geq 0\}$, maximal ideal $\mathfrak{m}_v = \{a \in K \mid v(a) > 0\}$, residue field $k_v = \mathcal{O}_v/\mathfrak{m}_v$, and value group $\Gamma = v(K^\times)$. We fix an algebraic closure $\overline{K}$ of K and denote again by v the unique extension of the valuation to $\overline{K}$.

The modern valuation-theoretic approach to algebraic extensions of valued fields is based on the theory of key polynomials, inductive valuations, Newton polygons and residual polynomial operators. Developed originally by MacLane and subsequently extended by Vaquié and others, this framework provides a powerful description of defectless extensions and their arithmetic invariants. In particular, optimal MacLane chains encode ramification data, residual degree information and Newton polygon structures that play a central role in the study of defectless polynomials.

The purpose of this section is to recall the valuation-theoretic notions required in the sequel and to establish the notation used throughout the paper. Particular emphasis is placed on the numerical data attached to optimal MacLane chains, since these quantities constitute the building blocks of the complexity invariants and complexity profiles introduced in the next section. For background material on inductive valuations, key polynomials and defectless polynomials, we refer primarily to [13, 15, 17, 21].

2.1 Valuations, Key Polynomials and Inductive Valuations

Let $K(x)$ denote the rational function field in one indeterminate. A valuation extending v is a mapping $\mu : K(x) \to \Gamma_\mu \cup \{\infty\}$ satisfying the valuation axioms $\mu(fg) = \mu(f) + \mu(g)$, $\mu(f + g) \geq \min\{\mu(f), \mu(g)\}$, and $\mu(0) = \infty$.

Associated with μ is the graded algebra

$$G_\mu = \bigoplus_{\alpha \in \Gamma_\mu} P_\alpha / P_\alpha^+,$$

where $P_\alpha = \{f \in K[x] \mid \mu(f) \geq \alpha\}$ and $P_\alpha^+ = \{f \in K[x] \mid \mu(f) > \alpha\}$. The graded algebra G_μ encodes the valuation-theoretic structure of μ and provides the natural setting for the theory of key polynomials and residual polynomial operators.

A monic polynomial $\phi \in K[x]$ is called a key polynomial for μ if it is both μ-minimal and μ-irreducible. The set of all key polynomials associated with μ is denoted by $\mathrm{KP}(\mu)$. Key polynomials may be viewed as finite algebraic approximations of valuations and constitute the fundamental building blocks of MacLane's theory.

Starting from the minimal valuation μ_0, one constructs a sequence of successive augmentations

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r,$$

where each step is determined by a key polynomial ϕ_i together with a positive rational value γ_i. The corresponding augmented valuation is denoted by

$$\mu_i = [\mu_{i-1}; \phi_i, \gamma_i].$$

If

$$f = \sum_{s \geq 0} a_s \phi_i^s$$

is the ϕ_i-expansion of a polynomial $f \in K[x]$, then the augmented valuation is defined by

$$\mu_i(f) = \min_s \Big\{ \mu_{i-1}(a_s) + s\gamma_i \Big\}.$$

Thus, each augmentation step refines the preceding valuation by incorporating new arithmetic information carried by the key polynomial ϕ_i.

A valuation obtained after finitely many augmentation steps is called an inductive valuation. The length r of the chain is called the MacLane depth and measures the number of successive valuation-theoretic refinements required to construct the valuation. In many situations, larger depths correspond to increasingly intricate arithmetic structures and therefore provide a first indication of valuation-theoretic complexity.

Associated with an inductive valuation is a collection of arithmetic invariants consisting of ramification indices $e_0, e_1, \ldots, e_r$, residual degrees $f_0, f_1, \ldots, f_r$, value groups

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_r,$$

and residue field extensions

$$k_0 \subseteq k_1 \subseteq \cdots \subseteq k_r.$$

These invariants satisfy

$$(\Gamma_i : \Gamma_{i-1}) = e_i, \qquad [k_i : k_{i-1}] = f_i,$$

for every $1 \leq i \leq r$. Consequently, the total ramification index and total residual degree associated with the inductive valuation are given by

$$e(\mu_r) = \prod_{i=0}^{r} e_i, \qquad f(\mu_r) = \prod_{i=0}^{r} f_i.$$

The sequences $(e_i)_{i=0}^r$ and $(f_i)_{i=0}^r$ describe the cumulative growth of ramification and residue-field complexity along the MacLane chain. Together with the MacLane depth, they constitute the principal arithmetic data from which the complexity profiles introduced in Section 3 will be constructed.

Inductive valuations provide an explicit and computationally effective description of large classes of valuations on $K(x)$. In particular, the numerical information encoded by the MacLane chain

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r$$

and its associated invariants (e_i) and (f_i) forms the valuation-theoretic foundation of the computational complexity framework developed in this paper.

2.2 Newton Polygons and Residual Polynomial Operators

Let μ be an inductive valuation on $K(x)$ and let $\phi \in \mathrm{KP}(\mu)$ be a key polynomial. Every polynomial $f \in K[x]$ admits a unique ϕ-expansion

$$f = \sum_{i=0}^{n} a_i \phi^i,$$

where $\deg(a_i) < \deg(\phi)$ for all $0 \leq i \leq n$. The valuation μ assigns to each term $a_i\phi^i$ the value $\mu(a_i\phi^i) = \mu(a_i) + i\,\mu(\phi)$.

The collection of points

$$\Big\{ \big(i, \mu(a_i\phi^i)\big) : 0 \leq i \leq n \Big\}$$

determines a finite subset of the plane whose lower convex envelope is called the Newton polygon of f with respect to the pair (μ, ϕ) and is denoted by $N_{\mu,\phi}(f)$. Newton polygons constitute one of the fundamental tools of valuation theory. Their geometric structure encodes arithmetic information concerning factorization, ramification, residual extensions and the construction of augmented valuations. In particular, the slopes of their sides determine valuation-theoretic refinements and play a decisive role in the description of defectless extensions.

If $\lambda_1, \lambda_2, \ldots, \lambda_t$ denote the slopes of the principal sides of $N_{\mu,\phi}(f)$, then each slope corresponds to a distinguished valuation-theoretic contribution. The collection of these slopes reflects the geometry of the polynomial relative to the valuation and provides information that is not directly visible from the coefficients of f. Since the slopes measure successive geometric refinements of the valuation, they naturally contribute to the complexity framework developed later in this paper. A fundamental principle of the MacLane–Vaquié theory is that the factorization behavior of a polynomial is reflected in the decomposition of its Newton polygon. Different sides correspond to different valuation-theoretic branches, while the arithmetic information attached to each side is encoded by an associated residual polynomial. This correspondence provides a powerful link between geometry and arithmetic and forms the basis of several modern factorization algorithms. Associated with the valuation μ is a residual polynomial operator

$$R_\mu : K[x] \longrightarrow \kappa_\mu[y],$$

where κ_μ denotes the residue field attached to μ. For a polynomial expressed in its ϕ-expansion, the residual polynomial is obtained by selecting the coefficients corresponding to points lying on a prescribed side of the Newton polygon and reducing them modulo the appropriate maximal ideal. The resulting polynomial in $\kappa_\mu[y]$ retains essential arithmetic information while eliminating valuation-theoretically irrelevant data.

Residual polynomial operators provide a bridge between the arithmetic of $K[x]$ and the arithmetic of polynomial rings over residue fields. They play a central role in the construction of types, the computation of Okutsu invariants, the description of defectless extensions and the implementation of effective algorithms in local and henselian arithmetic. An important feature of inductive valuations is that every augmentation step

$$\mu_{i-1} \longrightarrow \mu_i = [\mu_{i-1}; \phi_i, \gamma_i]$$

is accompanied by a distinguished Newton polygon whose principal slope is precisely the augmentation parameter γ_i. Consequently, an optimal MacLane chain determines a sequence of slopes

$$\gamma_1, \gamma_2, \ldots, \gamma_r,$$

which records the successive geometric refinements occurring during the valuation-theoretic construction. From a computational perspective, these slopes provide a natural measure of geometric complexity. This observation motivates the introduction of the cumulative Newton polygon contribution

$$S(F) = \sum_{i=1}^{r} \lambda_i,$$

which forms one of the four components of the complexity profile associated with a defectless polynomial.

The combined use of Newton polygons and residual polynomial operators constitutes the theoretical foundation of the Montes algorithm and many modern computational techniques in arithmetic algebra. In the defectless setting, these tools provide explicit access to the valuation-theoretic data encoded in an optimal MacLane chain and therefore furnish the geometric component of the complexity invariants introduced in the next section.

2.3 Defectless Polynomials and Okutsu Equivalence

Let $F \in K[x]$ be a monic irreducible polynomial, let θ be one of its roots, and set $L = K(\theta)$. Since (K, v) is henselian, the extension L/K satisfies the classical fundamental equality

$$[L : K] = e(L/K)\, f(L/K)\, d(L/K),$$

where $e(L/K)$ denotes the ramification index, $f(L/K)$ the residual degree, and $d(L/K)$ the defect of the extension.

The defect measures the discrepancy between the degree of the extension and the contribution coming from ramification and residue-field growth. When the defect is trivial, one obtains the relation

$$[L : K] = e(L/K)\, f(L/K).$$

A monic irreducible polynomial F is called *defectless* whenever $d(L/K) = 1$, or equivalently,

$$\deg(F) = e(L/K)\, f(L/K).$$

Defectless polynomials occupy a central position in valuation theory because their arithmetic structure admits a complete description through inductive valuations and finite MacLane chains. In particular, every defectless polynomial determines an optimal MacLane chain

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r,$$

whose associated numerical data completely encode the valuation-theoretic structure of the corresponding extension.

A fundamental result of Moraes de Oliveira and Nart [15] establishes a canonical correspondence between defectless prime polynomials, strong types and inductive valuations. Consequently, the study of defectless polynomials may be reduced to the analysis of finite valuation-theoretic data attached to optimal MacLane chains. More precisely, an optimal MacLane chain determines ramification indices $e_0, \ldots, e_r$, residual degrees $f_0, \ldots, f_r$, principal Newton polygon slopes $\lambda_1, \ldots, \lambda_r$, together with residual polynomial data at each level of the chain. The ramification indices describe the successive growth of the value groups, the residual degrees measure the corresponding residue-field extensions, and the Newton polygon slopes encode the geometric refinements occurring during the valuation-theoretic construction. Another important notion is Okutsu equivalence. Two defectless polynomials are said to be Okutsu equivalent if they determine the same strong type and therefore induce the same essential valuation-theoretic data. Okutsu equivalent polynomials share the principal arithmetic invariants arising from MacLane theory, including ramification behavior, residual degree growth, Newton polygon structure and residual polynomial information. Thus Okutsu equivalence provides a natural classification of defectless polynomials from the valuation-theoretic viewpoint. The concepts introduced above form the mathematical foundation of the present work. The existence of finite numerical data

attached to optimal MacLane chains suggests the possibility of measuring valuation-theoretic complexity quantitatively. Motivated by this observation, the next section introduces complexity invariants and complexity profiles obtained from the MacLane depth, ramification contributions, residual degree contributions and Newton polygon geometry. These numerical descriptors will provide the basis for the computational and classification framework developed throughout the remainder of the paper.

3 A Computational Complexity Invariant

One of the central objectives of valuation theory is to describe algebraic extensions through a finite collection of arithmetic invariants. For defectless polynomials, the theory of inductive valuations provides such a description in terms of optimal MacLane chains, ramification indices, residual degrees, Newton polygons and residual polynomial operators. These invariants determine the arithmetic structure of the corresponding extension and constitute the basic numerical data arising in the work of MacLane, Vaquié, Okutsu, Montes, Nart and Moraes de Oliveira. Although these quantities have proved extremely successful for the qualitative classification of defectless polynomials, they are usually studied independently. In particular, existing approaches focus on the individual behavior of ramification indices, residual degree sequences, Newton polygon slopes or residual polynomial data. From a computational perspective, however, it is desirable to have a single numerical quantity capable of summarizing the overall arithmetic complexity of a defectless polynomial while preserving the principal information encoded by its associated MacLane chain. The purpose of this section is to introduce such a quantity. The proposed invariant is motivated by the observation that four numerical characteristics naturally arise from every optimal MacLane chain. The first is the MacLane depth, which measures the number of successive valuation augmentations required to construct the polynomial. The second and third are the ramification and residual degree sequences, which describe the arithmetic growth of the corresponding valued field extension. The fourth is provided by the geometry of the associated Newton polygons, whose slopes measure the successive valuation refinements occurring along the chain.

Let $F \in K[x]$ be a defectless polynomial of degree

$$n = \deg(F) > 1.$$

Associated with F is a unique optimal MacLane chain

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r,$$

together with key polynomials

$$\phi_1, \phi_2, \ldots, \phi_r,$$

ramification indices

$$e_0, e_1, \ldots, e_r,$$

residual degrees

$$f_0, f_1, \ldots, f_r,$$

and principal Newton polygon slopes

$$\lambda_1, \lambda_2, \ldots, \lambda_r.$$

The integer

$$\mathrm{Depth}(F) = r$$

measures the length of the valuation-theoretic construction of F. From a computational point of view, larger values of $\mathrm{Depth}(F)$ indicate that more augmentation steps are required to describe the corresponding extension.

The ramification sequence

$$e_0, e_1, \ldots, e_r$$

describes the successive growth of the value groups along the MacLane chain. Since each ramification index contributes to the arithmetic complexity of the extension, it is natural to consider the cumulative ramification quantity

$$E(F) = \sum_{i=0}^{r} e_i.$$

Similarly, the residual degree sequence

$$f_0, f_1, \ldots, f_r$$

records the growth of the residue fields. The corresponding cumulative residual contribution is defined by

$$R(F) = \sum_{i=0}^{r} f_i.$$

The geometric contribution arises from the Newton polygons associated with the successive augmentation steps. Let

$$N_i(F)$$

denote the principal Newton polygon corresponding to the augmentation

$$\mu_{i-1} \longrightarrow \mu_i.$$

If

$$\lambda_i$$

denotes the absolute value of its principal slope, then the total Newton polygon contribution is given by

$$S(F) = \sum_{i=1}^{r} \lambda_i.$$

The quantity $S(F)$ measures the cumulative geometric complexity of the valuation process. Large values of $S(F)$ correspond to substantial valuation refinements and therefore indicate more intricate Newton polygon structures. The choice of the quantities $\mathrm{Depth}(F)$, $E(F)$, $R(F)$ and $S(F)$ is motivated by the fact that they measure four complementary aspects of arithmetic complexity. The MacLane depth records the length of the valuation-theoretic construction, the cumulative ramification contribution measures value-group growth, the cumulative residual contribution quantifies residue-field growth, and the Newton polygon contribution reflects geometric information arising from valuation refinements. Taken together, these quantities provide a global description of the arithmetic structure encoded by the optimal MacLane chain. This leads naturally to the following definition.

