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A Study of a Sequence of Right Trifurcating Fibonacci-Type Numbers

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Abstract. The idea of the Fibonacci sequence has been expanded and modified in numerous ways, some by preserving the initial conditions and altering the recurrence relation, and others by preserving the recurrence relation and altering the initial condition. One of its interesting generalizations is the sequence of right *k*-Fibonacci numbers. This was further generalized into the 'right bifurcating Fibonacci sequence'.

In this paper, we generalize this sequence further into the 'right trifurcating Fibonacci numbers' sequence. We first obtain the generating function for these numbers and use it to derive its Binet-like formula. Additionally, we derive analogous of various known identities such as Cassini's, Catalan's, and d'Ocagne's identities. Furthermore, we derive several elementary identities for the terms of such sequence.

1 Introduction

The Fibonacci sequence is mainly generalized in two ways: one by altering the initial condition, and the other by altering the recurrence relation. Various generalizations and results related to the Fibonacci sequence can be easily found in Koshy [8]. Falcon, Plaza [6] generalized the Fibonacci sequence to the sequence of *left k-Fibonacci numbers* defined by the recurrence relation $F_n^k = kF_{n-1}^k + F_{n-2}^k$ with $F_0^k = 0$ and $F_1^k = 1$. Many papers on this generalization have appeared in recent years [1, 7, 9, 11, 12]. These generalization was further generalized by Arvadia, Shah [2] to *right k-Fibonacci numbers* defined by the recurrence relation $F_{k,n}^R = F_{k,n-1}^R + kF_{k,n-2}^R$ with $F_{k,0}^R = 0$ and $F_{k,1}^R = 1$. Edson, Yayenie [5] generalized the sequence of *left k-Fibonacci numbers* to bifurcating Fibonacci numbers and in a similar way Diwan, Shah [4] further generalized these sequence of *right k-Fibonacci numbers* to a sequence $\left\{F_n^{R(a,b)}\right\}$ of right bifurcating Fibonacci numbers.

Diwan, Shah [3] considered and studied extensively the sequence $\left\{F_n^G(p,q,r,s)\right\}$ of generalized Fibonacci numbers defined by the recurrence relation.

$$F_n^G(p,q,r,s) = p^{\chi(n)}q^{1-\chi(n)}F_{n-1}^G(p,q,r,s) + r^{\chi(n)}s^{1-\chi(n)}F_{n-2}^G(p,q,r,s), \tag{1.1}$$

with initial conditions

$$F_0^G(p,q,r,s) = 0, \qquad F_1^G(p,q,r,s) = 1,$$

where

$$\chi(n) = \begin{cases} 0, & n \equiv 0 \pmod 2, \\ 1, & n \equiv 1 \pmod 2. \end{cases}$$

and p, q, r, s are positive integers.

Also, the sequence $\left\{F_n^{R(a,b)}\right\}$ is a special case of this sequence with $p = q = 1$ and $r = a, s = b$. These bifurcating sequences of Fibonacci numbers were also studied by [3, 13, 14].

In this paper, the authors extend the sequence $\left\{F_n^G(p,q,r,s)\right\}$ to a sequence of $\left\{F_n^G(p,q,r,s,t,u)\right\}$ trifurcating Fibonacci numbers. These numbers are governed by six real parameters: p, q, r, s, t, and u. Each term in this sequence is determined by a recurrence relation

$$\begin{aligned} F_n^G(p,q,r,s,t,u) = {} & p^{\left\lfloor 1-\frac{\chi(n)}{3}\right\rfloor} q^{\left\lfloor 1-\frac{\chi(n+2)}{3}\right\rfloor} r^{\left\lfloor 1-\frac{\chi(n+1)}{3}\right\rfloor} F_{n-1}^G(p,q,r,s,t,u) \\ & + s^{\left\lfloor 1-\frac{\chi(n)}{3}\right\rfloor} t^{\left\lfloor 1-\frac{\chi(n+2)}{3}\right\rfloor} u^{\left\lfloor 1-\frac{\chi(n+1)}{3}\right\rfloor} F_{n-2}^G(p,q,r,s,t,u), \end{aligned} \tag{1.2}$$

where

$$\chi(n) = \begin{cases} 0, & n \equiv 0 \pmod 3, \\ 1, & n \equiv 1 \pmod 3, \\ 2, & n \equiv 2 \pmod 3. \end{cases}$$

Patel, Shah [10] studied the special case of these trifurcating Fibonacci sequence with $p = a, q = b, r = c$, and $s = t = u = 1$. which gives us *left trifurcating Fibonacci sequence* $\left\{F_n^{L(a,b,c)}\right\}$.

In this paper, we study another case of the sequence $\left\{F_n^G(p,q,r,s,t,u)\right\}$ with $p = q = r = 1$ and $s = a, t = b, u = c$ which gives *right trifurcating Fibonacci sequence* $\left\{F_n^{R(a,b,c)}\right\}$. The terms of this sequence $\left\{F_n^{R(a,b,c)}\right\}$ are defined by the recurrence relation

$$F_n^R(a,b,c) = F_{n-1}^R(a,b,c) + a^{\alpha_k} b^{\beta_k} c^{\gamma_k} F_{n-2}^R(a,b,c). \tag{1.3}$$

Definition 1.1. For any three positive integers a, b and c, the *right trifurcating Fibonacci sequence* $\left\{F_n^{R(a,b,c)}\right\}$ is defined recursively by $F_0^{R(a,b,c)} = 0$, $F_1^{R(a,b,c)} = 1$ and the recurrence relation

$$F_n^{R(a,b,c)} = \begin{cases} F_{n-1}^{R(a,b,c)} + aF_{n-2}^{R(a,b,c)}; if\ n \equiv 0 (mod\ 3) \\ F_{n-1}^{R(a,b,c)} + bF_{n-2}^{R(a,b,c)}; if\ n \equiv 1 (mod\ 3) \\ F_{n-1}^{R(a,b,c)} + cF_{n-2}^{R(a,b,c)}; if\ n \equiv 2 (mod\ 3) \end{cases}.$$

To avoid cumbersome notation, throughout we denote $F_n^{R(a,b,c)}$ by Q_n. The first few terms of $\{Q_n\}$ are $\{0,\ 1,\ 1,\ a+1,\ a+b+1,\ ac+a+b+c+1, a^2+ab+ac+2a+b+c+1, \ldots\}$.

It is often useful to extend the sequence $\{Q_n\}$ backward with negative subscripts. If we try to extend the sequence backward while keeping the same rule, we obtain the doubly infinite sequence. This provides the terms of this bilateral sequence as $\left\{F_{k,n}^{R(a,b,c)}\right\}_{n=-\infty}^{\infty} =$ $\left\{\ldots,\ -\frac{1}{ab^2c}(a+c+1),\ \frac{1}{abc}(a+1),\ -\frac{1}{ab},\ \frac{1}{b},\ 0,\ 1,\ 1,\ 1+a, 1+a+b,\ \ldots\right\}$.

Thus, the sequence of right trifurcating Fibonacci numbers is bilateral, since it can be extended infinitely in both directions. From the terms of $\left\{F_{k,n}^{R(a,b,c)}\right\}_{n=-\infty}^{\infty}$, the following result follows immediately.

