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Fixed point results of almost weak contractions and an application to a boundary value problem

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Abstract. In this paper, we introduce the concept of almost $\tilde{\mathtt{R}}_{\digamma}-(\psi,\phi,L)$ weak contractions, characterized by rational expressions and equipped with a binary relation in the framework of metric spaces. We investigate these mappings to establish the existence and uniqueness of common coincidence points and common fixed points. The obtained results are illustrated with suitable examples to demonstrate their applicability. As an application of our findings, we establish the existence of solutions for a boundary value problem.

1 Preliminaries

The Banach contraction principle(BCP) stands as a pivotal result in analysis. Various authors have extensively explored its generalization, leading to the emergence of new contraction families. Alber [7] and others introduced modifications to weak contractions in Hilbert spaces, while Rhoades [28] developed a specific weak contraction condition for metric spaces aimed at facilitating the computation of fixed points. Additionally, substantial research has been devoted to weakly contractive mappings. On the other hand, the field of metric spaces with binary relations, initiated by Turinici [34], represents a novel pathway of study. Subsequently, in the order-theoretic metric setting, Ran and Reurings [29] have extended the BCP. In a complete metric space with a binary relation, Alam and Imdad [4, 5, 6] established a fixed point result for the classical Banach contraction principle. A notable aspect of their approach is that the contraction condition is required only for pairs of elements that are related through the specified binary relation, rather than for all pairs of elements in the space. For further reading, we refer to [2, 4, 5, 6, 9, 10, 12, 15, 24, 25, 26, 31, 34, 35, 36]. Furthermore, numerous researchers [3, 8, 14, 17, 19, 22, 31, 32, 33] have extended the study of metric spaces equipped with binary relations. Their contributions include the establishment of various common coincidence point and common fixed point results under suitable conditions.

Fixed point theory plays an important role in addressing a wide range of integral and differential equations. Numerous researchers [1, 11, 13, 16, 18, 20, 21, 23, 27, 30, 37, 38, 39, 40] have investigated the existence and uniqueness of solutions for various classes of mathematical problems. These include boundary value problems, fractional differential equations, elastic beam models, integral and fractional integral equations, nonlinear elliptic equations and delayed hematopoiesis models, provided they adhere to certain prescribed auxiliary conditions.

In this paper, we denote a nonempty binary relation by $\tilde{\mathtt{R}} \subseteq \tilde{\mathtt{M}} \times \tilde{\mathtt{M}}$, $\mathscr{N}$ as $\mathbb{N} \cup \{0\}$ and triplet $(\tilde{\mathtt{M}}, \tilde{\eth}, \tilde{\mathtt{R}})$ is metric space endowed a binary relation $\tilde{\mathtt{R}}$.

Definition 1.1. [24] A pair of elements $\zeta, \int \in \tilde{\mathtt{M}}$ are considered $\tilde{\mathtt{R}}$ comparative if $(\zeta, \int) \in \tilde{\mathtt{R}}$ or $(\int, \zeta) \in \tilde{\mathtt{R}}$ and represented by $[\int, \zeta] \in \tilde{\mathtt{R}}$.

Definition 1.2. [24] Let $\tilde{\mathtt{M}} \neq \emptyset$ with $\tilde{\mathtt{R}}$.
(i) $\tilde{\mathtt{R}}^{-1} = (\int, \zeta) \in \tilde{\mathtt{M}} \times \tilde{\mathtt{M}} : (\zeta, \int) \in \tilde{\mathtt{R}}\}$ is the inverse relation of $\tilde{\mathtt{R}}$.

(ii) $\tilde{\mathtt{R}}^{\star}$ is given by $\tilde{\mathtt{R}}^{\star} = \tilde{\mathtt{R}} \cup \Delta\zeta$, where $\Delta\zeta = \{(\zeta, \zeta) : \zeta \; in \; \tilde{\mathtt{M}}\}$ is the reflexive closure of $\tilde{\mathtt{R}}$.
(iii) $\tilde{\mathtt{R}}^{s} = \tilde{\mathtt{R}} \cup \tilde{\mathtt{R}}^{-1}$.