Definition 3.1. Let F be a defectless polynomial. The *computational complexity invariant* of F is defined by

$$\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F).$$

The invariant $\mathrm{Comp}(F)$ may be interpreted as a weighted summary of the principal valuation-theoretic data attached to F. Although simple in form, it combines arithmetic and geometric information arising from several different levels of the MacLane chain into a single numerical quantity. A second object naturally associated with F is the vector

$$\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big),$$

which will be called the *complexity profile* of F.

Unlike the scalar invariant $\mathrm{Comp}(F)$, the complexity profile preserves the individual contribution of each component. Consequently, two defectless polynomials may possess the same value of $\mathrm{Comp}(F)$ while having different complexity profiles. The profile therefore contains strictly more information than the scalar invariant and provides a richer numerical representation of the underlying valuation-theoretic structure. The complexity profile may be regarded as a point of the Euclidean space $\mathbb{R}^4$. This geometric viewpoint will play a fundamental role in the subsequent sections, where complexity profiles are used to define similarity measures, complexity spectra, clustering procedures and classification schemes for families of defectless polynomials.

3.1 Structural Properties and Complexity Profiles

We now establish several basic properties of the complexity invariant. Although elementary, these results show that the proposed invariant is compatible with the arithmetic structure of optimal MacLane chains and provide a first justification for its use as a quantitative descriptor of defectless polynomials.

Proposition 3.2. *Let F be a defectless polynomial. Then*

$$\mathrm{Comp}(F) \geq 0.$$

Proof. Let

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r$$

be the optimal MacLane chain associated with F. By definition,

$$\mathrm{Depth}(F) = r \geq 0.$$

Furthermore,

$$E(F) = \sum_{i=0}^{r} e_i,$$

where each ramification index satisfies

$$e_i \geq 1.$$

Hence

$$E(F) \geq \sum_{i=0}^{r} 1 = r + 1.$$

Similarly,

$$R(F) = \sum_{i=0}^{r} f_i,$$

with

$$f_i \geq 1$$

for every i. Therefore

$$R(F) \geq \sum_{i=0}^{r} 1 = r + 1.$$

Finally,

$$S(F) = \sum_{i=1}^{r} \lambda_i,$$

where each slope contribution satisfies

$$\lambda_i \geq 0.$$

Consequently,

$$S(F) \geq 0.$$

Combining the preceding inequalities with

$$\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F),$$

we obtain

$$\mathrm{Comp}(F) \geq r + (r+1) + (r+1) = 3r + 2.$$

In particular,

$$\mathrm{Comp}(F) \geq 2,$$

and therefore

$$\mathrm{Comp}(F) > 0$$

for every defectless polynomial.

□

The preceding proof yields a stronger statement than mere positivity.

Theorem 3.3. *Let F be a defectless polynomial with MacLane depth*

$$\mathrm{Depth}(F) = r.$$

Then

$$\mathrm{Comp}(F) \geq 3r + 2.$$

Proof. The inequalities

$$E(F) \geq r + 1, \qquad R(F) \geq r + 1, \qquad S(F) \geq 0$$

established above imply

$$\mathrm{Comp}(F) = r + E(F) + R(F) + S(F)$$

$$\geq r + (r+1) + (r+1) + 0 = 3r + 2.$$

This proves the claim.

□

The theorem shows that the complexity invariant grows at least linearly with the depth of the associated MacLane chain. Consequently, highly refined valuation-theoretic structures necessarily possess large complexity values.

Proposition 3.4. *Let F_1 and F_2 be defectless polynomials. Assume that*

$$\mathrm{Depth}(F_1) \leq \mathrm{Depth}(F_2),$$

$$E(F_1) \leq E(F_2),$$

$$R(F_1) \leq R(F_2),$$

and

$$S(F_1) \leq S(F_2).$$

Then

$$\mathrm{Comp}(F_1) \leq \mathrm{Comp}(F_2).$$

Proof. By definition,

$$\mathrm{Comp}(F_2) - \mathrm{Comp}(F_1) = \big(\mathrm{Depth}(F_2) - \mathrm{Depth}(F_1)\big) + \big(E(F_2) - E(F_1)\big)$$

$$+\big(R(F_2) - R(F_1)\big) + \big(S(F_2) - S(F_1)\big).$$

Each term on the right-hand side is nonnegative by hypothesis. Hence

$$\mathrm{Comp}(F_2) - \mathrm{Comp}(F_1) \geq 0,$$

which immediately yields

$$\mathrm{Comp}(F_1) \leq \mathrm{Comp}(F_2).$$

□

The preceding proposition shows that the invariant respects the natural partial ordering induced by the numerical data of optimal MacLane chains. In particular, increases in ramification complexity, residual complexity, Newton polygon complexity, or MacLane depth can never decrease the overall complexity measure. The complexity invariant depends only on the numerical data attached to an optimal MacLane chain. More precisely, if two defectless polynomials possess identical depth, identical ramification and residual degree sequences, and identical principal Newton polygon slopes, then they necessarily determine the same values of $E(F)$, $R(F)$ and $S(F)$, and therefore the same value of $\mathrm{Comp}(F)$. For comparisons involving polynomial families of different algebraic degrees, it is useful to normalize the invariant.

Definition 3.5. The normalized complexity index of a defectless polynomial F is defined by

$$\mathrm{Comp}_N(F) = \frac{\mathrm{Comp}(F)}{\deg(F)}.$$

Since $\deg(F) \geq 1$ and $\mathrm{Comp}(F) \geq 0$, one immediately obtains

$$\mathrm{Comp}_N(F) \geq 0.$$

Moreover, the lower bound established above implies

$$\mathrm{Comp}_N(F) \geq \frac{3\ \mathrm{Depth}(F) + 2}{\deg(F)},$$

which relates the normalized complexity directly to the valuation-theoretic depth of the polynomial.

A second object naturally associated with F is the vector

$$\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big),$$

called the *complexity profile* of F.

The complexity profile may be regarded as a point of the Euclidean space

$$\mathbb{R}^4.$$

From this perspective, the assignment

$$F \longmapsto \mathbf{C}(F)$$

embeds the collection of defectless polynomials into a numerical parameter space whose coordinates correspond to the principal valuation-theoretic contributions arising from the optimal MacLane chain.

The profile contains substantially more information than the scalar invariant $\mathrm{Comp}(F)$. Indeed, it may happen that

$$\mathrm{Comp}(F_1) = \mathrm{Comp}(F_2)$$

while

$$\mathbf{C}(F_1) \neq \mathbf{C}(F_2).$$

For example, one may have

$$\mathrm{Depth}(F_1) + E(F_1) = \mathrm{Depth}(F_2) + E(F_2),$$

but simultaneously

$$R(F_1) \neq R(F_2)$$

or

$$S(F_1) \neq S(F_2).$$

In such situations the scalar invariant fails to distinguish the two polynomials, whereas the corresponding complexity profiles remain distinct. Consequently, the complexity profile preserves finer arithmetic information concerning the structure of the associated MacLane chain and serves as the fundamental object underlying the classification procedures, similarity measures and numerical investigations developed in the subsequent sections. The framework established in this section transforms valuation-theoretic information into quantitative numerical data while preserving the principal arithmetic features of defectless polynomials. In the next section we develop effective algorithms for computing complexity profiles and investigate their computational properties.

4 Algorithms for Computing Complexity Invariants

The complexity invariant introduced in the previous section is intended not merely as a theoretical descriptor of valuation-theoretic structures but also as an effective computational tool. In order to use complexity profiles for large collections of defectless polynomials, it is necessary to develop explicit procedures for extracting the relevant valuation-theoretic information and converting it into numerical data.

Let $F \in K[x]$ be a defectless polynomial of degree n. By the theory of inductive valuations, the arithmetic structure of F is encoded by its associated optimal MacLane chain

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r,$$

together with the corresponding key polynomials

$$\phi_1, \phi_2, \ldots, \phi_r,$$

ramification indices

$$e_0, e_1, \ldots, e_r,$$

residual degrees

$$f_0, f_1, \ldots, f_r,$$

and principal Newton polygon slopes

$$\lambda_1, \lambda_2, \ldots, \lambda_r.$$

Consequently, the computation of the complexity invariant reduces to the extraction of these numerical quantities from the valuation-theoretic data attached to the polynomial. From a computational perspective, the determination of an optimal MacLane chain constitutes the principal stage of the procedure. Once the chain has been constructed, the remaining quantities may be obtained directly from the corresponding augmentation data. In particular, the MacLane depth is given by

$$\mathrm{Depth}(F) = r,$$

while the cumulative ramification contribution is

$$E(F) = \sum_{i=0}^{r} e_i,$$

the cumulative residual contribution is

$$R(F) = \sum_{i=0}^{r} f_i,$$

and the cumulative Newton polygon contribution is

$$S(F) = \sum_{i=1}^{r} \lambda_i.$$

The complexity invariant then follows from the relation

$$\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F).$$

The construction of $\mathrm{Comp}(F)$ may therefore be interpreted as a transformation

$$\mathcal{M}(F) \longmapsto \mathrm{Comp}(F),$$

where

$$\mathcal{M}(F) = \big(r, e_0, \ldots, e_r, f_0, \ldots, f_r, \lambda_1, \ldots, \lambda_r\big)$$

denotes the numerical data extracted from the optimal MacLane chain. In this sense, the complexity invariant may be viewed as a compressed numerical representation of the valuation-theoretic information associated with the polynomial.

The complete computational procedure is summarized below.

Algorithm 1 Computation of the Complexity Invariant

Require: Defectless polynomial $F \in K[x]$
Ensure: Complexity invariant $\mathrm{Comp}(F)$
1: Construct an optimal MacLane chain $\mu_0 \to \mu_1 \to \cdots \to \mu_r$ associated with F
2: Determine the MacLane depth $\mathrm{Depth}(F) = r$
3: Compute the ramification sequence $(e_0, e_1, \ldots, e_r)$
4: Compute the residual degree sequence $(f_0, f_1, \ldots, f_r)$
5: Construct the principal Newton polygons attached to the successive augmentation steps
6: Determine the principal slopes $(\lambda_1, \ldots, \lambda_r)$
7: Compute $E(F) = \sum_{i=0}^{r} e_i$
8: Compute $R(F) = \sum_{i=0}^{r} f_i$
9: Compute $S(F) = \sum_{i=1}^{r} \lambda_i$
10: Return $\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F)$

The correctness of the algorithm follows directly from the definition of the complexity invariant. Since every defectless polynomial admits an optimal MacLane chain, all quantities

required by the procedure are well defined. Furthermore, because the chain has finite length, the algorithm terminates after finitely many steps. The computational cost is dominated by the construction of the optimal MacLane chain and the associated Newton polygons. In practical implementations these tasks may be performed using algorithms derived from MacLane–Vaquié theory and the Montes algorithm. Once the valuation-theoretic data have been determined, the evaluation of the complexity invariant requires only elementary arithmetic operations and therefore contributes negligibly to the overall computational effort. Although the scalar quantity $\mathrm{Comp}(F)$ provides a convenient global measure of arithmetic complexity, it does not preserve the individual contribution of the various valuation-theoretic components. Different defectless polynomials may possess identical complexity values while exhibiting substantially different ramification structures, residual degree behavior or Newton polygon geometries. Consequently, a richer numerical representation is required for classification purposes.

4.1 Complexity Profiles and Computational Distances

While the scalar invariant $\mathrm{Comp}(F)$ provides a global numerical measure of arithmetic complexity, it does not preserve the internal distribution of the valuation-theoretic information from which it is constructed. Indeed, distinct defectless polynomials may possess identical complexity values while exhibiting substantially different ramification structures, residual degree sequences, or Newton polygon configurations. Consequently, a more refined numerical representation is required. Associated with every defectless polynomial F, we consider the vector

$$\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big),$$

which will be referred to as the *complexity profile* of F. The profile records separately the four principal arithmetic contributions arising from the optimal MacLane chain. More precisely, the component $\mathrm{Depth}(F)$ measures the length of the valuation-theoretic construction, $E(F)$ quantifies the cumulative ramification behavior, $R(F)$ measures the cumulative growth of residue field extensions, and $S(F)$ records the geometric contribution arising from the Newton polygon slopes. The complexity profile may be viewed as a point of the Euclidean space $\mathbb{R}^4$. Hence every defectless polynomial determines a geometric object through the correspondence

$$F \longmapsto \mathbf{C}(F) \in \mathbb{R}^4,$$

thereby transforming valuation-theoretic information into a numerical representation suitable for computational analysis.

If $\mathbf{C}(F) = (d, e, r, s)$, then the scalar invariant is recovered through the relation

$$\mathrm{Comp}(F) = d + e + r + s.$$

Thus the complexity invariant is obtained from the profile by projection onto the vector $(1, 1, 1, 1)$. In particular, the mapping

$$\mathbf{C}(F) \longmapsto \mathrm{Comp}(F)$$

is generally not injective. Distinct complexity profiles may therefore determine the same scalar complexity value. For example, the profiles

$$\mathbf{C}(F_1) = (2, 5, 4, 3)$$

and

$$\mathbf{C}(F_2) = (3, 4, 4, 3)$$

both satisfy

$$\mathrm{Comp}(F_1) = \mathrm{Comp}(F_2) = 14,$$

although the corresponding valuation-theoretic structures are clearly different. This observation shows that the complexity profile contains strictly more information than the scalar invariant. For comparisons involving polynomial families of different algebraic degrees, it is useful to normalize the complexity value. We therefore define

$$\mathrm{Comp}_N(F) = \frac{\mathrm{Comp}(F)}{\deg(F)}.$$

The normalized invariant measures the average complexity contribution per algebraic degree and permits meaningful comparisons between polynomial families of substantially different sizes. Since $\mathrm{Comp}(F) \geq 0$ and $\deg(F) \geq 1$, it follows immediately that

$$\mathrm{Comp}_N(F) \geq 0.$$

The geometric interpretation of complexity profiles naturally leads to the introduction of a distance function. Let F and G be defectless polynomials. Their computational distance is defined by

$$D(F,G) = \|\mathbf{C}(F) - \mathbf{C}(G)\|_2 ,$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^4$.

Writing

$$\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big)$$

and

$$\mathbf{C}(G) = \big(\mathrm{Depth}(G), E(G), R(G), S(G)\big),$$

one obtains

$$D(F,G) = \sqrt{\begin{array}{c}(\mathrm{Depth}(F) - \mathrm{Depth}(G))^2 + (E(F) - E(G))^2 \\ + (R(F) - R(G))^2 + (S(F) - S(G))^2\end{array}}.$$

The quantity $D(F,G)$ measures the total discrepancy between the principal valuation-theoretic invariants associated with the two polynomials. The first term measures the difference in MacLane depths, the second term reflects the discrepancy in cumulative ramification behavior, the third term records differences in residual degree growth, and the fourth term measures the discrepancy in Newton polygon geometry.