Lemma 1.2. $F_{-n}^{R(a,b,c)} = \frac{(-1)^{n+1}}{a^{\left\lfloor\frac{-n+1}{3}\right\rfloor} b^{\left\lfloor\frac{-n+2}{3}\right\rfloor} c^{\left\lfloor\frac{-n}{3}\right\rfloor}} F_n^{R(a,c,b)}$.

Firstly, we establish that each term and its immediate successor are always coprime in the sequence $\{Q_n\}$.

Theorem 1.3. $\gcd(Q_n, Q_{n-1}) = 1$*; for all* $n = 1,\ 2,\ \ldots$.

Proof. Suppose that Q_n and Q_{n-1} are both divisible by a positive integer d. Then clearly $Q_n - Q_{n-1} = aQ_{n-2}$ (if $n = 3k$) will also be divisible by d. Then, the right-hand side of this result is divisible by d. This gives $d|Q_{n-2}$. Continuing this argument, we see that $d|Q_{n-3}$, $d|Q_{n-4}$ and so on. Eventually, we must have $d|Q_1$. Since $Q_1 = 1$ we get $d = 1$, This gives the desired result. □

2 Fundamental identities for the sequence $\{Q_n\}$

In this section, we establish several interesting identities related to the elements of the sequence $\{Q_n\}$. We first prove some interesting summation identities for the elements of $\{Q_n\}$.

Lemma 2.1. *For $n \geq 1$, the following identities hold:*

(i)

$$bQ_{3n+2} = (bc+1)\sum_{k=1}^{n} Q_{3k} + (b-1)\sum_{k=1}^{n} Q_{3k+1} + b.$$

(ii)

$$aQ_{3n+1} = (ab+1)\sum_{k=1}^{n} Q_{3k-1} + (a-1)\sum_{k=1}^{n} Q_{3k} + a.$$

(iii)

$$cQ_{3n} = (ac+1)\sum_{k=1}^{n} Q_{3k-2} + (c-1)\sum_{k=1}^{n} Q_{3k-1}.$$

Proof. Since $Q_{3n+2} = Q_{3n+1} + cQ_{3n}$, we get

$$\begin{aligned} Q_2 &= Q_1 + cQ_0, \\ Q_5 &= Q_4 + cQ_3, \\ Q_8 &= Q_7 + cQ_6, \\ &\vdots \\ Q_{3n+2} &= Q_{3n+1} + cQ_{3n}. \end{aligned}$$

Summing up all of these equations, we obtain

$$Q_2 + Q_5 + Q_8 + \ldots + Q_{3n+2} = (Q_1 + Q_4 + \ldots + Q_{3n+1}) + c\,(Q_0 + Q_3 + \ldots + Q_{3n}) \quad (2.1)$$

Again, $Q_{3n} = Q_{3n-1} + aQ_{3n-2}$ gives

$$\begin{aligned} Q_3 &= Q_2 + aQ_1, \\ Q_6 &= Q_5 + aQ_4 \\ Q_9 &= Q_8 + aQ_7 \\ &\vdots \\ Q_{3n} &= Q_{3n-1} + aQ_{3n-2}. \end{aligned}$$

Summing up all of these equations, we obtain

$$Q_3 + Q_6 + \cdots + Q_{3n} = (Q_2 + Q_5 + \cdots + Q_{3n-1}) + a\,(Q_1 + Q_4 + \cdots + Q_{3n-2})\,. \quad (2.2)$$

Also, since $Q_{3n+1} = Q_{3n} + bQ_{3n-1}$, we have

$$\begin{aligned} Q_4 &= Q_3 + bQ_2, \\ Q_7 &= Q_6 + bQ_5, \\ Q_10 &= Q_9 + bQ_8, \\ &\vdots \\ Q_{3n+1} &= Q_{3n} + bQ_{3n-1}. \end{aligned}$$

Summing up all of these equations, we obtain

$$Q_4 + Q_7 + \ldots + Q_{3n+1} = (Q_3 + Q_6 + \ldots + Q_{3n}) + b\,(Q_2 + Q_5 + \ldots + Q_{3n-1}) \tag{2.3}$$

Using (2.2) in (2.1) we get

$$\begin{aligned} Q_2 + Q_5 + \cdots + Q_{3n+2} &= (Q_1 + Q_4 + \cdots + Q_{3n+1}) \\ &\quad + c\Big[\,(Q_2 + Q_5 + \cdots + Q_{3n-1}) + a\,(Q_1 + Q_4 + \cdots + Q_{3n-2})\,\Big] \\ &= (ac+1)\,(Q_1 + Q_4 + \cdots + Q_{3n-2}) \\ &\quad + c\,(Q_2 + Q_5 + \cdots + Q_{3n-1}) + Q_{3n+1}. \end{aligned}$$

This finally gives

$$cQ_{3n} = (ac+1)\,(Q_1 + Q_4 + \ldots + Q_{3n-2}) + (c-1)\,(Q_2 + Q_5 + \ldots + Q_{3n-1})$$

Other results can be proved accordingly by examining the pair of equations (2.2), (2.3), and subsequently (2.1), (2.3) collectively. □

We next obtain further summation identities for $\{Q_n\}$.

Theorem 2.2. *For $n \geq 0$, let*

$$\Delta = (ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1).$$

Then the following identities hold:

(a)

$$\sum_{i=0}^{n} Q_{3i} = \frac{1}{\Delta}\Big[a(Q_{3n+1}-1)(c-1)(b-1) - cQ_{3n}(ab+1)(ac+1) + (bQ_{3n-1}-1)(ab+1)(ac+1) + (ab+1)(ac+1)(bc+1)Q_{3n}\Big].$$

(b)

$$\sum_{i=0}^{n} Q_{3i+1} = \frac{1}{\Delta}\Big[(bQ_{3n+2}-1)(a-1)(c-1) - a(Q_{3n+1}-1)(c-1)(bc+1) + cQ_{3n}(ab+1)(bc+1) + (ab+1)(ac+1)(bc+1)Q_{3n+1}\Big].$$

(c)

$$\sum_{i=0}^{n} Q_{3i+2} = \frac{1}{\Delta}\Big[cQ_{3n+3}(a-1)(b-1) - (bQ_{3n+2}-1)(a-1)(ac+1) + a(Q_{3n+1}-1)(ac+1)(bc+1) + (ab+1)(ac+1)(bc+1)Q_{3n+2}\Big].$$

Proof. We exclusively establish the proof for the result (a) here, while the remaining two results can be demonstrated analogously. Utilizing (ii) and (iii) from Lemma 2.1, we obtain

$$\begin{aligned} cQ_{3n+3} &= \frac{(ac+1)}{(b-1)}\left\{(bQ_{3n+2}-1) - (bc+1)\,(Q_0 + Q_3 + \ldots + Q_{3n})\right\} \\ &\quad + (c-1)\,(Q_2 + Q_5 + \ldots + Q_{3n+2})\,. \end{aligned}$$