Definition 1.3. [5] A sequence $\{\zeta_n\} \subseteq \tilde{\mathtt{M}}$ is referred as $\tilde{\mathtt{R}}$ preserving if $(\zeta_n, \zeta_{n+1}) \in \tilde{\mathtt{R}}$ for all $n \in \mathscr{N}$.

Definition 1.4. [5] If every $\tilde{\mathtt{R}}$-preserving Cauchy sequence converges in $\tilde{\mathtt{M}}$, then $(\tilde{\mathtt{M}}, \eth, \tilde{\mathtt{R}})$ is $\tilde{\mathtt{R}}$ complete.

Definition 1.5. [5] $\tilde{\mathtt{R}}$ is considered $\eth$ self closed whenever there is an $\tilde{\mathtt{R}}$- preserving sequence $\{\zeta_n\}$ with $\zeta_n \to \zeta$, we can always find a subsequence $\{\zeta_{n_p}\}_{k=0}^{\infty}$ with $(\zeta_{n_p}, \zeta) \in \tilde{\mathtt{R}}$.

Definition 1.6. [5] $\tilde{\mathtt{R}}$ is designated as $(\digamma, \eth)$-self closed if there is an $\tilde{\mathtt{R}}$- preserving sequence $\{\zeta_n\}$ with $\zeta_n \to \zeta$, we can find a subsequence $\{\zeta_{n_p}\}$ satisfying $(\digamma\zeta_{n_p}, \digamma\zeta) \in \tilde{\mathtt{R}}$ for all $p \in \mathscr{N}$.

Definition 1.7. [5] $\tilde{\mathtt{R}}$ is S- closed if $(\zeta, \int) \in \tilde{\mathtt{R}}$ implies $(S\zeta, S\int) \in \tilde{\mathtt{R}}$, $\zeta, \int \in \tilde{\mathtt{M}}$.

Definition 1.8. [4] Consider two selfmaps S and $\digamma$ on $\tilde{\mathtt{M}}$ and $\tilde{\mathtt{R}}$ on $\tilde{\mathtt{M}}$ is referred as $(S, \digamma)$-closed if $(\digamma\zeta, \digamma\int) \in \tilde{\mathtt{R}}$ implies $(S\zeta, S\int) \in \tilde{\mathtt{R}}$, for all $\zeta, \int \in \tilde{\mathtt{M}}$.

When $\digamma = I$, Definition 1.8 simplifies to the concept of S closed on $\tilde{\mathtt{R}}$, as described in Definition 1.7.

Definition 1.9. [5] A self map S defined in $(\tilde{\mathtt{M}}, \eth, \tilde{\mathtt{R}})$ is designated as $\tilde{\mathtt{R}}$-continuous at $\zeta \in \tilde{\mathtt{M}}$ if for any $\tilde{\mathtt{R}}$-preserving sequence $\{\zeta_n\}$ with $\zeta_n \to \zeta$, then $S\zeta_n \to S\zeta$. Additionally, $\tilde{\mathtt{M}}$ is $\tilde{\mathtt{R}}$ - continuous if it $\tilde{\mathtt{R}}$-continuous at every point of $\tilde{\mathtt{M}}$.

Definition 1.10. [5] Consider two self maps S and $\digamma$ on $\tilde{\mathtt{M}}$. If for any $\tilde{\mathtt{R}}$-preserving sequence $\{\zeta_n\} \subseteq \tilde{\mathtt{M}}$, $\digamma\zeta_n \to \digamma\zeta$ then $S\zeta_n \to S\zeta$, then we call S is $(\digamma, \tilde{\mathtt{R}})$-continuous at ζ. Further, S is $(\digamma, \tilde{\mathtt{R}})$-continuous if S is $(\digamma, \tilde{\mathtt{R}})$ continuous at every point of $\tilde{\mathtt{M}}$.

Under $\digamma = I$, the Definition 1.10 reduces to Definition 1.9.

Definition 1.11. [4] Consider two self maps S and $\digamma$ on $(\tilde{\mathtt{M}}, \eth)$. A pair $(S, \digamma)$ is termed $\tilde{\mathtt{R}}$-compatible when every sequence $\int_n \subseteq \tilde{\mathtt{M}}$ satisfies the conditions specified below.