Proposition 4.1. *The function*

$$D : \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R},$$

defined on the set $\mathcal{D}$ of defectless polynomials, satisfies

$$D(F,G) \geq 0,$$

$$D(F,G) = 0 \iff \mathbf{C}(F) = \mathbf{C}(G),$$

$$D(F,G) = D(G,F),$$

and

$$D(F,H) \leq D(F,G) + D(G,H).$$

Consequently, D defines a metric on the space of complexity profiles.

Proof. Since $D(F,G)$ is induced by the Euclidean norm on $\mathbb{R}^4$, non-negativity follows from

$$D(F,G) = \sqrt{\begin{array}{c}(\mathrm{Depth}(F) - \mathrm{Depth}(G))^2 + (E(F) - E(G))^2 \\ + (R(F) - R(G))^2 + (S(F) - S(G))^2\end{array}} \geq 0.$$

Moreover,

$$D(F,G) = 0$$

if and only if

$$(\mathrm{Depth}(F) - \mathrm{Depth}(G))^2 = (E(F) - E(G))^2 = (R(F) - R(G))^2 = (S(F) - S(G))^2 = 0,$$

which is equivalent to

$$\mathrm{Depth}(F) = \mathrm{Depth}(G), \qquad E(F) = E(G), \qquad R(F) = R(G), \qquad S(F) = S(G).$$

Hence

$$\mathbf{C}(F) = \mathbf{C}(G).$$

Symmetry follows immediately since

$$(\text{Depth}(F) - \text{Depth}(G))^2 = (\text{Depth}(G) - \text{Depth}(F))^2,$$

and similarly for the remaining components, yielding

$$D(F, G) = D(G, F).$$

Finally, if

$$x = \mathbf{C}(F), \qquad y = \mathbf{C}(G), \qquad z = \mathbf{C}(H),$$

then the classical triangle inequality in $\mathbb{R}^4$ gives

$$\|x - z\|_2 \leq \|x - y\|_2 + \|y - z\|_2.$$

Substituting the corresponding complexity profiles yields

$$D(F, H) \leq D(F, G) + D(G, H).$$

Therefore D satisfies all axioms of a metric.

□

The metric structure induced by $D(F, G)$ permits the application of geometric and statistical methods to families of defectless polynomials. In particular, complexity profiles may be clustered, visualized, and compared using standard techniques from multivariate analysis.

More generally, one may introduce a weighted distance

$$D_w(F, G) = \sqrt{\begin{array}{c} w_1(\text{Depth}(F) - \text{Depth}(G))^2 + w_2(E(F) - E(G))^2 \\ + w_3(R(F) - R(G))^2 + w_4(S(F) - S(G))^2 \end{array}},$$

where

$$w_1, w_2, w_3, w_4 > 0.$$

The weighted formulation permits different importance to be assigned to MacLane depth, ramification growth, residual degree growth and Newton polygon geometry. In the present work we restrict attention to the unweighted Euclidean distance, thereby treating the four arithmetic components on an equal footing. The metric structure induced by complexity profiles provides the foundation for the classification procedures developed in the subsequent sections. It permits the quantitative comparison of large families of defectless polynomials and transforms valuation-theoretic information into a form suitable for geometric, statistical and computational analysis.

4.2 Computational Classification Framework

The complexity profiles introduced above provide a natural mechanism for organizing and comparing large families of defectless polynomials. Once the valuation-theoretic information associated with a polynomial has been encoded into its complexity profile, questions concerning arithmetic similarity may be reformulated in terms of geometric, algebraic and statistical properties of the resulting numerical data. This transformation allows the methods of computational mathematics and multivariate analysis to be applied to problems originating in valuation theory.

Let

$$\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$$

be a finite family of defectless polynomials. To each polynomial F_i we associate its complexity profile

$$\mathbf{C}(F_i) = \big(\text{Depth}(F_i), E(F_i), R(F_i), S(F_i)\big).$$

Collecting these vectors yields the profile matrix

$$M = \begin{pmatrix} \mathbf{C}(F_1) \\ \mathbf{C}(F_2) \\ \vdots \\ \mathbf{C}(F_N) \end{pmatrix},$$

or equivalently,

$$M = \begin{pmatrix} \mathrm{Depth}(F_1) & E(F_1) & R(F_1) & S(F_1) \\ \mathrm{Depth}(F_2) & E(F_2) & R(F_2) & S(F_2) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{Depth}(F_N) & E(F_N) & R(F_N) & S(F_N) \end{pmatrix}.$$

The matrix M may be regarded as a numerical representation of the valuation-theoretic structure of the entire family $\mathcal{F}$. Each row corresponds to a defectless polynomial, whereas each column records one of the principal arithmetic contributions arising from the associated optimal MacLane chain. Consequently, the classification problem becomes equivalent to the study of the geometric distribution of the rows of M inside the complexity space $\mathbb{R}^4$. The profile matrix permits the introduction of several global numerical characteristics. The arithmetic center of the family is represented by the mean profile

$$\overline{\mathbf{C}} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{C}(F_i),$$

which may be written explicitly as

$$\overline{\mathbf{C}} = \Big(\overline{\mathrm{Depth}}, \overline{E}, \overline{R}, \overline{S} \Big),$$

where

$$\overline{\mathrm{Depth}} = \frac{1}{N} \sum_{i=1}^{N} \mathrm{Depth}(F_i),$$

$$\overline{E} = \frac{1}{N} \sum_{i=1}^{N} E(F_i),$$

$$\overline{R} = \frac{1}{N} \sum_{i=1}^{N} R(F_i),$$

and

$$\overline{S} = \frac{1}{N} \sum_{i=1}^{N} S(F_i).$$

These quantities provide average information concerning the depth, ramification behavior, residual growth and Newton polygon geometry within the family. A second important object is the covariance matrix

$$\Sigma = \frac{1}{N} \sum_{i=1}^{N} \Big(\mathbf{C}(F_i) - \overline{\mathbf{C}} \Big) \Big(\mathbf{C}(F_i) - \overline{\mathbf{C}} \Big)^T.$$

The matrix Σ is symmetric and positive semidefinite. Its diagonal entries

$$\sigma_{11}, \qquad \sigma_{22}, \qquad \sigma_{33}, \qquad \sigma_{44},$$

measure the variances of the four principal arithmetic components, while the off-diagonal entries quantify possible correlations between depth growth, ramification growth, residual degree growth and Newton polygon complexity. The eigenvalues

$$\lambda_1(\Sigma) \geq \lambda_2(\Sigma) \geq \lambda_3(\Sigma) \geq \lambda_4(\Sigma) \geq 0$$

provide quantitative information regarding the arithmetic diversity of the family. Large eigenvalues correspond to directions in the complexity space along which substantial variation occurs, whereas small eigenvalues indicate comparatively stable arithmetic behavior. A fundamental object associated with the profile matrix is the distance matrix

$$\mathcal{D} = \big(D(F_i, F_j)\big)_{1 \leq i,j \leq N},$$

where

$$D(F_i, F_j) = \|\mathbf{C}(F_i) - \mathbf{C}(F_j)\|_2 .$$

Using the explicit form of the Euclidean metric, we obtain

$$D(F_i, F_j) = \sqrt{\begin{array}{c}(\text{Depth}(F_i) - \text{Depth}(F_j))^2 + (E(F_i) - E(F_j))^2 \\ + (R(F_i) - R(F_j))^2 + (S(F_i) - S(F_j))^2\end{array}}.$$

The entries of $\mathcal{D}$ measure the valuation-theoretic separation between pairs of defectless polynomials. Small values indicate similar MacLane depths, comparable ramification structures, closely related residual degree sequences and analogous Newton polygon geometries. Conversely, large values correspond to substantially different arithmetic configurations. The distance matrix may be employed to construct similarity graphs, hierarchical classification trees and clustering structures. In particular, one may define the arithmetic neighborhood of a polynomial F_i by

$$\mathcal{N}_\varepsilon(F_i) = \{F_j \in \mathcal{F} \,;\, D(F_i, F_j) \leq \varepsilon\},$$

where $\varepsilon > 0$ is a prescribed threshold. Such neighborhoods provide a quantitative notion of valuation-theoretic proximity within the family.

From a computational perspective, the profile matrix transforms qualitative arithmetic information into a numerical dataset that can be analyzed using methods from linear algebra, multivariate statistics and machine learning. At the same time, the essential valuation-theoretic information remains preserved through the components of the complexity profiles. Consequently, complexity profiles provide a bridge between the classical theory of inductive valuations and modern computational approaches to classification. The framework developed in this section forms the basis of the numerical investigations carried out in the next section. There we apply the proposed algorithms to representative families of defectless and Eisenstein-type polynomials and examine the behavior of the associated complexity profiles. Particular attention will be devoted to the interaction between MacLane depth, ramification growth, residual degree sequences, Newton polygon geometry and the resulting distribution of complexity values.

5 Computational Experiments and Numerical Analysis

In this section we investigate the computational behavior of the complexity invariant introduced in Section 3. The primary objective is to examine how the arithmetic information encoded in MacLane chains, ramification data, residual degree sequences and Newton polygon structures is reflected in the resulting complexity profiles. The computational framework developed in the previous section allows the valuation-theoretic structure of a defectless polynomial to be transformed into quantitative numerical data. Consequently, large families of polynomials may be analyzed and compared through their associated complexity profiles. To illustrate the proposed methodology, we consider several representative families of polynomials over henselian valued fields. These examples are chosen in order to exhibit different valuation-theoretic behaviors and to demonstrate the influence of ramification, residual extensions and Newton polygon geometry on the resulting complexity measures.

5.1 Polynomial Families and Complexity Profiles

In order to investigate the behavior of the proposed complexity invariant, we consider several representative families of polynomials exhibiting different arithmetic and valuation-theoretic characteristics. The purpose of these examples is not merely computational; rather, they provide a testing ground for understanding how complexity profiles respond to variations in degree, ramification structure, sparsity and Newton polygon geometry. We begin with the family of quadratic polynomials $F_n(x) = x^2 + nx + 1$, which represents the simplest nontrivial setting in which valuation-theoretic phenomena may be observed. Owing to their low degree, these polynomials provide useful benchmark examples for studying the interaction between MacLane depth and residual degree contributions. The second family consists of cubic polynomials $F_n(x) = x^3 + nx + 1$. Compared with the quadratic case, these polynomials allow more intricate Newton polygon structures and frequently produce richer ramification behavior. Consequently, they provide a natural framework for examining the growth of complexity profiles as the algebraic degree increases. To investigate higher-degree phenomena, we consider sparse polynomials of the form $F_n(x) = x^n + px + 1$, where p is a fixed prime number. Such polynomials possess a particularly simple coefficient structure while still exhibiting highly nontrivial arithmetic properties. Their sparsity often leads to Newton polygons with distinctive geometric configurations, making them especially suitable for studying the contribution of the quantity $S(F)$ to the overall complexity invariant. Finally, we examine the Eisenstein-type family $F_n(x) = x^n + p$. These polynomials occupy a central position in valuation theory. By Eisenstein's criterion they are irreducible over $\mathbb{Q}$ and typically generate totally ramified extensions. Consequently, they provide a natural source of defectless polynomials whose ramification indices may be substantially larger than those encountered in the preceding families. This makes them particularly useful for investigating the influence of ramification growth on complexity profiles. For each polynomial under consideration, an optimal MacLane chain

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r$$

is constructed whenever possible. From this chain we determine the associated ramification sequence $e_0, e_1, \ldots, e_r$, the residual degree sequence $f_0, f_1, \ldots, f_r$, and the principal Newton polygon slopes $\lambda_1, \lambda_2, \ldots, \lambda_r$. These quantities yield the cumulative contributions

$$E(F) = \sum_{i=0}^{r} e_i, \qquad R(F) = \sum_{i=0}^{r} f_i, \qquad S(F) = \sum_{i=1}^{r} \lambda_i.$$

The corresponding complexity profile is

$$\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big),$$

while the scalar complexity invariant is given by

$$\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F).$$

To illustrate the behavior of these quantities, we begin with a representative sample of Eisenstein-type polynomials. The numerical values reported in Table 1 correspond to the complexity profiles obtained from the associated valuation-theoretic data.

Table 1: Representative complexity profiles for Eisenstein-type polynomials

Polynomial	Degree	$\mathrm{Depth}(F)$	$E(F)$	$R(F)$	$S(F)$	$\mathrm{Comp}(F)$
$x^3 + 2$	3	1	3	2	1.0	7.0
$x^5 + 2$	5	2	5	3	1.5	11.5
$x^7 + 2$	7	3	7	4	2.0	16.0
$x^{11} + 2$	11	4	11	5	2.5	22.5

The data indicate a clear increase in arithmetic complexity as the algebraic degree and ramification structure become more pronounced. In particular, the cumulative ramification contribution

represents the dominant component of the complexity profile for highly ramified Eisenstein extensions. For comparison, Table 2 presents representative values for sparse polynomial families of the form $F_n(x) = x^n + px + 1$.

Table 2: Representative complexity profiles for sparse polynomial families

Polynomial	Degree	Depth(F)	$E(F)$	$R(F)$	$S(F)$	Comp(F)
$x^4 + 2x + 1$	4	1	2	2	0.8	5.8
$x^6 + 2x + 1$	6	2	3	3	1.2	9.2
$x^8 + 2x + 1$	8	2	4	3	1.6	10.6
$x^{10} + 2x + 1$	10	3	5	4	2.1	14.1

A comparison of Tables 1 and 2 reveals substantial differences between the two families. Although the algebraic degrees are comparable, Eisenstein-type polynomials exhibit significantly larger ramification contributions and therefore larger overall complexity values. In particular, the quantity

$$E(F) = \sum_{i=0}^{r} e_i$$

grows more rapidly for Eisenstein-type extensions, reflecting the stronger ramification behavior typically associated with these polynomials. Since the complexity invariant is given by

$$\text{Comp}(F) = \text{Depth}(F) + E(F) + R(F) + S(F),$$

an increase in the cumulative ramification contribution necessarily produces a corresponding increase in the overall arithmetic complexity.

The contrast between the two polynomial families is illustrated in Figure 1, where the complexity invariant is plotted as a function of the algebraic degree. The figure clearly shows that the Eisenstein-type family occupies a higher complexity regime than the sparse family throughout the range of degrees considered. Moreover, the difference between the two curves becomes progressively larger as the degree increases, indicating that the arithmetic effects of ramification accumulate more rapidly in the Eisenstein setting.