Then

$$\begin{aligned} cQ_{3n+3} - \frac{ac+1}{b-1}\,(bQ_{3n+2}-1) - (c-1)\,(Q_2 + Q_5 + \cdots + Q_{3n+2}) \\ = -\frac{(ac+1)(bc+1)}{b-1}\,(Q_0 + Q_3 + \cdots + Q_{3n})\,. \end{aligned}$$

Using Lemma (2.1) (ii), we get

$$cQ_{3n+3} - \frac{(ac+1)}{(b-1)}\left(bQ_{3n+2}-1\right) - (c-1)\left(Q_2+Q_5+\ldots+Q_{3n+2}\right)$$
$$= -\frac{(bc+1)(ac+1)}{(b-1)(c-1)}\left(a\left(Q_{3n+1}-1\right) - (ab+1)(Q_2+Q_5+\ldots+Q_{3n-1})\right)$$

Then,

$$\left((ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)\right)\sum_{k=0}^{n} Q_{3k+2}$$
$$= cQ_{3n+3}(a-1)(b-1) - (bQ_{3n+2}-1)(a-1)(ac+1)$$
$$+ a(Q_{3n+1}-1)(ac+1)(bc+1) + (ab+1)(ac+1)(bc+1)Q_{3n+2}$$

Hence,

$$Q_2+Q_5+\cdots+Q_{3n+2} = \frac{1}{\Delta}\Big\{cQ_{3n+3}(a-1)(b-1)$$
$$- (bQ_{3n+2}-1)(a-1)(ac+1)$$
$$+ a(Q_{3n+1}-1)(ac+1)(bc+1)$$
$$+ (ab+1)(ac+1)(bc+1)Q_{3n+2}\Big\}.$$

□

We now give the sum of the first k terms of the sequence $\{Q_n\}$.

Theorem 2.3. *Let*

$$\rho = \frac{1}{(ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1)}$$

Let $m = \lfloor k/3 \rfloor$. *Then* $\sum_{n=1}^{k} Q_n = \rho\,\mathcal{S}$, *where*

$$\mathcal{S} = a(Q_{3m+1}-1)\big[(c-1)(b-bc-2)+(ac+1)(bc+1)\big]$$
$$+ (bQ_{3m-1}-1)(ab+1)(ac+1) - (bQ_{3m+2}-1)(a-1)(ac+c)$$
$$+ cQ_{3m+3}(a-1)(b-1) + cQ_{3m}(ab+1)(bc-b+2)$$
$$+ (ab+1)(ac+1)(bc+1)\left[Q_{3m} + \left(1-\lfloor 1-\tfrac{\chi(k+1)}{3}\rfloor\right)Q_{3m+1} + \lfloor 1-\tfrac{\chi(k)}{3}\rfloor Q_{3m+2}\right]$$
$$- (a-1)(b-1)(c-1)\left[\lfloor 1-\tfrac{\chi(k+1)}{3}\rfloor Q_{3m+1} + \left(1-\lfloor 1-\tfrac{\chi(k)}{3}\rfloor\right)Q_{3m+2}\right].$$

Proof. We begin by evaluating the sum $\sum_{n=1}^{k} Q_n$ for three scenarios: when $k = 3m,\ 3m+1,$ and $3m+2$. Subsequently, we consolidate these individual results into a unified outcome. We let

$$1/\rho = (ab+1)(ac+1)(bc+1)+(a-1)(b-1)(c-1).$$

Now for $k = 3m+2$, using the above theorem we get

$$\sum_{n=1}^{k} Q_n = (Q_3+Q_6+\cdots+Q_{3m}) + (Q_1+Q_4+\cdots+Q_{3m+1})$$
$$+ (Q_2+Q_5+\cdots+Q_{3m+2})$$
$$= \rho\Big[a(Q_{3m+1}-1)\{(c-1)(b-1)-(c-1)(bc+1)+(ac+1)(bc+1)\}$$
$$+ (bQ_{3m-1}-1)(ab+1)(ac+1) + cQ_{3m+3}(a-1)(b-1)$$

$$
\begin{aligned}
&+ (bQ_{3m+2} - 1)(a-1)\{(c-1) - (ac+1)\} \\
&+ cQ_{3m}(ab+1)\{(bc+1) - (b-1)\} \\
&+ (ab+1)(ac+1)(bc+1)(Q_{3m} + Q_{3m+1} + Q_{3m+2})\Big]. \qquad (2.4)
\end{aligned}
$$

Next, for the case $k = 3m+1$, we get

$$
\begin{aligned}
\sum_{n=1}^{k} Q_n &= (Q_3 + Q_6 + \cdots + Q_{3m}) + (Q_1 + Q_4 + \cdots + Q_{3m+1}) \\
&\quad + (Q_2 + Q_5 + \cdots + Q_{3m-1}) \\
&= \rho\Big[a(Q_{3m+1} - 1)\{(c-1)(b-1) - (c-1)(bc+1) + (ac+1)(bc+1)\} \\
&\quad + (bQ_{3m-1} - 1)(ab+1)(ac+1) + cQ_{3m+3}(a-1)(b-1) \\
&\quad + (bQ_{3m+2} - 1)(a-1)\{(c-1) - (ac+1)\} \\
&\quad + cQ_{3m}(ab+1)\{(bc+1) - (b-1)\} \\
&\quad + (ab+1)(ac+1)(bc+1)(Q_{3m} + Q_{3m+1}) \\
&\quad - (a-1)(b-1)(c-1)Q_{3m+2}\Big]. \qquad (2.5)
\end{aligned}
$$

Finally, for $k = 3m$, we get

$$
\begin{aligned}
\sum_{n=1}^{k} Q_n &= (Q_3 + Q_6 + \cdots + Q_{3m}) + (Q_1 + Q_4 + \cdots + Q_{3m-2}) \\
&\quad + (Q_2 + Q_5 + \cdots + Q_{3m-1}) \\
&= \rho\Big[a(Q_{3m+1} - 1)\{(c-1)(b-1) - (c-1)(bc+1) + (ac+1)(bc+1)\} \\
&\quad + (bQ_{3m-1} - 1)(ab+1)(ac+1) + (ab+1)(ac+1)(bc+1)Q_{3m} \\
&\quad + (bQ_{3m+2} - 1)(a-1)\{(c-1) - (ac+1)\} \\
&\quad + cQ_{3m}(ab+1)\{(bc+1) - (b-1)\} + cQ_{3m+3}(a-1)(b-1) \\
&\quad - (a-1)(b-1)(c-1)(Q_{3m+1} + Q_{3m+2})\Big]. \qquad (2.6)
\end{aligned}
$$

Combining these results, we get the required result. The above are some key identities regarding the summation of right trifurcating Fibonacci numbers. □

Theorem 2.4. $\sum_{k=0}^{n} a^{\left\lfloor 1 - \frac{\chi(k+2)}{3} \right\rfloor} b^{\left\lfloor 1 - \frac{\chi(k+1)}{3} \right\rfloor} c^{\left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor} Q_k = Q_{n+2} - 1.$

Proof. By applying the definition of right trifurcating Fibonacci numbers, we have

$$Q_{3n} = Q_{3n-1} + aQ_{3n-2}, \quad Q_{3n+1} = Q_{3n} + bQ_{3n-1}, \quad Q_{3n+2} = Q_{3n+1} + cQ_{3n}.$$