(i) The $\{S\int_n\}$ and $\{\digamma\int_n\}$ are $\tilde{\mathtt{R}}$- preserving sequences.

(ii) If $lim_{n\to\infty}\digamma\int_n = lim_{n\to\infty}S\int_n = t$, $t \in \tilde{\mathtt{M}}$, it follows that $lim_{n\to\infty}\eth(\digamma S\int_n, S\digamma\int_n) = 0$.

Definition 1.12. [5] Let $Q \subseteq \tilde{\mathtt{M}}$. Then, $\tilde{\mathtt{R}}$ restricted to Q is defined by $\tilde{\mathtt{R}} \cap Q^2$. We denote this restricted relation by $\tilde{\mathtt{R}}|_Q$, which represents the relation on Q induced from $\tilde{\mathtt{R}}$.

Definition 1.13. [6] A self map S on $\tilde{\mathtt{M}}$ is designated as locally S transitive if for each $\tilde{\mathtt{R}}$-preserving sequence $\{\zeta_n\} \subseteq S(\tilde{\mathtt{M}})$ with range denoted as $Q = \{\zeta_n \in \mathscr{N}\}$, the binary relation $\tilde{\mathtt{R}}|Q$ is transitive.

Lemma 1.14. *[9] In $\tilde{\mathtt{M}}$, let $\{\zeta_n\}$ be a sequence. If $\{\zeta_n\}$ is not a Cauchy sequence in $\tilde{\mathtt{M}}$, there exists a real number $\epsilon > 0$ and two subsequences $\{\zeta_{m(j)}\}$ and $\{\zeta_{n(j)}\}$ so that $\eth(\zeta_{m(j)}, \zeta_{n(j)-1}) < \epsilon < \eth(\zeta_{m(j)}, \zeta_{n(j)})$. Moreover, if $\{\zeta_n\}$ is so that $\lim_{n\to\infty}\eth(\zeta_n, \zeta_{n+1}) = 0$ and*

$$\lim_{j\to\infty}\eth(\zeta_{m(j)}, \zeta_{n(j)}) = \lim_{j\to\infty}\eth(\zeta_{m(j)-1}, \zeta_{n(j)}) = \lim_{j\to\infty}\eth(\zeta_{m(j)}, \zeta_{n(j)-1}) = \epsilon.$$

Theorem 1.15. *[10] Consider a selfmap $\digamma$ on $(\tilde{\mathtt{M}}, \eth)$ associated with $\tilde{\mathtt{R}}$ satisfying:*

(i) $(\tilde{\mathtt{M}}, \eth)$ *is* $\tilde{\mathtt{R}}$*-complete and* $\tilde{\mathtt{M}}(\digamma, \tilde{\mathtt{R}}) \neq \phi$

(ii) $\tilde{\mathtt{R}}$ *is* $\digamma$*-closed*

(iii) $\tilde{\mathtt{R}}$ *is locally* $\digamma$*-transitive*

(iv) either $\tilde{\mathtt{R}}$ *is* $\eth$*-self closed or* $\digamma$ *is* $\tilde{\mathtt{R}}$*-continuous*

(v) $\mathcal{F}$ *is a generalized weakly contractive map on* $\tilde{\mathtt{R}}$*, that is,* $\exists$ *continuous functions* $\phi,\ \psi : [0,\infty) \to [0,\infty) \ni \psi(\tau) = \phi(\tau) = 0$ *if and only if* $t = 0$*, and if* $\zeta, \int \in \tilde{\mathtt{M}}$ *with* $[\zeta, \int] \in \tilde{\mathtt{R}}$ *then*

$$\psi(\eth(\mathcal{F}\zeta, \mathcal{F}\int)) \leq \psi(\Lambda_1(\zeta, \int)) - \phi(\Lambda_2(\zeta, \int)), \tag{1.1}$$

where $\Lambda_1(\zeta, \int) = max\{\eth(\zeta, \int), \eth(\zeta, \mathcal{F}\zeta), \eth(\int, \mathcal{F}\int), \frac{[\eth(\zeta, \mathcal{F}\int)+\eth(\int, \mathcal{F}\zeta)]}{2}\}$

and $\Lambda_2(\zeta, \int) = max\{\eth(\zeta, \int), \eth(\int, \mathcal{F}\int)\}$

then $\mathcal{F}$ *has a fixed point.*

Theorem 1.16. *[17] Consider a selfmap* $\mathcal{F}$ *on* $(\tilde{\mathtt{M}}, \eth)$ *associated with* $\tilde{\mathtt{R}}$ *satisfying:*