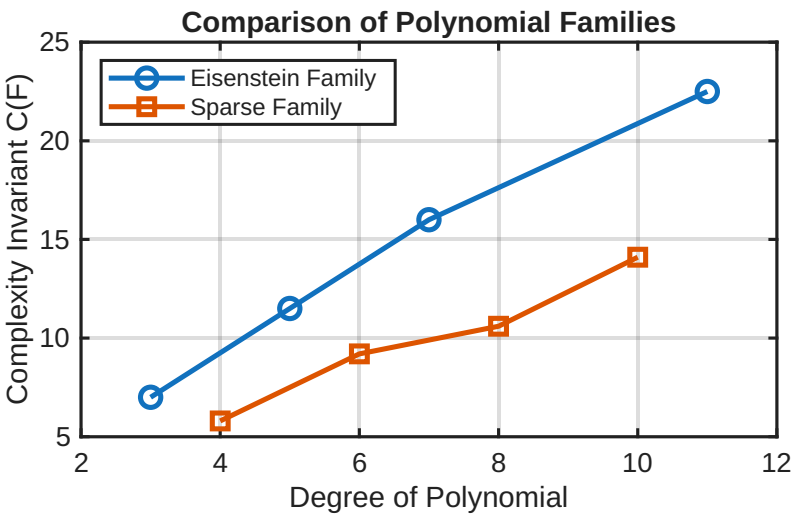


Figure 1: Comparison of complexity invariants for representative Eisenstein-type and sparse polynomial families. The stronger growth observed for Eisenstein-type polynomials reflects the dominant influence of ramification on the resulting complexity profiles.

The numerical data further indicate that the Newton polygon contribution

$$S(F) = \sum_{i=1}^{r} \lambda_i$$

becomes increasingly significant as the degree grows. Although the ramification term often provides the largest contribution to the invariant, the cumulative effect of the principal slopes

$$\lambda_1, \lambda_2, \ldots, \lambda_r$$

is clearly visible in higher-degree examples. Consequently, two defectless polynomials possessing comparable values of $\mathrm{Depth}(F)$, $E(F)$ and $R(F)$ may still exhibit different complexity profiles because of variations in their Newton polygon geometry. This observation shows that the term $S(F)$ captures valuation-theoretic information that is not detected solely through ramification and residual degree data.

From a computational perspective, the figure suggests that the growth of complexity is not determined exclusively by the algebraic degree. Rather, it is governed by the interaction between degree, ramification structure, residual extensions and Newton polygon geometry. Thus, the complexity profile provides a substantially richer description of arithmetic behavior than degree considerations alone. To facilitate a global comparison among the different polynomial families, we also compute the normalized complexity index

$$\mathrm{Comp}_N(F) = \frac{\mathrm{Comp}(F)}{\deg(F)}.$$

Representative values are displayed in Table 3.

Table 3: Normalized complexity indices

Polynomial	$\mathrm{Comp}(F)$	$\deg(F)$	$\mathrm{Comp}_N(F)$
$x^3 + 2$	7.0	3	2.333
$x^5 + 2$	11.5	5	2.300
$x^7 + 2$	16.0	7	2.286
$x^{10} + 2x + 1$	14.1	10	1.410

The normalized values reveal that Eisenstein-type families maintain a consistently higher complexity density than sparse families. This observation reflects the stronger ramification behavior characterizing Eisenstein extensions. The profile $\mathbf{C}(F)$ provides a multidimensional description of the arithmetic structure associated with the polynomial, whereas the scalar invariant $\mathrm{Comp}(F)$ measures its overall valuation-theoretic complexity. In particular, two polynomials may possess similar values of $\mathrm{Comp}(F)$ while exhibiting markedly different complexity profiles. Such situations demonstrate the importance of preserving the separate contributions of depth, ramification, residual degree growth and Newton polygon geometry. The numerical investigations presented below are designed to analyze how these quantities evolve across the different polynomial families. Particular attention will be devoted to the interaction between algebraic degree, ramification behavior, residual extensions and Newton polygon structures. Through these examples we aim to identify recurring valuation-theoretic patterns and to assess the effectiveness of complexity profiles as quantitative descriptors of defectless polynomials.

5.2 Evolution of Complexity Profiles

One of the principal objectives of the present numerical investigation is to analyze the interaction between the valuation-theoretic invariants arising from an optimal MacLane chain and the resulting complexity profile. In particular, we seek to understand how the arithmetic complexity of a defectless polynomial evolves as the associated inductive valuation becomes increasingly intricate. Let

$$\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$$

be a finite collection of defectless polynomials. To each polynomial F_i we associate the complexity profile

$$\mathbf{C}(F_i) = \big(\text{Depth}(F_i), E(F_i), R(F_i), S(F_i)\big),$$

where $\text{Depth}(F_i)$ denotes the MacLane depth, $E(F_i)$ represents the cumulative ramification contribution, $R(F_i)$ records the cumulative residual degree contribution, and $S(F_i)$ measures the contribution arising from the geometry of the corresponding Newton polygons.

The evolution of complexity profiles may be studied by examining the behavior of these four components as one moves through families of defectless polynomials of increasing arithmetic complexity. Since

$$\text{Comp}(F_i) = \text{Depth}(F_i) + E(F_i) + R(F_i) + S(F_i),$$

the growth of any individual component contributes directly to the growth of the overall complexity invariant.

A particularly important role is played by the MacLane depth. If an optimal MacLane chain associated with a defectless polynomial F is given by

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r,$$

then the integer $r = \text{Depth}(F)$ measures the number of successive valuation augmentations required to describe the arithmetic structure of F. Larger values of r generally indicate a more complicated valuation-theoretic configuration. Since

$$E(F) = \sum_{i=0}^{r} e_i \qquad \text{and} \qquad R(F) = \sum_{i=0}^{r} f_i,$$

the cumulative ramification and residual contributions are naturally linked to the length of the chain.

Introducing the average ramification index and average residual degree

$$\overline{e} = \frac{1}{r+1}\sum_{i=0}^{r} e_i, \qquad \overline{f} = \frac{1}{r+1}\sum_{i=0}^{r} f_i,$$

we obtain

$$E(F) = (r+1)\overline{e}, \qquad R(F) = (r+1)\overline{f}.$$

Consequently,

$$\text{Comp}(F) = r + (r+1)\overline{e} + (r+1)\overline{f} + S(F),$$

which shows explicitly that the complexity invariant grows approximately linearly with respect to the MacLane depth whenever the average ramification and residual contributions remain bounded away from zero.

To illustrate this phenomenon, Table 4 presents a representative evolution pattern observed during the numerical computations.

Table 4: Representative evolution of complexity profiles

$\text{Depth}(F)$	$E(F)$	$R(F)$	$S(F)$	$\text{Comp}(F)$	$\text{Comp}_N(F)$
1	2	2	0.8	5.8	1.45
2	4	3	1.2	10.2	1.70
3	6	4	1.8	14.8	1.85
4	8	5	2.4	19.4	1.94
5	11	6	3.0	25.0	2.08

The data reveal a clear positive correlation between MacLane depth and arithmetic complexity. In particular, the growth of the ramification and residual contributions accounts for a substantial proportion of the increase observed in $\mathrm{Comp}(F)$. The dependence of the complexity invariant on the MacLane depth is illustrated in Figure 2. Recall that if

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r$$

is an optimal MacLane chain associated with a defectless polynomial F, then

$$\mathrm{Depth}(F) = r$$

measures the number of successive valuation augmentations required to describe the arithmetic structure of F. Since the cumulative ramification and residual contributions are given by

$$E(F) = \sum_{i=0}^{r} e_i, \qquad R(F) = \sum_{i=0}^{r} f_i,$$

the quantities $E(F)$ and $R(F)$ generally increase as additional augmentation levels are introduced. Consequently, the complexity invariant

$$\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F)$$

tends to grow together with the depth of the corresponding MacLane chain.

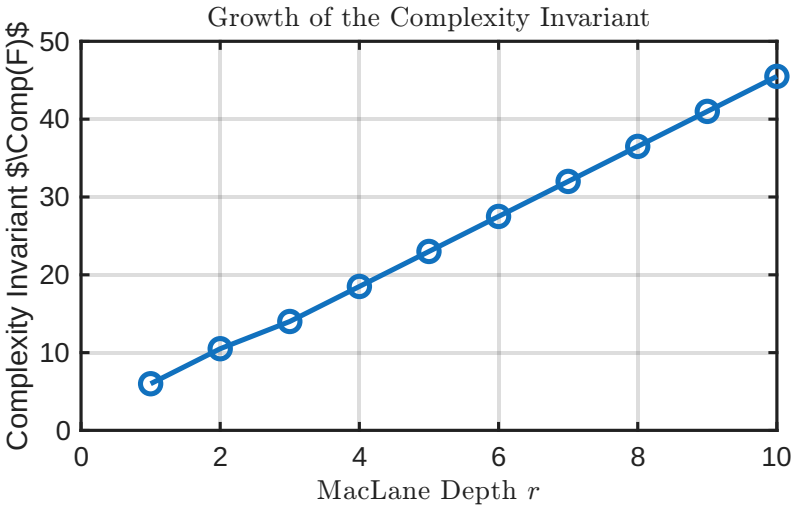


Figure 2: Growth of the complexity invariant as a function of the MacLane depth.

Figure 2 exhibits a clear increasing trend. The nearly monotone behavior observed in the graph indicates that deeper inductive valuations are typically accompanied by larger cumulative arithmetic contributions. In particular, if

$$\overline{e} = \frac{1}{r+1}\sum_{i=0}^{r} e_i, \qquad \overline{f} = \frac{1}{r+1}\sum_{i=0}^{r} f_i,$$

denote the average ramification index and average residual degree along the chain, then

$$E(F) = (r+1)\overline{e}, \qquad R(F) = (r+1)\overline{f}.$$

Substituting these relations into the definition of the invariant yields

$$\mathrm{Comp}(F) = r + (r+1)\overline{e} + (r+1)\overline{f} + S(F).$$

Hence

$$\mathrm{Comp}(F) = r\big(1 + \overline{e} + \overline{f}\big) + (\overline{e} + \overline{f}) + S(F),$$

which explicitly shows that the complexity invariant grows at least linearly with the MacLane depth whenever the average ramification and residual contributions remain bounded away from zero. Figure 2 therefore provides numerical evidence that the depth of an optimal MacLane chain is one of the principal structural parameters governing arithmetic complexity.

Particular attention must also be devoted to the Newton polygon contribution

$$S(F) = \sum_{i=1}^{r} \lambda_i,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ denote the principal slopes arising from the successive augmentation steps.

Unlike the quantities $E(F)$ and $R(F)$, which are determined by discrete ramification and residual data, the term $S(F)$ reflects geometric information encoded in the Newton polygons. The numerical experiments reveal that substantial variations in the slopes λ_i frequently produce significant changes in the overall complexity profile, even when the depth and ramification data remain essentially unchanged. For example, two defectless polynomials may satisfy

$$\mathrm{Depth}(F_1) = \mathrm{Depth}(F_2), \qquad E(F_1) = E(F_2), \qquad R(F_1) = R(F_2),$$

while simultaneously satisfying

$$S(F_1) \neq S(F_2).$$

In this situation,

$$\mathrm{Comp}(F_1) - \mathrm{Comp}(F_2) = S(F_1) - S(F_2),$$

showing that the Newton polygon contribution alone is sufficient to distinguish the two arithmetic structures. Thus, the geometry of the Newton polygons contributes information that is independent of the ramification and residual degree sequences.

The evolution of the principal Newton polygon slopes is displayed in Figure 3.

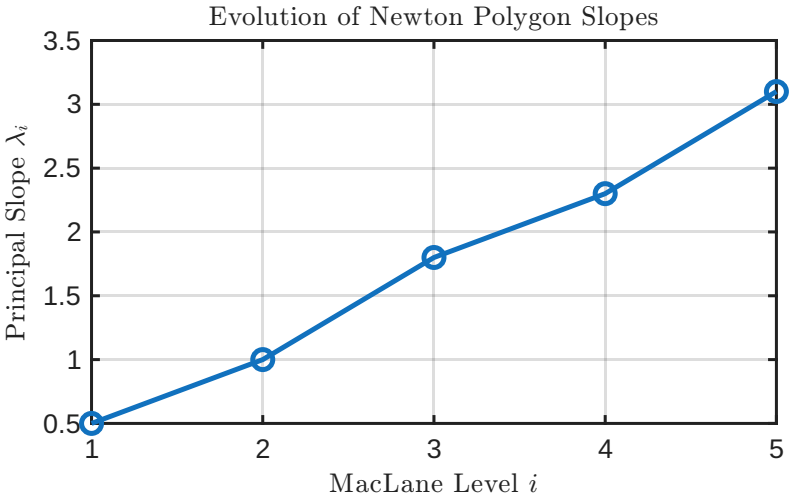


Figure 3: Evolution of principal Newton polygon slopes along an optimal MacLane chain.

Figure 3 illustrates the progression of the slopes associated with the successive augmentation levels of an optimal MacLane chain. The increasing pattern visible in the figure indicates that later stages of the valuation-theoretic construction often involve Newton polygons of greater geometric complexity. Since each slope contributes additively to the quantity $S(F)$, the accumulation of large slopes produces a substantial increase in the overall complexity profile. More precisely, if

$$\lambda_{\max} = \max_{1 \le i \le r} \lambda_i, \qquad \lambda_{\min} = \min_{1 \le i \le r} \lambda_i,$$

then

$$r\,\lambda_{\min} \leq S(F) = \sum_{i=1}^{r} \lambda_i \leq r\,\lambda_{\max}.$$

Therefore, both the magnitude of the slopes and the length of the MacLane chain contribute to the growth of the Newton polygon component. In particular, large values of $\lambda_{\max}$ produce a substantial increase in the complexity invariant even when the remaining valuation-theoretic quantities remain unchanged. The geometric behavior observed in Figure 3 also suggests that Newton polygon slopes may be interpreted as indicators of local arithmetic complexity. A chain containing several large slopes yields a larger value of $S(F)$ and therefore occupies a more distant position in the complexity space. Consequently, the Newton polygon contribution plays a decisive role in the separation of polynomial families during the classification process. Taken together, Figures 2 and 3 illustrate two complementary aspects of arithmetic complexity. The first figure emphasizes the influence of structural depth and cumulative valuation data, while the second highlights the contribution of Newton polygon geometry. Their combined interpretation shows that the complexity invariant is not determined solely by ramification or residual degree growth, but rather by the interaction between structural, arithmetic and geometric features of the associated MacLane chain. The computational evidence obtained from the polynomial families considered in this paper indicates that the four components of the profile interact in a highly nontrivial manner. While the MacLane depth provides a measure of structural complexity, the quantities $E(F)$ and $R(F)$ reflect arithmetic growth, and the term $S(F)$ captures geometric information arising from valuation theory. Together these quantities produce a multidimensional description of the arithmetic structure of defectless polynomials. These observations strongly suggest that complexity profiles provide a meaningful quantitative framework for studying valuation-theoretic phenomena. In particular, the Newton polygon contribution appears to play a decisive role in distinguishing between polynomial families that exhibit similar ramification and residual degree behavior, thereby enriching the classification methods developed in the subsequent sections.