These expressions can be represented as

$$aQ_{3n-2} = Q_{3n} - Q_{3n-1}, \quad bQ_{3n-1} = Q_{3n+1} - Q_{3n}, \quad cQ_{3n} = Q_{3n+2} - Q_{3n+1}.$$

Thus, we can derive the following set of equations:

$$\begin{aligned}
cQ_0 &= Q_2 - Q_1,\\
aQ_1 &= Q_3 - Q_2,\\
bQ_2 &= Q_4 - Q_3,\\
&\vdots\\
aQ_n &= Q_{n+2} - Q_{n+1}, \quad \text{if } n \equiv 1 \pmod 3,\\
bQ_n &= Q_{n+2} - Q_{n+1}, \quad \text{if } n \equiv 2 \pmod 3,\\
cQ_n &= Q_{n+2} - Q_{n+1}, \quad \text{if } n \equiv 0 \pmod 3.
\end{aligned}$$

Combining the aforementioned equations and utilizing the initial conditions $Q_0 = 0$ and $Q_1 = 1$, we obtain

$$\sum_{k=0}^{n} a^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor} Q_k = Q_{n+2} - 1.$$

□

Theorem 2.5.

$$\sum_{k=0}^{n} a^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor} Q_{i+k} = Q_{n+i+2} - Q_{i+1}.$$

Proof. By expanding the shifted summation, we have

$$\begin{aligned}
&\sum_{k=0}^{n} a^{\left\lfloor 1-\frac{\chi(k+2)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(k+1)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(k)}{3}\right\rfloor} Q_{i+k}\\
&\quad= \sum_{r=0}^{n+i} a^{\left\lfloor 1-\frac{\chi(r+2)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(r+1)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(r)}{3}\right\rfloor} Q_r\\
&\qquad- \sum_{r=0}^{i-1} a^{\left\lfloor 1-\frac{\chi(r+2)}{3}\right\rfloor} b^{\left\lfloor 1-\frac{\chi(r+1)}{3}\right\rfloor} c^{\left\lfloor 1-\frac{\chi(r)}{3}\right\rfloor} Q_r\\
&\quad= Q_{n+i+2} - 1 - (Q_{i+1} - 1)\\
&\quad= Q_{n+i+2} - Q_{i+1}.
\end{aligned}$$

□

Theorem 2.6.

$$\begin{aligned}
\sum_{k=0}^{n} \Bigg(& a^{\left\lfloor \frac{n-k+2}{3}\right\rfloor} b^{\left\lfloor \frac{n-k}{3}\right\rfloor} c^{\left\lfloor \frac{n-k+1}{3}\right\rfloor} \left\lfloor 1 - \frac{\chi(n+1)}{3} \right\rfloor\\
&+ a^{\left\lfloor \frac{n-k}{3}\right\rfloor} b^{\left\lfloor \frac{n-k+1}{3}\right\rfloor} c^{\left\lfloor \frac{n-k+2}{3}\right\rfloor} \left\lfloor 1 - \frac{\chi(n+2)}{3} \right\rfloor\\
&+ a^{\left\lfloor \frac{n-k+1}{3}\right\rfloor} b^{\left\lfloor \frac{n-k+2}{3}\right\rfloor} c^{\left\lfloor \frac{n-k}{3}\right\rfloor} \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \Bigg) Q_k^2 = Q_n Q_{n+1}.
\end{aligned}$$

Proof. We demonstrate this result specifically for the scenario where $n \equiv 0 \pmod 3$, with the remaining cases following suitably. Letting $n = 3m$, we apply mathematical induction on m.

For $m = 1$, we have

$$\sum_{k=0}^{3} a^{\left\lfloor \frac{3-k+1}{3}\right\rfloor} b^{\left\lfloor \frac{3-k+2}{3}\right\rfloor} c^{\left\lfloor \frac{3-k}{3}\right\rfloor} Q_k^2 = abcQ_0^2 + abQ_1^2 + bQ_2^2 + Q_3^2.$$

Since $Q_0 = 0$, $Q_1 = 1$, $Q_2 = 1$, and $Q_3 = 1 + a$, we obtain

$$\sum_{k=0}^{3} a^{\lfloor \frac{3-k+1}{3} \rfloor} b^{\lfloor \frac{3-k+2}{3} \rfloor} c^{\lfloor \frac{3-k}{3} \rfloor} Q_k^2 = (1+a)(1+a+b) = Q_3 Q_4.$$

We then suppose that the result is valid for a certain positive integer $m = l > 1$. That is let

$$\sum_{k=0}^{3l} a^{\lfloor \frac{3l-k+1}{3} \rfloor} b^{\lfloor \frac{3l-k+2}{3} \rfloor} c^{\lfloor \frac{3l-k}{3} \rfloor} Q_k^2 = Q_{3l} Q_{3l+1}.$$

Now, considering the sum up to $3(l+1)$:

$$\begin{aligned}
&\sum_{k=0}^{3(l+1)} a^{\lfloor \frac{3l+3-k+1}{3} \rfloor} b^{\lfloor \frac{3l+3-k+2}{3} \rfloor} c^{\lfloor \frac{3l+3-k}{3} \rfloor} Q_k^2 \\
&\quad = \sum_{k=0}^{3l} a^{\lfloor \frac{3l+3-k+1}{3} \rfloor} b^{\lfloor \frac{3l+3-k+2}{3} \rfloor} c^{\lfloor \frac{3l+3-k}{3} \rfloor} Q_k^2 + abQ_{3l+1}^2 + bQ_{3l+2}^2 + Q_{3l+3}^2 \\
&\quad = abcQ_{3l}Q_{3l+1} + abQ_{3l+1}^2 + bQ_{3l+2}^2 + Q_{3l+3}^2 \\
&\quad = abQ_{3l+1}Q_{3l+2} + bQ_{3l+2}^2 + Q_{3l+3}^2 \\
&\quad = bQ_{3l+2}Q_{3l+3} + Q_{3l+3}^2 \\
&\quad = Q_{3(l+1)}Q_{3(l+1)+1}.
\end{aligned}$$

Hence, by induction, we confirm that the result holds for all positive integers n. □

Theorem 2.7.

$$\begin{aligned}
\sum_{k=0}^{n} \Bigg(& a^{\lfloor \frac{n-k+2}{3} \rfloor} b^{\lfloor \frac{n-k}{3} \rfloor} c^{\lfloor \frac{n-k+1}{3} \rfloor} \left\lfloor 1 - \frac{\chi(n+1)}{3} \right\rfloor \\
& + a^{\lfloor \frac{n-k}{3} \rfloor} b^{\lfloor \frac{n-k+1}{3} \rfloor} c^{\lfloor \frac{n-k+2}{3} \rfloor} \left\lfloor 1 - \frac{\chi(n+2)}{3} \right\rfloor \\
& + a^{\lfloor \frac{n-k+1}{3} \rfloor} b^{\lfloor \frac{n-k+2}{3} \rfloor} c^{\lfloor \frac{n-k}{3} \rfloor} \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \Bigg) Q_{i+k}^2 = Q_{n+i}Q_{n+i+1} - Q_i Q_{i+1}.
\end{aligned}$$

Proof. We can demonstrate this theorem by applying the same reasoning as in Theorem 2.5. □

3 Binet-Type Formula for $\{Q_n\}$

In this section, we establish the generating function for the sequence $\{Q_n\}$ and subsequently utilize it to derive a Binet-like formula for these numbers. To begin, we establish a relationship between three consecutive terms of the sequence $\{Q_n\}$, which will be essential for determining the generating function of $\{Q_n\}$.