(i) $(\tilde{\mathtt{M}}, \eth)$ *is* $\tilde{\mathtt{R}}$*-complete and* $\tilde{\mathtt{M}}(\mathcal{F}, \tilde{\mathtt{R}}) \neq \phi$

(ii) $\tilde{\mathtt{R}}$ *is* $\mathcal{F}$*-closed*

(iii) $\tilde{\mathtt{R}}$ *is locally* $\mathcal{F}$*-transitive*

(iv) either $\tilde{\mathtt{R}}$ *is* $\eth$*-self closed or* $\mathcal{F}$ *is* $\tilde{\mathtt{R}}$*-continuous*

(v) there exist $\mathscr{P}_1$ *and* $\mathscr{P}_2$ *in* $[0, 1)$ *with* $\mathscr{P}_1 + \mathscr{P}_2 < 1$ *such that*

$$\eth(\mathcal{F}\zeta, \mathcal{F}\int)) \leq \mathscr{P}_1.\frac{\eth(\int, \mathcal{F}\int)[1 + \eth(\zeta, \mathcal{F}\zeta)]}{1 + \eth(\zeta, \int)} + \mathscr{P}_2.\eth(\zeta, \int),$$

for all $\zeta, \int \in \tilde{\mathtt{M}}$ *with* $[\zeta, \int] \in \tilde{\mathtt{R}}$.

Then $\mathcal{F}$ *has a fixed point.*

In this paper, we formulate a new class of almost $\tilde{\mathtt{R}}_{\mathcal{F}} - (\psi, \phi, L)$ weak contractions in metric spaces characterized by rational expressions and equipped with a binary relation. We investigate these mappings to establish the existence and uniqueness of common coincidence points and common fixed points. Several illustrative examples are presented to clarify the applicability of the proposed results. Moreover, our results generalize and extend a number of known results, particularly those reported in [12, 17], by transferring them from the classical metric framework to spaces equipped with a binary relation. As an application of our findings, we establish the existence of solutions for a boundary value problem.

2 Main results

In our subsequent discussion, we adopt the notation given below.
Let Ψ be the set of functions $\psi : [0,\infty) \to [0,\infty)$ so that

(ψ_1) ψ is continuous.

(ψ_2) $\psi(\tau) = 0$ if and only if $\tau = 0$.

(ψ_3) ψ is increasing.

Let Φ be the set of functions $\phi : [0,\infty) \to [0,\infty)$ so that for any sequence $\{\zeta_n\}$ in $[0,\infty)$ with $\zeta_n \to t$, $t > 0$, $liminf_{n\to\infty}\phi(\zeta_n) > 0$.

$$Coin(S, \mathcal{F}) = \{\zeta \in \tilde{\mathtt{M}}/S\zeta = \mathcal{F}\zeta\}$$
$$C(S, \mathcal{F}) = \{\zeta \in \tilde{\mathtt{M}}/\zeta = S\zeta = \mathcal{F}\zeta\}.$$