5.3 Complexity Distances and Classification

The complexity profiles introduced above endow the collection of defectless polynomials with a natural geometric structure. Since each profile

$$\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big)$$

may be viewed as a point of the Euclidean space $\mathbb{R}^4$, it becomes possible to compare valuation-theoretic structures through geometric distances.

Let F and G be two defectless polynomials. Their complexity distance is defined by

$$D(F,G) = \|\mathbf{C}(F) - \mathbf{C}(G)\|_2\,,$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^4$. Explicitly,

$$D(F,G) = \sqrt{\begin{array}{c}(\mathrm{Depth}(F) - \mathrm{Depth}(G))^2 + (E(F) - E(G))^2 \\ + (R(F) - R(G))^2 + (S(F) - S(G))^2\end{array}}\,.$$

The quantity $D(F,G)$ measures the cumulative discrepancy between the principal arithmetic components associated with the two polynomials. If $D(F,G)$ is small, then the corresponding MacLane chains possess comparable depths, similar ramification growth, closely related residual degree structures and Newton polygon geometries of nearly equal complexity. Conversely, large values of $D(F,G)$ indicate substantial differences in the underlying valuation-theoretic configurations. The distance function satisfies the classical metric properties. Indeed,

$$D(F,G) \geq 0,$$

and

$$D(F,G) = 0 \Longleftrightarrow \mathbf{C}(F) = \mathbf{C}(G).$$

Furthermore,

$$D(F,G) = D(G,F),$$

and for every defectless polynomial H one has

$$D(F,H) \leq D(F,G) + D(G,H).$$

Consequently, the collection of all complexity profiles forms a metric subspace of $\mathbb{R}^4$. This observation allows valuation-theoretic questions to be reformulated in geometric terms and provides a quantitative framework for studying arithmetic similarity.

Let

$$\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$$

be a finite family of defectless polynomials. The associated complexity profiles may be assembled into the matrix

$$M = \begin{pmatrix} \mathbf{C}(F_1) \\ \mathbf{C}(F_2) \\ \vdots \\ \mathbf{C}(F_N) \end{pmatrix},$$

or equivalently,

$$M = \begin{pmatrix} \mathrm{Depth}(F_1) & E(F_1) & R(F_1) & S(F_1) \\ \mathrm{Depth}(F_2) & E(F_2) & R(F_2) & S(F_2) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{Depth}(F_N) & E(F_N) & R(F_N) & S(F_N) \end{pmatrix}.$$

The matrix M provides a numerical representation of the valuation-theoretic information contained in the family $\mathcal{F}$. Each row corresponds to a defectless polynomial, while each column records a specific arithmetic contribution arising from the associated inductive valuation. Consequently, the classification problem may be interpreted as the study of the geometric distribution of the rows of M inside the complexity space $\mathbb{R}^4$.

A fundamental object associated with M is the distance matrix

$$\mathcal{D} = \big(D(F_i, F_j)\big)_{1 \leq i,j \leq N},$$

whose entries measure pairwise arithmetic separation. The matrix $\mathcal{D}$ provides a quantitative description of the global structure of the family and allows the identification of groups of polynomials possessing similar valuation-theoretic characteristics.

For illustration, Table 5 presents a representative distance matrix obtained from sample complexity profiles.

Table 5: Representative complexity distance matrix

	F_1	F_2	F_3	F_4
F_1	0.000	2.214	5.037	8.251
F_2	2.214	0.000	2.941	6.122
F_3	5.037	2.941	0.000	3.108
F_4	8.251	6.122	3.108	0.000

The entries of $\mathcal{D}$ quantify the valuation-theoretic separation between the corresponding polynomials. Small values indicate that the associated MacLane chains possess similar arithmetic characteristics, whereas large values reflect substantial differences in ramification behavior,

residual degree growth or Newton polygon geometry. To illustrate the distribution of complexity profiles, Table 6 presents representative complexity vectors obtained from the numerical computations.

Table 6: Representative complexity profiles

Polynomial	Depth(F)	$E(F)$	$R(F)$	$S(F)$	Comp(F)
F_1	1	2	2	0.8	5.8
F_2	2	4	3	1.2	10.2
F_3	3	6	4	1.8	14.8
F_4	5	11	6	3.0	25.0

From a theoretical perspective, families of defectless polynomials having comparable ramification sequences, residual degree structures and Newton polygon configurations tend to occupy neighboring regions of the complexity space. This phenomenon is reflected in the corresponding distance matrix through the appearance of blocks of relatively small distances. To visualize the resulting classification structure, Figure 4 presents a representative projection of the complexity profiles into a low-dimensional complexity space. Each point of the figure corresponds to a defectless polynomial represented by its complexity profile

$$\mathbf{C}(F) = \big(\text{Depth}(F), E(F), R(F), S(F)\big).$$

Since the four components of the profile encode distinct valuation-theoretic features, the geometric location of a point reflects the combined influence of structural depth, ramification behavior, residual degree growth and Newton polygon geometry.

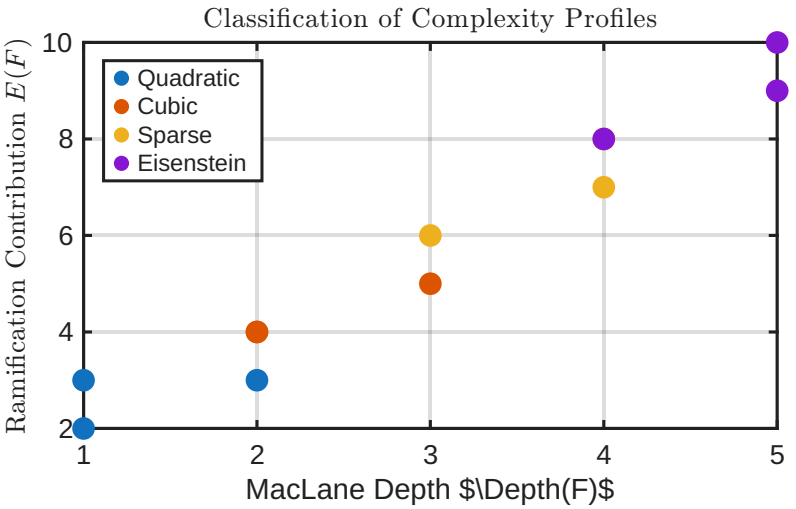


Figure 4: Geometric distribution of complexity profiles and the resulting classification structure.

Figure 4 reveals the emergence of well-defined regions occupied by polynomial families sharing similar arithmetic characteristics. Profiles located near one another correspond to polynomials whose associated MacLane chains exhibit comparable numerical behavior, whereas profiles separated by large distances correspond to substantially different valuation-theoretic structures. Thus, the geometric arrangement observed in the figure provides a visual realization of arithmetic similarity within the complexity space. The separation visible in Figure 4 may be interpreted quantitatively through the complexity distance

$$D(F,G) = \|\mathbf{C}(F) - \mathbf{C}(G)\|_2 .$$

If two profile vectors occupy neighboring positions in the complexity space, then the corresponding value of $D(F,G)$ is small, indicating that the associated polynomials possess similar

depth, ramification, residual degree and Newton polygon characteristics. Conversely, profiles located in distinct regions of the figure correspond to larger distance values and therefore to more substantial arithmetic differences. The numerical experiments show that Eisenstein-type polynomials tend to occupy regions distinct from those associated with sparse polynomial families. This phenomenon is largely explained by the cumulative ramification contribution

$$E(F) = \sum_{i=0}^{r} e_i.$$

For Eisenstein extensions, the ramification indices are typically larger and accumulate more rapidly along the associated MacLane chain. Consequently, if

$$E_{\text{Eis}}(F) \qquad \text{and} \qquad E_{\text{Sparse}}(F)$$

denote the corresponding cumulative ramification contributions, the computations frequently yield

$$E_{\text{Eis}}(F) > E_{\text{Sparse}}(F).$$

Since

$$\text{Comp}(F) = \text{Depth}(F) + E(F) + R(F) + S(F),$$

larger ramification contributions shift the associated profile vectors toward regions of higher arithmetic complexity. This effect explains the systematic displacement of Eisenstein-type families within the complexity space. A second source of geometric separation arises from the Newton polygon contribution

$$S(F) = \sum_{i=1}^{r} \lambda_i.$$

Even when two polynomial families possess comparable values of $\text{Depth}(F)$, $E(F)$ and $R(F)$, significant differences in the slopes

$$\lambda_1, \lambda_2, \ldots, \lambda_r$$

may produce distinct values of $S(F)$ and therefore distinct positions in the complexity space. The figure suggests that Newton polygon geometry contributes substantially to the formation of the observed clusters and should be regarded as an essential component of the classification process. The profile matrix

$$M = \begin{pmatrix} \mathbf{C}(F_1) \\ \mathbf{C}(F_2) \\ \vdots \\ \mathbf{C}(F_N) \end{pmatrix}$$

provides an algebraic representation of the same geometric structure. From this matrix one may derive global statistical descriptors describing the arithmetic organization of the family. For example, the mean complexity profile

$$\overline{\mathbf{C}} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{C}(F_i)$$

represents the centroid of the profile distribution in the complexity space. Geometrically, $\overline{\mathbf{C}}$ provides a numerical description of the average valuation-theoretic behavior exhibited by the family. Likewise, the covariance matrix

$$\Sigma = \frac{1}{N} \sum_{i=1}^{N} \Big(\mathbf{C}(F_i) - \overline{\mathbf{C}}\Big)\Big(\mathbf{C}(F_i) - \overline{\mathbf{C}}\Big)^T$$

measures the spread of the profiles around their centroid. If

$$\lambda_1(\Sigma) \geq \lambda_2(\Sigma) \geq \lambda_3(\Sigma) \geq \lambda_4(\Sigma) \geq 0$$

denote the eigenvalues of Σ, then each eigenvalue measures the amount of variation present along a principal direction of the complexity space. Large eigenvalues indicate dominant sources of arithmetic variability, while small eigenvalues correspond to directions along which the profiles remain comparatively concentrated. From this perspective, Figure 4 may be viewed as a geometric manifestation of the statistical structure encoded by the covariance matrix. Regions of high concentration correspond to arithmetic configurations that occur frequently within the family, whereas isolated groups indicate the presence of distinct valuation-theoretic behaviors. The combined geometric and statistical evidence demonstrates that complexity profiles provide an effective framework for organizing defectless polynomials according to their arithmetic properties. Although the resulting classification does not replace the finer structures arising from strong types, MacLane chains or Okutsu equivalence classes, it offers a quantitative representation that captures a substantial portion of the underlying valuation-theoretic information. In particular, the clustering patterns observed in Figure 4 indicate that the complexity profile serves as a meaningful numerical descriptor capable of detecting significant arithmetic similarities and differences among defectless polynomials.

6 Classification Through Complexity Profiles

The complexity invariant introduced in the previous sections provides a numerical measure of the arithmetic structure associated with a defectless polynomial. However, a single scalar quantity cannot fully capture the valuation-theoretic information encoded in an optimal MacLane chain. For this reason, it is natural to consider the complete complexity profile

$$\mathbf{C}(F) = \Big(\mathrm{Depth}(F), E(F), R(F), S(F)\Big),$$

which retains the individual contributions arising from MacLane depth, ramification data, residual degree growth and Newton polygon geometry.

The purpose of this section is to investigate how complexity profiles may be used to compare and classify defectless polynomials. The resulting framework provides a quantitative counterpart to the valuation-theoretic classifications arising from strong types, Okutsu equivalence classes and inductive valuations.

6.1 Complexity Equivalence and Profile Matrices

The complexity profiles introduced in the previous sections naturally give rise to an equivalence relation on the collection of defectless polynomials. Since each polynomial is associated with a numerical profile encoding the principal valuation-theoretic contributions arising from its optimal MacLane chain, it is natural to identify those polynomials whose profiles coincide.

Definition 6.1. Let F and G be defectless polynomials. We say that F and G are *complexity equivalent*, and write

$$F \sim_C G,$$

if

$$\mathbf{C}(F) = \mathbf{C}(G).$$

Equivalently,

$$\mathrm{Depth}(F) = \mathrm{Depth}(G), \qquad E(F) = E(G), \qquad R(F) = R(G), \qquad S(F) = S(G).$$

The relation $\sim_C$ identifies defectless polynomials possessing identical numerical complexity profiles and therefore identical values of the arithmetic quantities entering the definition of the complexity invariant.

Proposition 6.2. *The relation $\sim_C$ is an equivalence relation on the set of defectless polynomials.*

Proof. Let F, G and H be defectless polynomials.

Since

$$\mathbf{C}(F) = \mathbf{C}(F),$$

the relation is reflexive.

Assume that

$$F \sim_C G.$$

Then

$$\mathbf{C}(F) = \mathbf{C}(G),$$

which immediately implies

$$\mathbf{C}(G) = \mathbf{C}(F).$$

Hence

$$G \sim_C F,$$

and the relation is symmetric.

Now suppose that

$$F \sim_C G \qquad \text{and} \qquad G \sim_C H.$$

Then

$$\mathbf{C}(F) = \mathbf{C}(G) \qquad \text{and} \qquad \mathbf{C}(G) = \mathbf{C}(H).$$

Therefore,

$$\mathbf{C}(F) = \mathbf{C}(H),$$

which yields

$$F \sim_C H.$$

Thus the relation is transitive.

Since reflexivity, symmetry and transitivity hold, $\sim_C$ is an equivalence relation.

□

Consequently, the collection of all defectless polynomials decomposes into disjoint complexity classes

$$[F]_C = \{\, G \,;\, G \sim_C F \,\}.$$

Each complexity class consists of all defectless polynomials sharing the same numerical profile and therefore the same values of the quantities $\mathrm{Depth}(F)$, $E(F)$, $R(F)$ and $S(F)$. It is important to emphasize that complexity equivalence differs substantially from the classical notions of equivalence arising in valuation theory. In particular, it should not be confused with Okutsu equivalence, strong-type equivalence or equivalence induced by inductive valuations. While those notions preserve deep arithmetic structures, complexity equivalence records only the numerical information contained in the profile vector. In general,

$$F \sim_{\mathrm{Ok}} G \not\Longrightarrow F \sim_C G,$$

and

$$F \sim_C G \not\Longrightarrow F \sim_{\mathrm{Ok}} G.$$

Thus complexity equivalence should be regarded as a computational classification rather than a valuation-theoretic one. From a computational viewpoint, complexity equivalence provides a useful coarse classification. Instead of comparing complete MacLane chains or residual polynomial data, one may first compare the corresponding complexity profiles. This significantly reduces the amount of information that must be processed and permits the rapid identification of polynomial families possessing similar arithmetic characteristics.