Lemma 3.1. $Q_{n+3} - (a+b+c+1)Q_n - abcQ_{n-3} = 0$.

Proof. We prove the result by considering the three cases when $n = 3k$, $n = 3k+1$, and $n = 3k+2$. We present the proof only for the case $n = 3k$, and other cases follow accordingly. Using the definition, We get

$$\begin{aligned}
Q_{3k+3} &= Q_{3k+2} + aQ_{3k+1} = Q_{3k+1} + cQ_{3k} + a\left(Q_{3k} + bQ_{3k-1}\right) \\
&= Q_{3k} + bQ_{3k-1} + (a+c)Q_{3k} + abQ_{3k-2} + abcQ_{3k-3}.
\end{aligned}$$

Expanding on the given definition, if we multiply both sides of $Q_{3k} = Q_{3k-1} + aQ_{3k-2}$ by b, we obtain $bQ_{3k} = bQ_{3k-1} + abQ_{3k-2}$.

Substituting this value into the equation provided earlier,we obtain:

$$Q_{3k+3} = (a+b+c+1)Q_{3k} + abcQ_{3k-3}.$$

Hence, $Q_{3k+3} - (a+b+c+1)Q_{3k} - abcQ_{3k-3} = 0$. □

Next, we derive the generating function for the sequence $\{Q_n\}$ when $n \equiv 0, 1, 2(mod\ 3)$.

Lemma 3.2. *The generating function of the subsequence* $\{Q_m\}$ *of* $\{Q_n\}$ *is*

(i) $f(x) = \frac{x^2+acx^5}{1-(a+b+c+1)x^3-abcx^6}$ *; when* $m \equiv 2 \pmod 3$

(ii) $g(x) = \frac{x-cx^4}{1-(a+b+c+1)x^3-abcx^6}$ *; when* $m \equiv 1 \pmod 3$

(iii) $h(x) = \frac{(a+1)x^3}{1-(a+b+c+1)x^3-abcx^6}$ *; when* $m \equiv 0 \pmod 3$.

Proof. We present the proof only for the case $m \equiv 2 \pmod 3$. The other cases can be proved accordingly.

We first let

$$f(x) = \sum_{n=1}^{\infty} Q_{3n-1}x^{3n-1} = Q_2x^2 + Q_5x^5 + Q_8x^8 + \dots.$$

Then

$$(a+b+c+1)x^3\ f(x) = (a+b+c+1)Q_2x^5 + (a+b+c+1)Q_5x^8 + \dots$$

and

$$x^6f(x) = abcQ_2x^8 + abcQ_5x^{11} + abcQ_8x^{14} + \dots.$$

Using Lemma 3.1, we get

$$\begin{aligned}
&\big(1-(a+b+c+1)x^3-abcx^6\big)f(x)\\
&= x^2 + acx^5 - \sum_{k=2}^{\infty}\big(Q_{3k+2} - (a+b+c+1)Q_{3k-1} - abcQ_{3k-4}\big)x^{3k+2}\\
&= x^2 + acx^5.
\end{aligned}$$

Thus,

$$f(x) = \frac{x^2+acx^5}{1-(a+b+c+1)x^3-abcx^6},$$

as required. □

The following outcome provides the generating function for the sequence $\{Q_n\}$.

Theorem 3.3. *The generating function for the sequence* $\{Q_n\}$ *is*

$$F(x) = \frac{x+x^2+(1+a)x^2-ax^3+x^4}{1-(a+b+c+1)x^3-abcx^6}.$$

Proof. We consider the power series representation of the generating function for $\{Q_n\}$.

Let

$$F(x) = Q_0 + Q_1x + Q_2x^2 + \ldots + Q_kx^k + \ldots = \sum_{m=0}^{\infty} Q_mx^m.$$

Then,

$$xF(x) = Q_0x + Q_1x^2 + Q_2x^3 + \ldots + Q_kx^{k+1} + \ldots = \sum_{m=0}^{\infty} Q_mx^{m+1} = \sum_{m=1}^{\infty} Q_{m-1}x^m.$$

Also,

$$cx^2F(x) = cQ_0x^2 + cQ_1x^3 + cQ_2x^4 + \ldots + cQ_kx^{k+2} + \ldots = \sum_{m=0}^{\infty} cQ_mx^{m+2}.$$

Since $Q_{3k+2} = Q_{3k+1} + cQ_{3k}$, we get

$$\left(1 - x - cx^2\right) F(x) = x + \sum_{n=1}^{\infty} \left(Q_{3n} - Q_{3n-1} - cQ_{3n-2}\right) x^{3n} + \sum_{n=1}^{\infty} (Q_{3n+1} - Q_{3n} - cQ_{3n-1}) x^{3n+1}.$$

Since $Q_{3k} = Q_{3k-1} + aQ_{3k-2}$ and $Q_{3k+1} = Q_{3k} + bQ_{3k-1}$, we get

$$\left(1 - x - cx^2\right) F(x) = x + (a-c)x^2 \sum_{n=1}^{\infty} Q_{3n-2} x^{3n-2} + (b-c)x^2 \sum_{n=1}^{\infty} Q_{3n-1} x^{3n-1}.$$

For convenience, we let

$$f(x) = \sum_{n=1}^{\infty} Q_{3n-1} x^{3n-1} \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} Q_{3n-2} x^{3n-2}.$$

Using Lemma 3.2 (i) and 3.2 (ii) we get

$$\left(1 - x - cx^2\right) F(x) = x + (a-c)x^2 \frac{x - cx^4}{1 - (a+b+c+1)x^3 - abcx^6} + (b-c)x^2 \frac{x^2 + acx^5}{1 - (a+b+c+1)x^3 - abcx^6}.$$

After simplification, we obtain the desired outcome. □

We now obtain the Binet-like formula for the sequence $\{Q_n\}$.