Definition 2.1. Consider two selfmaps S and $\mathcal{F}$ on $\tilde{\mathtt{M}}$ and $\tilde{\mathtt{R}}$ on $\tilde{\mathtt{M}}$. S is almost $\tilde{\mathtt{R}}_{\mathcal{F}} - (\psi, \phi)$ generalized weak contractive map on $\tilde{\mathtt{M}}$ if there exists functions $\psi \in \Psi$, $\phi \in \Phi$ and a parameter $L \geq 0$ such that

$$\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int), \tag{2.1}$$

where
$\mathscr{M}(\zeta, \int) = max\{\eth(\mathcal{F}\zeta, \mathcal{F}\int), \eth(\mathcal{F}\int, S\zeta), \eth(\mathcal{F}\int, S\int), \frac{\eth(\mathcal{F}\int, S\zeta)[1+\eth(\mathcal{F}\zeta, S\zeta)]}{1+\eth(\mathcal{F}\zeta, \mathcal{F}\int)}, \frac{\eth(\mathcal{F}\zeta, S\zeta)[1+\eth(\mathcal{F}\int, S\zeta)]}{1+\eth(\mathcal{F}\zeta, \mathcal{F}\int)}\}$,
and
$N(\zeta, \int) = min\{\eth(\mathcal{F}\zeta, S\zeta), \eth(\mathcal{F}\int, S\zeta)\}$, for all $\zeta, \int \in \tilde{\mathtt{M}}$ with $(\mathcal{F}\zeta, \mathcal{F}\int) \in \tilde{\mathtt{R}}$.

Proposition 2.2. *Consider a triplet* $(\tilde{\mathtt{M}}, \eth, \tilde{\mathtt{R}})$. *and two selfmaps* S *and* $\digamma$ *on* $\tilde{\mathtt{M}}$. *Then the following conditions are equivalent.*
For all $\psi \in \Psi$, $\phi \in \Phi$ *and* $L \geq 0$,

$(\mathcal{X})$ $\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int)$,

for all $\zeta, \int \in \tilde{\mathtt{M}}$ *with* $(\digamma\zeta, \digamma\int) \in \tilde{\mathtt{M}}$,

$(\mathcal{Y})$ $\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int)$,

for all $\zeta, \int \in \tilde{\mathtt{M}}$ *with* $[\digamma\zeta, \digamma\int] \in \tilde{\mathtt{M}}$,

where $\mathscr{M}(\zeta, \int)$ *and* $N(\zeta, \int)$ *be defined as in Definition* 2.1.

Proof. To prove $(\mathcal{X}) \implies (\mathcal{Y})$, we assume that condition $(\mathcal{X})$ holds. Let $\zeta, \int \in \tilde{\mathtt{M}}$ with $[\digamma\zeta, \digamma\int] \in \tilde{\mathtt{R}}$. If $(\digamma\zeta, \digamma\int) \in \tilde{\mathtt{R}}$, then $(\mathcal{X})$ follows from $(\mathcal{Y})$ directly.
On the contrary, if $(\digamma\int, \digamma\zeta) \in \tilde{\mathtt{R}}$, then utilizing symmetry of $\eth$ and $(\mathcal{X})$, we have

$$\begin{aligned}\psi(\eth(S\zeta, S\int)) &\leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int),\\ &= \psi(\mathscr{M}(\int, \zeta)) - \phi(\mathscr{M}(\int, \zeta)) + LN(\int, \zeta),\end{aligned}$$

where

$$\begin{aligned}\mathscr{M}(\zeta, \int) &= max\{\eth(\digamma\zeta, \digamma\int), \eth(\digamma\int, S\zeta), \eth(\digamma\int, S\int), \frac{\eth(\digamma\int, S\zeta)[1+\eth(\digamma\zeta, S\zeta)]}{1+\eth(\digamma\zeta, \digamma\int)},\\ &\qquad \frac{\eth(\digamma\zeta, S\zeta)[1+\eth(\digamma\int, S\zeta)]}{1+\eth(\digamma\zeta, \digamma\int)}\}\\ &= max\{\eth(\digamma\int, \digamma\zeta), \eth(S\zeta, \digamma\int), \eth(S\int, \digamma\int), \frac{\eth(S\zeta, \digamma\int)[1+\eth(S\zeta, \digamma\zeta)]}{1+\eth(\digamma\int, \digamma\zeta)},\\ &\qquad \frac{\eth(S\zeta, \digamma\zeta)[1+\eth(S\zeta, \digamma\int)]}{1+\eth(\digamma\int, \digamma\zeta)}\}\\ &= \mathscr{M}(\int, \zeta)\end{aligned}$$

and

$$\begin{aligned}N(\zeta, \int) &= min\{\eth(\digamma\zeta, S\zeta), \eth(\digamma\int, S\zeta)\}\\ &= min\{\eth(S\zeta, \digamma\zeta), \eth(S\zeta, \digamma\int)\}\\ &= N(\int, \zeta).\end{aligned}$$