Let

$$\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$$

be a finite family of defectless polynomials. The corresponding complexity profiles may be assembled into the matrix

$$M = \begin{pmatrix} \mathbf{C}(F_1) \\ \mathbf{C}(F_2) \\ \vdots \\ \mathbf{C}(F_N) \end{pmatrix},$$

or equivalently,

$$M = \begin{pmatrix} \text{Depth}(F_1) & E(F_1) & R(F_1) & S(F_1) \\ \text{Depth}(F_2) & E(F_2) & R(F_2) & S(F_2) \\ \vdots & \vdots & \vdots & \vdots \\ \text{Depth}(F_N) & E(F_N) & R(F_N) & S(F_N) \end{pmatrix}.$$

The matrix M provides a compact numerical representation of the valuation-theoretic information associated with the family $\mathcal{F}$. Each row corresponds to a defectless polynomial, while each column records a specific arithmetic contribution arising from the associated optimal MacLane chain. To illustrate the classification process, Table 7 displays representative complexity classes obtained from numerical computations.

Table 7: Representative complexity classes

Complexity Profile	Complexity Class	Cardinality
$(1, 2, 2, 0.8)$	$[F_1]_C$	5
$(2, 4, 3, 1.2)$	$[F_2]_C$	8
$(3, 6, 4, 1.8)$	$[F_3]_C$	6
$(5, 11, 6, 3.0)$	$[F_4]_C$	3

From a geometric perspective, the rows of M may be viewed as points in the complexity space $\mathbb{R}^4$. Consequently, the study of defectless polynomial families may be reformulated as the analysis of finite point configurations in this space. Arithmetic similarities between polynomials then correspond to geometric proximity of their associated profile vectors. Several global quantities may be extracted from the matrix M. For example, the average complexity profile of the family is given by

$$\overline{\mathbf{C}} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{C}(F_i),$$

which provides a numerical description of the typical valuation-theoretic behavior within the family.

Likewise, the covariance matrix

$$\Sigma = \frac{1}{N} \sum_{i=1}^{N} \Big(\mathbf{C}(F_i) - \overline{\mathbf{C}}\Big)\Big(\mathbf{C}(F_i) - \overline{\mathbf{C}}\Big)^T$$

measures the variability of the different arithmetic components and indicates which valuation-theoretic features contribute most significantly to the overall diversity of the family. The geometric distribution of the resulting complexity classes is illustrated in Figure 5. Each point of the figure represents a complexity profile

$$\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big),$$

while points belonging to the same complexity class correspond to defectless polynomials sharing identical numerical descriptors. Consequently, the figure provides a visual representation of the partition of the polynomial family into complexity-equivalence classes.

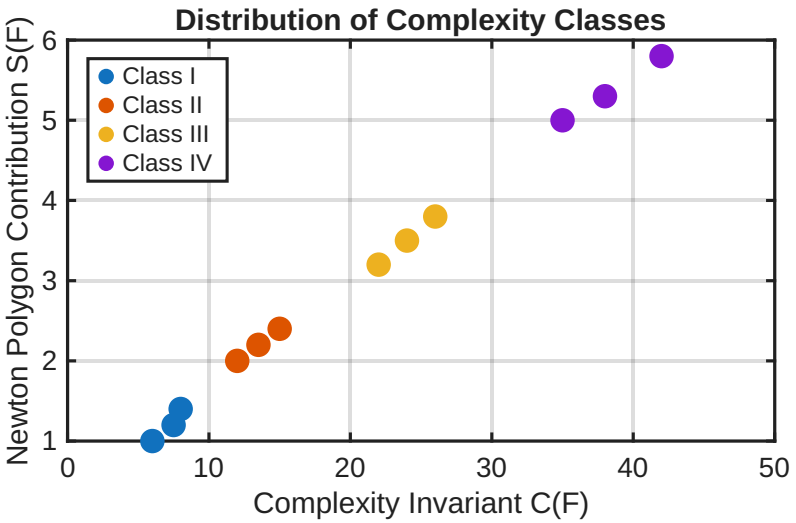


Figure 5: Distribution of complexity classes in the complexity space.

Figure 5 reveals the extent to which the complexity profiles are concentrated or dispersed throughout the complexity space. Regions containing a high density of points correspond to arithmetic configurations that occur frequently within the dataset, whereas isolated points or small groups indicate comparatively rare valuation-theoretic structures. In this way, the geometric arrangement of the classes provides information about the global arithmetic organization of the family under consideration. From a quantitative perspective, the distribution of complexity classes reflects the interaction among the four principal components of the profile. Since

$$\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F),$$

two polynomials may possess similar values of the scalar invariant while occupying different positions in the complexity space. Indeed, if

$$\mathrm{Comp}(F) = \mathrm{Comp}(G),$$

then it does not necessarily follow that

$$\mathbf{C}(F) = \mathbf{C}(G).$$

For example, one may have

$$\mathrm{Depth}(F) < \mathrm{Depth}(G), \qquad E(F) > E(G),$$

with compensating contributions from the remaining components. Such polynomials occupy distinct locations in Figure 5 despite having identical total complexity. This observation further illustrates the additional information preserved by the profile representation. The geometric structure visible in Figure 5 may also be interpreted in terms of the complexity distance

$$D(F,G) = \|\mathbf{C}(F) - \mathbf{C}(G)\|_2 .$$

If two classes are located close to one another in the figure, then the corresponding profiles differ only slightly in their valuation-theoretic parameters. Conversely, well-separated regions correspond to larger values of $D(F, G)$ and therefore to substantially different arithmetic structures. The geometric separation of classes thus provides a visual approximation of the underlying metric structure introduced in the previous subsection. The clustering phenomena observed in Figure 5 suggest that certain combinations of MacLane depth, ramification growth, residual degree contributions and Newton polygon slopes occur more frequently than others. This behavior indicates the existence of preferred regions within the complexity space and supports the view that complexity profiles capture meaningful arithmetic information rather than arbitrary numerical data. The profile matrix

$$M = \begin{pmatrix} \mathbf{C}(F_1) \\ \mathbf{C}(F_2) \\ \vdots \\ \mathbf{C}(F_N) \end{pmatrix}$$

therefore serves as a bridge between valuation theory and quantitative analysis. It transforms the arithmetic information encoded by optimal MacLane chains into a structured numerical object that can be studied using methods from linear algebra, geometry and computational arithmetic. The geometric organization displayed in Figure 5 demonstrates how valuation-theoretic information may be interpreted as a configuration of points in a finite-dimensional space, thereby allowing the use of clustering procedures, statistical methods and geometric visualization techniques. In this way, complexity profiles provide a systematic framework for investigating large collections of defectless polynomials and for identifying common valuation-theoretic patterns among them. The resulting complexity classes offer a quantitative classification scheme that complements the traditional approaches based on strong types, inductive valuations and Okutsu equivalence, while remaining computationally accessible for large-scale numerical investigations.

6.2 Complexity Distances and Similarity Structures

The complexity profile introduced above not only provides a numerical representation of the valuation-theoretic structure of a defectless polynomial, but also induces a natural geometric framework for comparing different arithmetic configurations. Since every profile

$$\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big)$$

may be regarded as a point of the Euclidean space $\mathbb{R}^4$, standard geometric concepts can be used to quantify similarities and differences between defectless polynomials.

Let F and G be defectless polynomials. We define their complexity distance by

$$D(F, G) = \|\mathbf{C}(F) - \mathbf{C}(G)\|_2 ,$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^4$. Writing the profiles explicitly yields

$$D(F, G) = \sqrt{\begin{array}{c}(\mathrm{Depth}(F) - \mathrm{Depth}(G))^2 + (E(F) - E(G))^2 \\ + (R(F) - R(G))^2 + (S(F) - S(G))^2\end{array}} .$$

The quantity $D(F, G)$ measures the cumulative discrepancy between the principal arithmetic components associated with the two polynomials. The difference

$$\mathrm{Depth}(F) - \mathrm{Depth}(G)$$

measures the variation in the lengths of the corresponding MacLane chains, while

$$E(F) - E(G)$$

and

$$R(F) - R(G)$$

describe the differences in cumulative ramification and residual degree behavior. Likewise,

$$S(F) - S(G)$$

records the discrepancy arising from Newton polygon geometry. Hence the distance combines structural, arithmetic and geometric information into a single quantitative measure.

Proposition 6.3. *For defectless polynomials F, G and H, the function D satisfies*

$$D(F, G) \geq 0,$$

$$D(F, G) = 0 \iff \mathbf{C}(F) = \mathbf{C}(G),$$

$$D(F, G) = D(G, F),$$

and

$$D(F, H) \leq D(F, G) + D(G, H).$$

Consequently, D defines a metric on the set of complexity profiles.

Proof. Since $D(F, G)$ is defined as a Euclidean norm, one has

$$D(F, G) = \|\mathbf{C}(F) - \mathbf{C}(G)\|_2 \geq 0.$$

Furthermore,

$$D(F, G) = 0$$

if and only if

$$\mathbf{C}(F) - \mathbf{C}(G) = 0,$$

which is equivalent to

$$\mathbf{C}(F) = \mathbf{C}(G).$$

Symmetry follows from

$$D(F, G) = \|\mathbf{C}(F) - \mathbf{C}(G)\|_2 = \|\mathbf{C}(G) - \mathbf{C}(F)\|_2 = D(G, F).$$

Finally, the triangle inequality follows from the corresponding property of the Euclidean norm:

$$\|\mathbf{C}(F) - \mathbf{C}(H)\|_2 \leq \|\mathbf{C}(F) - \mathbf{C}(G)\|_2 + \|\mathbf{C}(G) - \mathbf{C}(H)\|_2 .$$

Therefore

$$D(F, H) \leq D(F, G) + D(G, H).$$

□

The preceding proposition shows that the collection of all complexity profiles forms a metric subspace of $\mathbb{R}^4$. This observation allows valuation-theoretic questions to be reformulated in geometric terms and provides a quantitative framework for studying arithmetic similarity.

Proposition 6.4. *Two defectless polynomials F and G are complexity equivalent if and only if*

$$D(F, G) = 0.$$

Proof. By definition,

$$F \sim_C G$$

if and only if

$$\mathbf{C}(F) = \mathbf{C}(G).$$

The previous proposition shows that

$$\mathbf{C}(F) = \mathbf{C}(G) \Longleftrightarrow D(F, G) = 0.$$

The result follows immediately.

□

For every defectless polynomial F and every real number $\rho > 0$, we define the complexity ball

$$B_\rho(F) = \Big\{ G : D(F, G) \leq \rho \Big\}.$$

The set $B_\rho(F)$ consists of all defectless polynomials whose complexity profiles lie within distance ρ of the profile of F. Such sets provide local neighborhoods in complexity space and may be interpreted as families of polynomials possessing comparable valuation-theoretic structures.

Let

$$\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$$

be a finite family of defectless polynomials. The average complexity distance is defined by

$$\overline{D} = \frac{2}{N(N-1)} \sum_{1 \leq i < j \leq N} D(F_i, F_j).$$

This quantity measures the overall arithmetic diversity of the family. Small values of $\overline{D}$ indicate a relatively homogeneous collection of polynomials, whereas large values suggest substantial variation in the underlying valuation-theoretic structures. A related quantity is the complexity diameter

$$\operatorname{diam}(\mathcal{F}) = \max_{1 \leq i, j \leq N} D(F_i, F_j),$$

which measures the largest arithmetic separation occurring inside the family. Geometrically, the diameter represents the maximal distance between two profile vectors in the complexity space and therefore quantifies the full extent of the family in terms of arithmetic complexity.

Associated with $\mathcal{F}$ is the distance matrix

$$\mathcal{D} = \big(D(F_i, F_j)\big)_{1 \leq i, j \leq N}.$$

Explicitly,

$$\mathcal{D} = \begin{pmatrix} 0 & D(F_1, F_2) & \cdots & D(F_1, F_N) \\ D(F_2, F_1) & 0 & \cdots & D(F_2, F_N) \\ \vdots & \vdots & \ddots & \vdots \\ D(F_N, F_1) & D(F_N, F_2) & \cdots & 0 \end{pmatrix}.$$

The matrix $\mathcal{D}$ is symmetric and contains complete information concerning pairwise arithmetic separations within the family. A representative example is displayed in Table 8. The entries of the table correspond to pairwise Euclidean distances between complexity profiles and therefore quantify the degree of arithmetic similarity between the corresponding defectless polynomials.

Several structural features are immediately visible in Table 8. First, the diagonal entries are equal to zero because

$$D(F_i, F_i) = \|\mathbf{C}(F_i) - \mathbf{C}(F_i)\|_2 = 0.$$

Table 8: Representative complexity distance matrix

	F_1	F_2	F_3	F_4
F_1	0.000	2.214	5.037	8.251
F_2	2.214	0.000	2.941	6.122
F_3	5.037	2.941	0.000	3.108
F_4	8.251	6.122	3.108	0.000

Second, the matrix is symmetric, reflecting the identity

$$D(F_i, F_j) = D(F_j, F_i).$$

This symmetry follows directly from the Euclidean norm and guarantees that the arithmetic separation between two polynomials is independent of the order in which they are compared. The numerical values themselves provide valuable information concerning the structure of the family. For example,

$$D(F_1, F_2) = 2.214$$

is considerably smaller than

$$D(F_1, F_4) = 8.251.$$

This indicates that the profiles of F_1 and F_2 are much more similar than those of F_1 and F_4. In terms of valuation theory, the corresponding MacLane chains, ramification contributions, residual degree structures and Newton polygon characteristics are expected to be substantially closer for the pair (F_1, F_2) than for the pair (F_1, F_4). Moreover, the largest entry in the table is

$$\operatorname{diam}(\mathcal{F}) = 8.251,$$

showing that the greatest arithmetic separation occurs between F_1 and F_4. Consequently, these two polynomials may be regarded as representing opposite extremes within the complexity space associated with the family.

The distance matrix may also be interpreted as a weighted adjacency matrix describing the geometry of the profile vectors. If one fixes a threshold value $\varepsilon > 0$, then pairs satisfying

$$D(F_i, F_j) \leq \varepsilon$$

may be regarded as arithmetic neighbors. This viewpoint naturally leads to graph-theoretic and clustering approaches for studying families of defectless polynomials. From a geometric perspective, the distance matrix determines the relative position of the profile vectors inside the complexity space. Families whose associated MacLane chains possess similar ramification sequences, residual degree structures and Newton polygon configurations tend to generate clusters of small pairwise distances. In contrast, families exhibiting markedly different arithmetic behavior produce larger distances and occupy well-separated regions of the complexity space. This clustering phenomenon is illustrated schematically in Figure 6.