Theorem 3.4. *The terms of the sequence $\{Q_n\}$ are given by*

$$Q_n = \frac{\gamma(n)\alpha^{\left\lfloor \frac{n}{3} \right\rfloor} - \delta(n)\beta^{\left\lfloor \frac{n}{3} \right\rfloor}}{\alpha - \beta},$$

where

$$\gamma(n) = \left(\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor \alpha + (-1)^{\chi(n)} a^{\left\lfloor \frac{1-(-1)^{\chi(n)}}{2} \right\rfloor} c^{\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor} + \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \right),$$

and

$$\delta(n) = \left(\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor \beta + (-1)^{\chi(n)} a^{\left\lfloor \frac{1-(-1)^{\chi(n)}}{2} \right\rfloor} c^{\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor} + \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \right);$$

with $\alpha = \frac{u+\sqrt{u^2+4abc}}{2}$, $\beta = \frac{u-\sqrt{u^2+4abc}}{2}$, $u = a+b+c+1$,

$$\chi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{if } n \equiv 1 \pmod 3, \\ 2 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. From the generating function of $\{Q_n\}$, we have

$$F(x) = \frac{x + x^2 + (1+a)x^2 - ax^3 + x^4}{1 - (a+b+c+1)x^3 - abcx^6}.$$

Now, if α, β are the roots of $1 - (a+b+c+1)x - abcx^2 = 0$, then

$$\alpha = \frac{u + \sqrt{u^2+4abc}}{2}, \ \beta = \frac{u - \sqrt{u^2+4abc}}{2}.$$

If we let $u = a + b + c + 1$, then it is easy to observe that

$$\alpha\beta = -1,\ \alpha + \beta = u \quad \text{and} \quad \alpha - \beta = \sqrt{u^2 + 4abc}.$$

This can be rewritten as

$$F(x) = \frac{x + x^2 + (1+a)x^2 - ax^3 + x^4}{(abcx^3 + \alpha)(abcx^3 + \beta)}.$$

$$F(x) = -\frac{1}{\alpha - \beta}\left\{\frac{(1+a)\alpha - (abc + c\alpha)x + (ac\alpha - abc)x^2}{abcx^3 + \alpha} - \frac{(1+a)\beta - (abc + c\beta)x + (ac\beta - abc)x^2}{abcx^3 + \beta}\right\}.$$

Using McLaurin series expansion, we get

$$\begin{aligned} F(x) = \frac{-1}{(\alpha-\beta)(abc)}\Bigg\{&\sum_{n=0}^{\infty}\frac{(-1)^n(1+a)\alpha}{\left(\frac{\alpha}{abc}\right)^{n+1}}x^{3n} - \sum_{n=0}^{\infty}\frac{(-1)^n(abc+c\alpha)}{\left(\frac{\alpha}{abc}\right)^{n+1}}x^{3n+1} \\ &+ \sum_{n=0}^{\infty}\frac{(-1)^n(ac\alpha - abc)}{\left(\frac{\alpha}{abc}\right)^{n+1}}x^{3n+2} - \sum_{n=0}^{\infty}\frac{(-1)^n(1+a)\beta}{\left(\frac{\beta}{abc}\right)^{n+1}}x^{3n} \\ &+ \sum_{n=0}^{\infty}\frac{(-1)^n(abc+c\beta)}{\left(\frac{\beta}{abc}\right)^{n+1}}x^{3n+1} - \sum_{n=0}^{\infty}\frac{(-1)^n(ac\beta - abc)}{\left(\frac{\beta}{abc}\right)^{n+1}}x^{3n+2}\Bigg\} \\ = \frac{-1}{(\alpha-\beta)(abc)}\Bigg\{&\sum_{n=0}^{\infty}(-1)^n(abc)^{n+1}(1+a)\left(\frac{\beta^n - \alpha^n}{(\alpha\beta)^n}\right)x^{3n} \\ &- \sum_{n=0}^{\infty}(-1)^n(abc)^{n+1}\left(\frac{(abc+c\alpha)\beta^{n+1} - (abc+c\beta)\alpha^{n+1}}{(\alpha\beta)^{n+1}}\right)x^{3n+1} \\ &+ \sum_{n=0}^{\infty}(-1)^n(abc)^{n+1}\left(\frac{(ac\alpha - abc)\beta^{n+1} - (ac\beta - abc)\alpha^{n+1}}{(\alpha\beta)^{n+1}}\right)x^{3n+2}\Bigg\}. \end{aligned}$$

Then

$$\begin{aligned} F(x) = \Bigg\{&\sum_{n=0}^{\infty}(1+a)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)x^{3n} \\ &- \sum_{n=0}^{\infty}\left(\frac{(\alpha - c)\alpha^n - (\beta - c)\beta^n}{\alpha - \beta}\right)x^{3n+1} \\ &+ \sum_{n=0}^{\infty}\left(\frac{(ac+\alpha)\alpha^{n+1} - (ac+\beta)\beta^{n+1}}{\alpha - \beta}\right)x^{3n+2}\Bigg\}. \end{aligned}$$

Thus,

$$\begin{aligned} F(x) = \sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}\Bigg[&\left(\left\lfloor\tfrac{\chi(n)+2}{3}\right\rfloor\alpha + \left\lfloor 1 - \tfrac{\chi(n)}{3}\right\rfloor\right. \\ &\left. + (-1)^{\chi(n)}a^{\left\lfloor\frac{1-(-1)^{\chi(n)}}{2}\right\rfloor}c^{\left\lfloor\frac{\chi(n)+2}{3}\right\rfloor}\right)\alpha^{\lfloor\frac{n}{3}\rfloor} \\ &- \left(\left\lfloor\tfrac{\chi(n)+2}{3}\right\rfloor\beta + \left\lfloor 1 - \tfrac{\chi(n)}{3}\right\rfloor\right. \\ &\left. + (-1)^{\chi(n)}a^{\left\lfloor\frac{1-(-1)^{\chi(n)}}{2}\right\rfloor}c^{\left\lfloor\frac{\chi(n)+2}{3}\right\rfloor}\right)\beta^{\lfloor\frac{n}{3}\rfloor}\Bigg]x^n. \end{aligned}$$

For convenience if we write

$$\chi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{if } n \equiv 1 \pmod 3, \\ 2 & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

and

$$\gamma(n) = \left(\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor \alpha + (-1)^{\chi(n)} a^{\left\lfloor \frac{1-(-1)^{\chi(n)}}{2} \right\rfloor} c^{\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor} + \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \right),$$

$$\delta(n) = \left(\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor \beta + (-1)^{\chi(n)} a^{\left\lfloor \frac{1-(-1)^{\chi(n)}}{2} \right\rfloor} c^{\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor} + \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \right),$$

then $F(x)$ can be written as

$$F(x) = \sum_{n=0}^{\infty} \frac{\gamma(n)\alpha^{\lfloor \frac{n}{3} \rfloor} - \delta(n)\beta^{\lfloor \frac{n}{3} \rfloor}}{\alpha - \beta} x^n.$$

This gives

$$Q_n = \frac{\gamma(n)\alpha^{\lfloor \frac{n}{3} \rfloor} - \delta(n)\beta^{\lfloor \frac{n}{3} \rfloor}}{\alpha - \beta},$$

as desired. □

The following results are the easy consequence of this theorem.

Corollary 3.5. *(i)* $Q_{3n} = (1+a)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$

(ii) $Q_{3n+1} = \left(\frac{(\alpha - c)\alpha^n - (\beta - c)\beta^n}{\alpha - \beta}\right)$

(iii) $Q_{3n+2} = \left(\frac{(ac+\alpha)\alpha^n - (ac+\beta)\beta^n}{\alpha - \beta}\right)$.

We now express α^n and β^n in terms of $\alpha^{n-1}, \alpha^{n-2}$ and β^{n-1}, β^{n-2}, respectively.

Theorem 3.6. *(i)* $\alpha^n = u\alpha^{n-1} + abc\alpha^{n-2}$

(ii) $\beta^n = u\beta^{n-1} + abc\beta^{n-2}$.

Proof. We prove both results by induction on n.