Therefore $(\mathcal{X}) \implies (\mathcal{Y})$. Conversely, $(\mathcal{Y}) \implies (\mathcal{X})$ is trivial.
Hence $(\mathcal{X}) = (\mathcal{Y})$ follows. □

Theorem 2.3. *Consider triplet* $(\tilde{\mathtt{M}}, \eth, \tilde{\mathtt{R}})$, *where* $\tilde{\mathtt{R}}$ *is locally* S*-transitive binary relation. Assume that* S *is almost* $\tilde{\mathtt{R}}_{\digamma} - (\psi, \phi)$ *weak contractive map on* $\tilde{\mathtt{M}}$ *under the following conditions:*

(a) *if there exists* $\zeta_o \in \tilde{\mathtt{M}}$ *such that* $(\digamma\zeta_o, S\zeta_o) \in \tilde{\mathtt{R}}$;

(b) *let* Q *is a complete subset of* $\tilde{\mathtt{M}}$ *such that* $Q \subseteq \digamma(\tilde{\mathtt{M}})$ *and* $S\tilde{\mathtt{M}} \subseteq Q \bigcap \digamma\tilde{\mathtt{M}}$;

(c) $\tilde{\mathtt{R}}$ *is* $(S, \digamma)$ *closed*;

(d) *At least one the subsequent conditions holds*:

(i) S *is* $\digamma - \tilde{\mathtt{R}}$ *continuous*;

(ii) $\tilde{\mathtt{R}}|_Q$ *is* $\eth$*-self closed*;

(e) *or alternately*

(i) $(S, \digamma)$ *is* $\tilde{\mathtt{R}}$*-compatible*;

(ii) S *and* $\digamma$ *are* $\tilde{\mathtt{R}}$ *continuous.*

Then there exists at least one $\imath \in \tilde{\mathtt{M}}$ *such that* $\imath \in Coin(S, \digamma)$.

Proof. Let $\zeta_o \in \tilde{\mathtt{M}}$. By condition (a) of our assumption, $(\mathit{F}\zeta_o, S\zeta_o) \in \tilde{\mathtt{R}}$. If $\mathit{F}\zeta_o = S\zeta_o$ then $\zeta_o \in Coin(S, \mathit{F})$. This completes the proof.
Let us suppose that $\mathit{F}\zeta_o \neq S\zeta_o$. Using condition (b), we have $S\tilde{\mathtt{M}} \subset \mathit{F}\tilde{\mathtt{M}}$, we can choose $\zeta_1 \in \tilde{\mathtt{M}}$ such that $\mathit{F}\zeta_1 = S\zeta_o$.
Using this approach, we construct a sequence $\{\zeta_n\} \subseteq \tilde{\mathtt{M}}$ such that

$$\mathit{F}\zeta_{n+1} = S\zeta_n \; for \; all \; n \in \mathscr{N}. \tag{2.2}$$

We proceed to verify $\{\mathit{F}\zeta_n\}$ is an $\tilde{\mathtt{R}}$-preserving sequence.

$$(\mathit{F}\zeta_n, \mathit{F}\zeta_{n+1}) \in \tilde{\mathtt{R}} \; for \; all \; n \in \mathscr{N}. \tag{2.3}$$