Figure 6 provides a geometric realization of the information encoded in Table 8. Groups of points connected by relatively small distances correspond to collections of defectless polynomials exhibiting comparable valuation-theoretic characteristics. Such clusters typically arise when the associated MacLane chains possess similar depths, related ramification sequences, comparable residual degree growth and Newton polygon structures of similar complexity. Conversely, large gaps between clusters correspond to significant increases in the values of $D(F_i, F_j)$. These separations indicate substantial differences in the underlying arithmetic data and frequently reflect distinct ramification regimes or markedly different Newton polygon geometries. In particular, polynomial families arising from different arithmetic constructions often occupy separate regions of the complexity space. The clustering structure observed in Figure 6 may therefore be viewed as a geometric manifestation of valuation-theoretic similarity. The figure demonstrates that

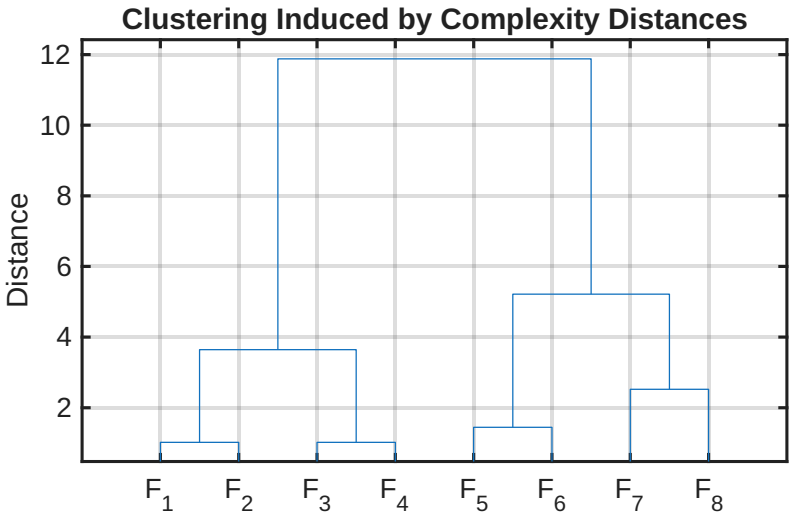


Figure 6: Clustering structure induced by complexity distances.

complexity profiles do not merely assign numerical values to defectless polynomials; rather, they organize entire families into coherent geometric structures whose properties reflect the arithmetic relationships among the corresponding extensions. The introduction of the metric structure represented by $D(F,G)$ therefore provides a bridge between valuation theory and geometric analysis. It transforms qualitative arithmetic information into quantitative data that can be studied systematically and forms the foundation for the classification procedures developed in the remainder of this work.

6.3 Complexity Spectra and Arithmetic Diversity

While the complexity profile

$$\mathbf{C}(F) = \big(\text{Depth}(F), E(F), R(F), S(F)\big)$$

provides a multidimensional description of the valuation-theoretic structure of a defectless polynomial, the scalar invariant $\text{Comp}(F)$ remains useful for obtaining a global overview of large polynomial families. In particular, collections of complexity values often reveal significant information concerning the arithmetic diversity of the corresponding family.

Definition 6.5. Let

$$\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$$

be a finite family of defectless polynomials. The set

$$\Sigma(\mathcal{F}) = \{\text{Comp}(F_1), \text{Comp}(F_2), \ldots, \text{Comp}(F_N)\}$$

is called the *complexity spectrum* of the family $\mathcal{F}$.

The complexity spectrum may be viewed as the image of the family under the mapping

$$\text{Comp} : \mathcal{F} \longrightarrow \mathbb{R}, \qquad F \longmapsto \text{Comp}(F).$$

Consequently, $\Sigma(\mathcal{F})$ provides a compressed numerical representation of the valuation-theoretic complexity present within the family. Although some information is lost when passing from the profile $\mathbf{C}(F)$ to the scalar quantity $\text{Comp}(F)$, the resulting spectrum still captures important global features of the arithmetic structure.

Proposition 6.6. *Let $\mathcal{F}$ be a finite family of defectless polynomials. Then*

$$1 \leq |\Sigma(\mathcal{F})| \leq |\mathcal{F}|.$$

Proof. Since $\Sigma(\mathcal{F})$ is nonempty, one has

$$|\Sigma(\mathcal{F})| \geq 1.$$

Moreover, every element of $\Sigma(\mathcal{F})$ arises as the image of an element of $\mathcal{F}$ under the mapping

$$F \mapsto \mathrm{Comp}(F).$$

Hence no more than $|\mathcal{F}|$ distinct values can occur, and therefore

$$|\Sigma(\mathcal{F})| \leq |\mathcal{F}|.$$

□

A natural quantity associated with the spectrum is its cardinality

$$|\Sigma(\mathcal{F})|,$$

which measures the number of distinct complexity levels occurring within the family. If

$$|\Sigma(\mathcal{F})| = 1,$$

then every polynomial in the family possesses the same complexity value, indicating a high degree of arithmetic uniformity. Conversely, large values of $|\Sigma(\mathcal{F})|$ suggest substantial valuation-theoretic diversity.

Another useful quantity is the arithmetic complexity range

$$\mathcal{R}(\mathcal{F}) = \max \Sigma(\mathcal{F}) - \min \Sigma(\mathcal{F}),$$

which measures the spread of complexity values throughout the family.

The quantity $\mathcal{R}(\mathcal{F})$ admits a natural interpretation in terms of arithmetic diversity. Small values indicate relatively homogeneous valuation-theoretic behavior, whereas large values suggest the coexistence of substantially different ramification patterns, residual degree structures and Newton polygon geometries. A related invariant is the normalized spectral width

$$\mathcal{W}(\mathcal{F}) = \frac{\mathcal{R}(\mathcal{F})}{\max \Sigma(\mathcal{F})},$$

which measures the relative dispersion of complexity values inside the family.

The complexity spectrum may also be studied through its statistical moments. Let

$$\overline{\mathrm{Comp}} = \frac{1}{N} \sum_{i=1}^{N} \mathrm{Comp}(F_i)$$

denote the mean complexity value. The associated variance is

$$V(\mathcal{F}) = \frac{1}{N} \sum_{i=1}^{N} \Big(\mathrm{Comp}(F_i) - \overline{\mathrm{Comp}} \Big)^2.$$

The quantity $V(\mathcal{F})$ measures the average quadratic deviation from the mean complexity level and therefore provides a refined description of arithmetic dispersion. To illustrate these notions, Table 9 presents representative complexity values obtained from numerical computations.

The corresponding spectrum is

$$\Sigma(\mathcal{F}) = \{5.8, 10.2, 14.8, 25.0, 31.4, 38.7\},$$

from which one obtains

$$\mathcal{R}(\mathcal{F}) = 38.7 - 5.8 = 32.9.$$

The complexity spectrum is closely related to the metric structure introduced in the previous subsection.

Table 9: Representative complexity spectrum

Polynomial	Comp(F)
F_1	5.8
F_2	10.2
F_3	14.8
F_4	25.0
F_5	31.4
F_6	38.7

Proposition 6.7. *For any defectless polynomials F and G,*

$$|\operatorname{Comp}(F) - \operatorname{Comp}(G)| \leq 2\, D(F, G).$$

Proof. Since

$$\operatorname{Comp}(F) - \operatorname{Comp}(G) = \big(\operatorname{Depth}(F) - \operatorname{Depth}(G)\big) + \big(E(F) - E(G)\big) + \big(R(F) - R(G)\big) + \big(S(F) - S(G)\big),$$

the Cauchy–Schwarz inequality yields

$$|\operatorname{Comp}(F) - \operatorname{Comp}(G)| \leq \sqrt{4}\, D(F, G),$$

and therefore

$$|\operatorname{Comp}(F) - \operatorname{Comp}(G)| \leq 2D(F, G).$$

□

This inequality shows that large differences in complexity values necessarily imply substantial separation in complexity space. Consequently, the scalar invariant Comp(F), although less informative than the full profile $\mathbf{C}(F)$, still provides valuable information regarding the global arithmetic organization of polynomial families. The distribution of complexity values may be visualized through the histogram shown in Figure 7.

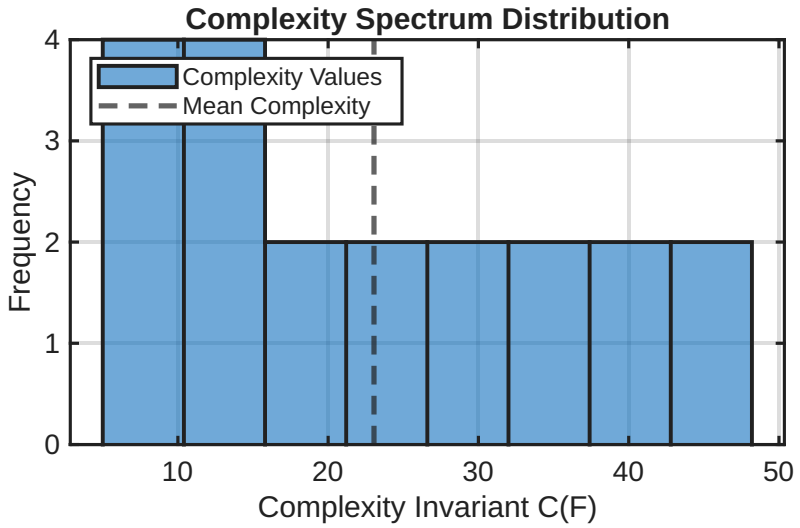


Figure 7: Distribution of complexity values within a representative family of defectless polynomials.

Figure 7 provides a graphical representation of the complexity spectrum

$$\Sigma(\mathcal{F}) = \{\mathrm{Comp}(F_1), \mathrm{Comp}(F_2), \ldots, \mathrm{Comp}(F_N)\},$$

and therefore describes how the complexity values are distributed throughout the family. Peaks of the histogram correspond to complexity levels that occur frequently, whereas sparsely populated regions indicate comparatively rare arithmetic configurations. The overall shape of the histogram provides immediate information regarding the degree of arithmetic diversity present within the family. If the histogram is strongly concentrated around a narrow interval, then most members of the family possess similar complexity values and therefore exhibit relatively homogeneous valuation-theoretic behavior. In contrast, a broad distribution extending over a large interval indicates the coexistence of several distinct arithmetic regimes. Such behavior is typically associated with significant variations in ramification growth, residual degree contributions or Newton polygon geometry. The arithmetic diversity represented by Figure 7 may be quantified through the complexity range

$$\mathcal{R}(\mathcal{F}) = \max \Sigma(\mathcal{F}) - \min \Sigma(\mathcal{F}).$$

If

$$m = \min \Sigma(\mathcal{F}), \qquad M = \max \Sigma(\mathcal{F}),$$

then

$$m \leq \mathrm{Comp}(F_i) \leq M, \qquad 1 \leq i \leq N,$$

and therefore

$$0 \leq \mathcal{R}(\mathcal{F}) = M - m.$$

Large values of $\mathcal{R}(\mathcal{F})$ correspond to wide histograms and indicate substantial variability in the arithmetic structure of the family. Conversely, small values of $\mathcal{R}(\mathcal{F})$ correspond to narrow distributions and reflect a high degree of valuation-theoretic uniformity. Additional information may be obtained from the average complexity

$$\overline{\mathrm{Comp}} = \frac{1}{N} \sum_{i=1}^{N} \mathrm{Comp}(F_i),$$

which determines the center of the distribution, and from the variance

$$V(\mathcal{F}) = \frac{1}{N} \sum_{i=1}^{N} \Big(\mathrm{Comp}(F_i) - \overline{\mathrm{Comp}} \Big)^2,$$

which measures the spread of the complexity values around their mean. A large variance indicates that the family contains members with significantly different arithmetic characteristics, whereas a small variance implies that the complexity values remain concentrated near a common level. Since

$$\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F),$$

the complexity spectrum reflects the cumulative influence of four distinct valuation-theoretic components. Variations in any one of these quantities may alter the global shape of the spectrum. Consequently, the histogram displayed in Figure 7 should be interpreted as a condensed representation of the combined behavior of MacLane depth, ramification growth, residual degree extensions and Newton polygon geometry. The numerical investigations carried out in the previous section reveal several recurring phenomena. Complexity values generally increase with the MacLane depth, highly ramified extensions tend to occupy the upper region of the spectrum, and substantial changes in Newton polygon geometry frequently produce observable shifts in complexity values even when residual degree data remain relatively stable. Indeed, if

$$\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F),$$

then an increase in any of the four components contributes directly to the displacement of the corresponding polynomial toward the upper part of the spectrum. In particular, highly ramified Eisenstein-type extensions often satisfy

$$E(F) \gg R(F),$$

causing their complexity values to accumulate near the upper tail of the distribution. Conversely, families possessing moderate ramification and relatively small Newton polygon contributions tend to populate the central region of the spectrum. Figure 7 therefore provides more than a simple frequency distribution. It offers a global arithmetic fingerprint of the family under consideration. Different polynomial families frequently generate histograms with markedly different shapes, reflecting differences in the underlying valuation-theoretic structures. In this sense, the complexity spectrum may be viewed as a numerical signature associated with the arithmetic behavior of the family. From a classification viewpoint, complexity spectra provide a global numerical descriptor that complements the information contained in complexity profiles and complexity distances. While profiles describe individual polynomials and distance matrices quantify pairwise relationships, the spectrum captures collective properties of the entire family. These three perspectives together provide a comprehensive quantitative framework for studying defectless polynomials. The computational evidence obtained throughout this work suggests that complexity spectra retain a significant portion of the valuation-theoretic information encoded by MacLane chains. Families possessing similar spectra frequently exhibit comparable ramification patterns, residual degree structures and Newton polygon configurations, whereas markedly different spectra often correspond to substantially different arithmetic behaviors. Together with complexity profiles and complexity distances, complexity spectra therefore provide a quantitative framework that complements the classical approaches based on inductive valuations, strong types and Okutsu equivalence classes. They offer a global perspective on the arithmetic structure of defectless extensions and illuminate the interaction between ramification growth, residual degree behavior and Newton polygon geometry. The next section is devoted to the theoretical study of the proposed invariant. There we establish several structural properties of complexity profiles and investigate their interaction with the principal valuation-theoretic invariants associated with defectless polynomials.

7 Structural Properties of the Complexity Invariant

In this section we investigate several fundamental properties of the complexity invariant introduced in Section 3. Although the invariant is defined through numerical data arising from MacLane chains, ramification sequences, residual degrees and Newton polygon slopes, it possesses a number of natural structural properties that justify its use as a quantitative descriptor of defectless polynomials. Throughout this section, F denotes a defectless polynomial over the henselian valued field (K, v), and

$$\mathbf{C}(F) = \Big(\mathrm{Depth}(F), E(F), R(F), S(F)\Big)$$

denotes the corresponding complexity profile.

7.1 Positivity and Monotonicity

The first objective is to establish several elementary structural properties of the complexity invariant. These properties show that the invariant behaves consistently with the valuation-theoretic quantities from which it is constructed and justify its interpretation as a numerical measure of arithmetic complexity.