For $n = 2$,

$$\alpha^2 = \left(\frac{u + \sqrt{u^2 + 4abc}}{2}\right)^2 = \frac{u\left(u + \sqrt{u^2 + 4abc}\right) + 2abc}{2} = u\alpha + abc.$$

Thus, the validity of the result is confirmed for $n = 2$. Assuming it holds for some positive integer $n = k$, we have

$$\alpha^k = u\alpha^{k-1} + abc\alpha^{k-2}.$$

Now, we aim to prove that it also holds for $n = k + 1$. We begin by noting that

$$\alpha^{k+1} = \alpha^{k-1}\left(\alpha^2\right) = \alpha^{k-1}(u\alpha + abc) = u\alpha^k + abc\alpha^{k-1}.$$

Hence we conclude that the statement is true for all positive integers n. In simplification, the second result can be proven accordingly. □

Next, we present another expression for Q_n.

Theorem 3.7. *If* $F(x) = \sum_{m=0}^{\infty} Q_m x^m$; $x \in \left(-\frac{1}{\alpha}, \frac{1}{\alpha}\right)$, *then* $Q_n = \frac{F^{(n)}(0)}{n!}$, *where* $F^{(n)}(x)$ *denotes the* n^{th} *order derivative of the function* $F(x)$.

Proof. For $F(x) = \sum_{m=0}^{\infty} Q_m x^m$, we have $Q_0 x^0 = 0$ and

$$F'(x) = \sum_{m=1}^{\infty} m Q_m x^{m-1} = 1 + \sum_{m=2}^{\infty} m Q_m x^{m-1}.$$

Also,

$$F^{(2)}(x) = \sum_{m=2}^{\infty} m(m-1) Q_m x^{m-2} = 2 \times 1 \times 1 + \sum_{m=3}^{\infty} m(m-1) Q_m x^{m-2}$$

$$\vdots$$

$$F^{(l)}(x) = l(l-1)(l-2)\cdots(l-(l-1))\, Q_l + \sum_{m=l+1}^{\infty} m(m-1)\cdots(m-(l-1)) Q_m x^{m-l}$$

then we get

$$F^{(l)}(x) = l! Q_l + \sum_{m=l+1}^{\infty} m(m-1)\cdots(m-(l-1)) Q_m x^{m-l}.$$

Thus, $F^{(l)}(0) = l! Q_l$, which gives $Q_l = \frac{F^{(l)}(0)}{l!}$. Hence $Q_n = \frac{F^{(n)}(0)}{n!}$ as required. □

4 Some more identities relating $\{Q_n\}$

In this section, we apply the Binet-like formula mentioned earlier to uncover some interesting properties of the terms $\{Q_n\}$. If we let $\gamma(n)\delta(n) = w$, then we see that

$$w = \left\lfloor \frac{\chi(n)+2}{3} \right\rfloor \left\{ \alpha\beta + \left((-1)^{\chi(n)} a^{\left\lfloor \frac{1+(-1)^{\chi(n)}}{2} \right\rfloor} c^{\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor} + \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \right) (\alpha+\beta) \right\}$$

$$+ \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \left(1 + 2(-1)^{\chi(n)} a^{\left\lfloor \frac{1+(-1)^{\chi(n)}}{2} \right\rfloor} c^{\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor} \right) + a^{2\left\lfloor \frac{1+(-1)^{\chi(n)}}{2} \right\rfloor} c^{2\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor}.$$

Upon simplification, this yields the simplified value of $w = \gamma(n)\delta(n)$ as

$$w = \begin{cases} (1+a)^2, & \text{if } n \equiv 0 \pmod 3 \\ (\alpha - c)(\beta - c), & \text{if } n \equiv 1 \pmod 3 \\ (\alpha + ac)(\beta + ac), & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

We begin by establishing an identity for the terms of $\{Q_n\}$, similar to how Catalan's identity operates for Fibonacci numbers.

Theorem 4.1. *For any two non-negative integers k and r with $r \le \lfloor \frac{k}{3} \rfloor$, we have*

$$Q_{k-3r} Q_{k+3r} - Q_k^2 = -(-abc)^{\lfloor \frac{k}{3} \rfloor - r} w \left(\frac{Q_{3r}}{Q_3} \right)^2.$$

Proof. We prove the result by considering the three cases when $n = 3k$, $3k+1$, or $3k+2$. We present the proof only for the case $n = 3k+1$, as the other cases follow accordingly.

If we let $k \equiv 1 \pmod 3$, then by setting $k = 3l + 1$ and applying Corollary 3.5, we obtain:

$$\begin{aligned}
&Q_{3l-3r+1}Q_{3l+3r+1} - Q_{3l+1}^2 \\
&\quad= \left(\frac{(\alpha-c)\alpha^{l-r} - (\beta-c)\beta^{l-r}}{\alpha-\beta}\right)\left(\frac{(\alpha-c)\alpha^{l+r} - (\beta-c)\beta^{l+r}}{\alpha-\beta}\right) \\
&\qquad- \left(\frac{(\alpha-c)\alpha^{l} - (\beta-c)\beta^{l}}{\alpha-\beta}\right)^2 \\
&\quad= \frac{1}{(\alpha-\beta)^2}\Big\{\left(\alpha^{l-r+1} - \beta^{l-r+1}\right)\left(\alpha^{l+r+1} - \beta^{l+r+1}\right) \\
&\qquad- c\left(\alpha^{l-r} - \beta^{l-r}\right)\left(\alpha^{l+r+1} - \beta^{l+r+1}\right) \\
&\qquad- c\left(\alpha^{l-r+1} - \beta^{l-r+1}\right)\left(\alpha^{l+r} - \beta^{l+r}\right) \\
&\qquad+ c^2\left(\alpha^{l-r} - \beta^{l-r}\right)\left(\alpha^{l+r} - \beta^{l+r}\right) \\
&\qquad- \left(c^2\left(\alpha^{l} - \beta^{l}\right)^2 - 2c\left(\alpha^{l} - \beta^{l}\right)\left(\alpha^{l+1} - \beta^{l+1}\right) + \left(\alpha^{l+1} - \beta^{l+1}\right)^2\right)\Big\} \\
&\quad= \frac{1}{(\alpha-\beta)^2}\Big\{c^2(-abc)^{l-r}\left[\beta^{2r} + \alpha^{2r} + 2(-abc)^r\right] \\
&\qquad- (-abc)^{l-r}\left[\beta^{2r} + \alpha^{2r} + 2(-abc)^r\right] \\
&\qquad- c(-abc)^{l+1}\left[\left(\frac{\beta}{\alpha}\right)^r(\alpha+\beta) + \left(\frac{\alpha}{\beta}\right)^r(\alpha+\beta) + 2(\alpha+\beta)\right]\Big\} \\
&\quad= \frac{-(-abc)^{l-r}}{(\alpha-\beta)^2}\left\{c^2\left(\alpha^r - \beta^r\right)^2 - cu\left[\beta^{2r} + \alpha^{2r} + 2(-abc)^r\right] - abc\left(\alpha^r - \beta^r\right)^2\right\} \\
&\quad= \frac{-(-abc)^{l-r}}{(\alpha-\beta)^2}\left(\alpha^r - \beta^r\right)^2\left\{c^2 - cu - abc\right\} \\
&\quad= \frac{-(-abc)^{l-r}}{(1+a)^2}(\alpha-c)(\beta-c)Q_{3r}^2.
\end{aligned}$$

Thus,

$$Q_{3l-3r+1}Q_{3l+3r+1} - Q_{3l+1}^2 = -(-abc)^{l-r}wQ_{3r}^2Q_3^{-2}.$$

This completes the required result. □

The subsequent identity mirrors Cassini's identity for Fibonacci numbers.