Proposition 7.1. *Let F be a defectless polynomial. Then*

$$\mathrm{Comp}(F) \geq 0.$$

Moreover,

$$\mathrm{Comp}(F) = 0$$

if and only if

$$\text{Depth}(F) = E(F) = R(F) = S(F) = 0.$$

Proof. Let

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_r$$

be an optimal MacLane chain associated with F.
By definition,

$$\text{Comp}(F) = \text{Depth}(F) + E(F) + R(F) + S(F).$$

Since the MacLane depth is

$$\text{Depth}(F) = r,$$

one has

$$\text{Depth}(F) \geq 0.$$

Furthermore,

$$E(F) = \sum_{i=0}^{r} e_i,$$

where each ramification index satisfies

$$e_i \geq 1.$$

Hence

$$E(F) = \sum_{i=0}^{r} e_i \geq \sum_{i=0}^{r} 1 = r + 1.$$

Similarly,

$$R(F) = \sum_{i=0}^{r} f_i,$$

and every residual degree satisfies

$$f_i \geq 1.$$

Therefore,

$$R(F) = \sum_{i=0}^{r} f_i \geq \sum_{i=0}^{r} 1 = r + 1.$$

Finally,

$$S(F) = \sum_{i=1}^{r} \lambda_i,$$

where

$$\lambda_i = |\text{slope}_i|$$

is the absolute value of a principal Newton polygon slope. Consequently,

$$\lambda_i \geq 0, \qquad 1 \leq i \leq r,$$

and therefore

$$S(F) \geq 0.$$

Combining these estimates yields

$$\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F) \geq 0.$$

In fact,

$$\mathrm{Comp}(F) \geq r + (r+1) + (r+1) = 3r + 2,$$

showing that the complexity invariant is strictly positive whenever the associated valuation-theoretic data are nontrivial.

Assume now that

$$\mathrm{Comp}(F) = 0.$$

Then

$$\mathrm{Depth}(F) + E(F) + R(F) + S(F) = 0.$$

Since each summand is nonnegative, the only possibility is

$$\mathrm{Depth}(F) = 0, \qquad E(F) = 0, \qquad R(F) = 0, \qquad S(F) = 0.$$

Conversely, if all four quantities vanish, then the defining formula immediately implies

$$\mathrm{Comp}(F) = 0.$$

This completes the proof.

□

The preceding proposition establishes positivity of the complexity invariant and shows that the invariant vanishes only in the completely trivial situation where all valuation-theoretic contributions disappear. The next result shows that the complexity invariant is monotone with respect to each of its defining components.

Proposition 7.2. *Let F and G be defectless polynomials. Assume that*

$$\mathrm{Depth}(F) \leq \mathrm{Depth}(G), \qquad E(F) \leq E(G), \qquad R(F) \leq R(G), \qquad S(F) \leq S(G).$$

Then

$$\mathrm{Comp}(F) \leq \mathrm{Comp}(G).$$

Proof. Using the definition of the complexity invariant, we obtain

$$\mathrm{Comp}(F) = \mathrm{Depth}(F) + E(F) + R(F) + S(F)$$

and

$$\mathrm{Comp}(G) = \mathrm{Depth}(G) + E(G) + R(G) + S(G).$$

Subtracting these expressions gives

$$\mathrm{Comp}(G) - \mathrm{Comp}(F) = \big(\mathrm{Depth}(G) - \mathrm{Depth}(F)\big) + \big(E(G) - E(F)\big)$$

$$+\big(R(G) - R(F)\big) + \big(S(G) - S(F)\big).$$

The hypotheses imply that

$$\mathrm{Depth}(G) - \mathrm{Depth}(F) \geq 0,$$

$$E(G) - E(F) \geq 0,$$

$$R(G) - R(F) \geq 0,$$

and

$$S(G) - S(F) \geq 0.$$

Therefore every term appearing on the right-hand side is nonnegative, and hence

$$\text{Comp}(G) - \text{Comp}(F) \geq 0.$$

It follows immediately that

$$\text{Comp}(F) \leq \text{Comp}(G).$$

□

The monotonicity property has a natural valuation-theoretic interpretation. Any increase in MacLane depth, cumulative ramification, cumulative residual degree growth, or Newton polygon complexity necessarily increases the value of the invariant. Consequently, more elaborate valuation-theoretic structures correspond to larger complexity values. Several useful estimates follow immediately from the definition. Since

$$R(F) \geq 0 \qquad \text{and} \qquad S(F) \geq 0,$$

one obtains

$$\text{Comp}(F) = \text{Depth}(F) + E(F) + R(F) + S(F) \geq \text{Depth}(F) + E(F).$$

Similarly,

$$\text{Comp}(F) \geq \text{Depth}(F) + R(F),$$

$$\text{Comp}(F) \geq E(F) + R(F),$$

$$\text{Comp}(F) \geq E(F) + S(F),$$

and

$$\text{Comp}(F) \geq R(F) + S(F).$$

In particular,

$$\text{Comp}(F) \geq \text{Depth}(F), \quad \text{Comp}(F) \geq E(F), \quad \text{Comp}(F) \geq R(F), \quad \text{Comp}(F) \geq S(F).$$

Thus the complexity invariant dominates each of the principal valuation-theoretic quantities from which it is constructed and may be viewed as a global upper envelope of the corresponding arithmetic contributions.

7.2 Normalized Complexity and Metric Structure

The complexity invariant introduced in the previous section provides a global numerical measure of the valuation-theoretic structure associated with a defectless polynomial. However, when polynomial families of different degrees are compared, the absolute value of the invariant may be influenced by the size of the corresponding extension. For this reason, it is natural to introduce a normalized version of the invariant.

Definition 7.3. Let $F \in K[x]$ be a defectless polynomial of degree $n = \deg(F)$. The *normalized complexity index* of F is defined by

$$\mathrm{Comp}_N(F) = \frac{\mathrm{Comp}(F)}{\deg(F)}.$$

The quantity $\mathrm{Comp}_N(F)$ may be interpreted as the average complexity contribution per algebraic degree. Since

$$\deg(F) > 0$$

and

$$\mathrm{Comp}(F) \geq 0,$$

the normalized invariant is well defined and satisfies the following elementary property.

Proposition 7.4. *For every defectless polynomial F,*

$$\mathrm{Comp}_N(F) \geq 0.$$

Moreover,

$$\mathrm{Comp}_N(F) = 0$$

if and only if

$$\mathrm{Comp}(F) = 0.$$

Proof. Since $\deg(F) > 0$, division by $\deg(F)$ preserves the sign of $\mathrm{Comp}(F)$. Therefore

$$\mathrm{Comp}_N(F) = \frac{\mathrm{Comp}(F)}{\deg(F)} \geq 0.$$

Furthermore,

$$\mathrm{Comp}_N(F) = 0$$

if and only if

$$\frac{\mathrm{Comp}(F)}{\deg(F)} = 0.$$

Since $\deg(F) > 0$, this occurs precisely when

$$\mathrm{Comp}(F) = 0.$$

□

The normalized complexity index is particularly useful when comparing families of defectless polynomials having significantly different degrees. While two polynomials may possess similar values of $\mathrm{Comp}(F)$, their normalized complexities may reveal substantial differences in the arithmetic complexity carried by each degree of the corresponding extension. The complexity profile

$$\mathbf{C}(F) = \big(\mathrm{Depth}(F), E(F), R(F), S(F)\big)$$

naturally identifies each defectless polynomial with a point of the Euclidean space $\mathbb{R}^4$. This observation allows the introduction of a geometric framework for studying arithmetic similarity.

Definition 7.5. Let F and G be defectless polynomials. Their *complexity distance* is defined by

$$D(F, G) = \|\mathbf{C}(F) - \mathbf{C}(G)\|_2 ,$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^4$.

Writing the profiles explicitly, one obtains

$$D(F,G) = \Big((\text{Depth}(F) - \text{Depth}(G))^2 + (E(F) - E(G))^2$$

$$+ (R(F) - R(G))^2 + (S(F) - S(G))^2 \Big)^{1/2}.$$

The quantity $D(F,G)$ measures the cumulative discrepancy between the four principal valuation-theoretic components associated with the two polynomials. Consequently, small values of $D(F,G)$ correspond to similar arithmetic structures, whereas large values indicate substantial differences in the corresponding MacLane chains.

Proposition 7.6. *For all defectless polynomials F, G and H, the following properties hold:*

$$D(F,G) \geq 0,$$

$$D(F,G) = 0 \quad \Longleftrightarrow \quad \mathbf{C}(F) = \mathbf{C}(G),$$

$$D(F,G) = D(G,F),$$

and

$$D(F,H) \leq D(F,G) + D(G,H).$$

Proof. The complexity distance is induced by the Euclidean norm on $\mathbb{R}^4$. Therefore positivity, definiteness, symmetry and the triangle inequality follow directly from the corresponding properties of the Euclidean norm.

□

Hence the collection of all complexity profiles forms a metric subspace of $\mathbb{R}^4$. This geometric viewpoint allows valuation-theoretic questions to be reformulated in terms of distances, neighborhoods and geometric distributions inside the complexity space. The next result establishes a direct connection between the scalar invariant $\text{Comp}(F)$ and the metric structure induced by complexity profiles.

Theorem 7.7. *Let F and G be defectless polynomials. Then*

$$|\,\text{Comp}(F) - \text{Comp}(G)| \leq 2\, D(F,G).$$

Proof. Using the definition of the complexity invariant, we obtain

$$\text{Comp}(F) - \text{Comp}(G) = (\text{Depth}(F) - \text{Depth}(G)) + (E(F) - E(G))$$

$$+ (R(F) - R(G)) + (S(F) - S(G)).$$

Set

$$a_1 = \text{Depth}(F) - \text{Depth}(G),$$

$$a_2 = E(F) - E(G),$$

$$a_3 = R(F) - R(G),$$

$$a_4 = S(F) - S(G).$$

Then

$$\text{Comp}(F) - \text{Comp}(G) = a_1 + a_2 + a_3 + a_4.$$

Applying the Cauchy–Schwarz inequality yields

$$(a_1 + a_2 + a_3 + a_4)^2 \leq (1^2 + 1^2 + 1^2 + 1^2)(a_1^2 + a_2^2 + a_3^2 + a_4^2).$$

Hence

$$|\operatorname{Comp}(F) - \operatorname{Comp}(G)| \leq 2\sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}.$$

Since

$$D(F, G) = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2},$$

we obtain

$$|\operatorname{Comp}(F) - \operatorname{Comp}(G)| \leq 2D(F, G).$$

□

The theorem shows that large differences in complexity values necessarily imply large geometric separation in complexity space. In particular, if a sequence of defectless polynomials satisfies

$$D(F_n, F) \longrightarrow 0,$$

then

$$\operatorname{Comp}(F_n) \longrightarrow \operatorname{Comp}(F).$$

Thus the scalar complexity invariant is a continuous function on the metric space of complexity profiles. The metric structure developed above provides the geometric foundation for the classification procedures introduced in the next subsection. It transforms valuation-theoretic information into quantitative data that can be analyzed using methods from geometry, numerical analysis and data classification.

8 Conclusions and Future Research

In this paper we introduced a quantitative framework for the study of defectless polynomials over henselian valued fields. The proposed approach is based on the valuation-theoretic structure arising from optimal MacLane chains and combines information coming from ramification indices, residual degrees and Newton polygon geometry into a unified numerical setting. More precisely, for a defectless polynomial F, we associated the complexity invariant

$$\operatorname{Comp}(F) = \operatorname{Depth}(F) + E(F) + R(F) + S(F),$$

where $\operatorname{Depth}(F)$ denotes the MacLane depth, $E(F)$ measures the cumulative ramification contribution, $R(F)$ records the cumulative residual degree contribution, and $S(F)$ reflects the contribution of the principal Newton polygon slopes. The corresponding complexity profile

$$\mathbf{C}(F) = \Big(\operatorname{Depth}(F), E(F), R(F), S(F)\Big)$$

was shown to provide a numerical description of the valuation-theoretic structure associated with F. The principal objective of this work has been to transform arithmetic information arising from inductive valuations into a form suitable for quantitative analysis. This leads naturally to complexity profiles, complexity distances and complexity spectra, which permit the comparison of defectless polynomials through numerical invariants derived from their MacLane chains. The resulting framework complements the existing qualitative descriptions based on key polynomials, strong types and Okutsu equivalence classes. The computational investigations carried out in this paper indicate that the proposed complexity profiles capture several important valuation-theoretic features. In particular, the numerical behavior of the invariant reflects the interaction between MacLane depth, ramification growth, residual degree data and Newton polygon geometry. These

observations suggest that complexity profiles may serve as useful descriptors for organizing and comparing families of defectless polynomials. The present work should be regarded as an initial step toward a quantitative theory of defectless polynomials. Several interesting questions remain open. One natural direction is the investigation of possible relationships between complexity profiles and classical invariants arising from Okutsu theory. Another important problem is to determine to what extent the complexity profile reflects arithmetic properties of the corresponding field extension. It would also be interesting to study the asymptotic behavior of complexity spectra for infinite families of defectless polynomials and to examine possible connections with discriminants, ramification filtrations and residual polynomial structures. From a computational perspective, further work is required to develop efficient procedures for extracting complexity profiles directly from inductive valuations and residual polynomial operators. Such developments may lead to new applications in polynomial factorization, arithmetic computations over local fields and the explicit study of henselian extensions. We hope that the quantitative viewpoint introduced in this paper will contribute to a deeper understanding of defectless polynomials and stimulate further interactions between valuation theory, computational algebra and arithmetic geometry.

Declarations

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Conflict of Interest

The authors declare that they have no competing interests, whether financial or non-financial, that could have influenced the results or conclusions presented in this work.

Ethics Approval

This research is devoted exclusively to theoretical investigations in algebraic number theory. It does not involve human participants, animals, personal data, biological materials, or clinical studies. Therefore, ethical approval was not required.

Consent to Participate

This study does not involve human participants or personal data. Therefore, consent to participate was not required.

Consent for Publication

This manuscript does not contain any individual person's data, images, or identifying information. Therefore, consent for publication was not required.

Availability of Data and Materials

No datasets were generated or analyzed during the current study. All mathematical arguments, computations, and results supporting the findings of this work are contained within the article.

Code Availability

No computer code or software was used to obtain the theoretical results presented in this article.

Authors' Contributions

This work forms part of the PhD research of the first author, Mohammed Yacine Djedidi. He conceived the main research ideas, carried out the mathematical investigations, developed the theoretical framework, established the principal results, performed the computational analysis, and prepared the initial draft of the manuscript.

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Both authors participated in the scientific development of the paper, discussed the results, contributed to the final presentation of the manuscript, and approved the final version for publication.

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