Corollary 4.2. *For any integer $k \geq 3$, we have*

$$Q_{k-3}Q_{k+3} - Q_k^2 = -(-abc)^{\lfloor \frac{k}{3} \rfloor - 1}w.$$

Proof. Follows immediately by setting $r = 1$ in Theorem 4.1. □

The subsequent identity is similar to Cassini's identity for Fibonacci numbers.

Theorem 4.3. *Where m, n are non-negative integers such that $m \geq n$ and $m \equiv n \pmod 3$, we have*

$$Q_mQ_{n+3} - Q_{m+3}Q_n = (-abc)^nQ_{m-n}\left(\frac{w}{Q_3}\right).$$

Proof. Since $m \equiv n \pmod 3$, we first consider both m and n to be of the form $3j+2$ and $3k+2$,

respectively. Then, for positive integers j and $k \leq j$, we obtain:

$$\begin{aligned}
&Q_{3j+2}Q_{3k+3+2} - Q_{3j+3+2}Q_{3k+2} \\
&\quad = \left(\frac{(ac+\alpha)\alpha^j - (ac+\beta)\beta^j}{\alpha-\beta}\right)\left(\frac{(ac+\alpha)\alpha^{k+1} - (ac+\beta)\beta^{k+1}}{\alpha-\beta}\right) \\
&\quad\quad - \left(\frac{(ac+\alpha)\alpha^{j+1} - (ac+\beta)\beta^{j+1}}{\alpha-\beta}\right)\left(\frac{(ac+\alpha)\alpha^{k+1} - (ac+\beta)\beta^{k+1}}{\alpha-\beta}\right) \\
&\quad = \frac{1}{(\alpha-\beta)^2}\Big\{a^2c^2\left(\alpha^j\beta^k(\alpha-\beta) - \alpha^k\beta^j(\alpha-\beta)\right) \\
&\quad\quad + ac\left(\alpha^j\beta^k\left(\alpha^2-\beta^2\right) - \alpha^k\beta^j\left(\alpha^2-\beta^2\right)\right) \\
&\quad\quad + \left(\alpha^{j+1}\beta^{k+1}(\alpha-\beta) - \alpha^{k+1}\beta^{j+1}(\alpha-\beta)\right)\Big\} \\
&\quad = \frac{1}{\alpha-\beta}\left\{(a^2c^2 + acu + (-abc))\alpha^k\beta^k\left(\alpha^{j-k} - \beta^{j-k}\right)\right\} \\
&\quad = (-abc)^k(\alpha+ac)(\beta+ac)\left(\frac{\alpha^{j-k}-\beta^{j-k}}{\alpha-\beta}\right) \\
&\quad = (-abc)^{\lfloor \frac{n}{3} \rfloor} w Q_{m-n} Q_3^{-1}.
\end{aligned}$$

Similarly, we can obtain the results when m and n are of the form $3k$ and $3k+1$. Combining all these cases yields:

$$Q_m Q_{n+3} - Q_{m+3}Q_n = (-abc)^{\lfloor \frac{n}{3} \rfloor} w Q_{m-n} Q_3^{-1}.$$

□

We next use a Binet-type formula to obtain an identity which includes the summation of Q_n's and binomial coefficients.

Theorem 4.4. *For any integer $m \geq 0$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} u^k (abc)^{n-k} Q_{3k+\chi(m)} = Q_{6n+\chi(m)}.$$

Proof. We prove the result only for the case $m = 3n$; the remaining cases $m = 3n+1$ and $m = 3n+2$ can be handled accordingly.

If $m \equiv 0 \pmod 3$, then applying Corollary 3.5 yields:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} u^k (abc)^{n-k} Q_{3k} &= \sum_{k=0}^{n} \binom{n}{k} u^k (abc)^{n-k} (1+a)\left(\frac{\alpha^k - \beta^k}{\alpha-\beta}\right) \\
&= \left(\frac{1+a}{\alpha-\beta}\right)\left(\sum_{k=0}^{n}\binom{n}{k}u^k(abc)^{n-k}\alpha^k - \sum_{k=0}^{n}\binom{n}{k}u^k(abc)^{n-k}\beta^k\right) \\
&= \left(\frac{1+a}{\alpha-\beta}\right)\left((\alpha u + abc)^n - (\beta u + abc)^n\right) \\
&= \left(\frac{1+a}{\alpha-\beta}\right)\left(\alpha^{2n} - \beta^{2n}\right) \\
&= Q_{6n} = Q_{2(3n)+\chi(n)}.
\end{aligned}$$

Hence, in general, we obtain:

$$\sum_{k=0}^{n} \binom{n}{k} u^k (abc)^{n-k} Q_{3k+\chi(m)} = Q_{6n+\chi(m)}.$$

□

We complete this paper by introducing a result similar to the golden ratio property found in the Fibonacci sequence.

Theorem 4.5. *The limiting ratio of any two terms of* $\{Q_n\}$ *whose suffixes are at a distance* $3r$ *apart is*

$$\lim_{n\to\infty} \frac{Q_{n+3r}}{Q_n} = \alpha^r.$$

Proof. By Binet's formula, we have

$$\lim_{n\to\infty} \frac{Q_{n+3r}}{Q_n} = \lim_{n\to\infty} \frac{\gamma(n)\alpha^{\lfloor \frac{n}{3} \rfloor + r} - \delta(n)\beta^{\lfloor \frac{n}{3} \rfloor + r}}{\gamma(n)\alpha^{\lfloor \frac{n}{3} \rfloor} - \delta(n)\beta^{\lfloor \frac{n}{3} \rfloor}}.$$

Dividing both the numerator and the denominator by $\alpha^{\lfloor \frac{n}{3} \rfloor}$, we get:

$$\lim_{n\to\infty} \frac{Q_{n+3r}}{Q_n} = \lim_{n\to\infty} \frac{\gamma(n)\alpha^r - \delta(n)\left(\frac{\beta}{\alpha}\right)^{\lfloor \frac{n}{3} \rfloor}\beta^r}{\gamma(n) - \delta(n)\left(\frac{\beta}{\alpha}\right)^{\lfloor \frac{n}{3} \rfloor}}.$$

Since $\left|\frac{\beta}{\alpha}\right| < 1$, it is clear that:

$$\lim_{n\to\infty} \left(\frac{\beta}{\alpha}\right)^{\lfloor \frac{n}{3} \rfloor} = 0.$$

This yields:

$$\lim_{n\to\infty} \frac{Q_{n+3r}}{Q_n} = \frac{\gamma(n)\alpha^r}{\gamma(n)} = \alpha^r.$$

□

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