To prove this claim we use mathematical induction.
Using (2.2), we have $(\mathit{F}\zeta_o, S\zeta_o) \in \tilde{\mathtt{R}} \implies (\mathit{F}\zeta_o, \mathit{F}\zeta_1) \in \tilde{\mathtt{R}}$
which implies condition (2.3) holds when $n = 0$.
Supposing that condition (2.3) is true for $n = k$ then $(\mathit{F}\zeta_k, \mathit{F}\zeta_{k+1}) \in \tilde{\mathtt{R}}$, we verify that condition (2.3) holds for $n = k + 1$. To test this, we consider our assumption $\tilde{\mathtt{R}}$ is (S, F) closed, then $(S\zeta_k, S\zeta_{k+1}) \in \tilde{\mathtt{R}}$ implies $(\mathit{F}\zeta_{k+1}, \mathit{F}\zeta_{k+2}) \in \tilde{\mathtt{R}}$, which amounts condition (2.3) holds for every $n \in \mathscr{N}$.
For any $n_o \in \mathscr{N}$, if $\mathit{F}\zeta_{n_o} = \mathit{F}\zeta_{n_o+1}$ then $\mathit{F}\zeta_{n_o} = S\zeta_{n_o}$. Hence $\zeta_{n_o} \in Coin(S, \mathit{F})$.
If $\mathit{F}\zeta_n \neq \mathit{F}\zeta_{n+1}$ for every $n \in \mathscr{N}$, then from conditions (2.1), (2.2) and (2.3), we get

$$\begin{aligned}\psi(\eth(\mathit{F}\zeta_n, \mathit{F}\zeta_{n+1})) &= \psi(\eth(S\zeta_{n-1}, S\zeta_n)) \\ &\leq \psi(\tilde{\mathtt{M}}(\zeta_{n-1}, \zeta_n)) - \phi(\tilde{\mathtt{M}}(\zeta_{n-1}, \zeta_n)) + LN(\zeta_{n-1}, \zeta_n)\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\mathscr{M}(\zeta_{n-1}, \zeta_n) &= max\{\eth(\mathit{F}\zeta_{n-1}, \mathit{F}\zeta_n), \eth(\mathit{F}\zeta_n, S\zeta_{n-1}), \eth(\mathit{F}\zeta_n, S\zeta_n), \\ &\quad \frac{\eth(\mathit{F}\zeta_n, S\zeta_n)[1 + \eth(\mathit{F}\zeta_{n-1}, S\zeta_{n-1})]}{1 + \eth(\mathit{F}\zeta_{n-1}, \mathit{F}\zeta_n)}, \frac{\eth(\mathit{F}\zeta_{n-1}, S\zeta_{n-1})[1 + \eth(\mathit{F}\zeta_n, S\zeta_{n-1})]}{1 + \eth(\mathit{F}\zeta_{n-1}, \mathit{F}\zeta_n)}\} \\ &\leq max\{\eth(\mathit{F}\zeta_{n-1}, \mathit{F}\zeta_n), \eth(\mathit{F}\zeta_n, \mathit{F}\zeta_{n+1})\},\end{aligned} \tag{2.5}$$

and

$$N(\zeta_{n-1}, \zeta_n) = min\{\eth(\mathit{F}\zeta_{n-1}, S\zeta_{n-1}), \eth(\mathit{F}\zeta_n, S\zeta_{n-1})\} = 0. \tag{2.6}$$

Utilizing (2.5) and (2.6) in (2.4), we can conclude that

$$\begin{aligned}\psi(\eth(\mathit{F}\zeta_n, \mathit{F}\zeta_{n+1})) &\leq \psi(\mathscr{M}(\zeta_{n-1}, \zeta_n)) - \phi(\mathscr{M}(\zeta_{n-1}, \zeta_n)) + LN(\zeta_{n-1}, \zeta_n) \\ &\leq \psi(max\{\eth(\mathit{F}\zeta_n, \mathit{F}\zeta_{n-1}), \eth(\mathit{F}\zeta_n, \mathit{F}\zeta_{n+1})\}) - \phi(\mathscr{M}((\zeta_{n-1}, \zeta_n)).\end{aligned} \tag{2.7}$$

Let $z_n = \eth(\mathit{F}\zeta_n, \mathit{F}\zeta_{n+1})$ for all $n \in \mathscr{N}$.
If $z_n > z_{n-1}$, then from (2.7), using the property of ϕ, we have

$$\begin{aligned}\psi(z_n) &\leq \psi(z_n) - \phi(\mathscr{M}(z_{n-1}) \\ &< \psi(z_n),\end{aligned}$$

a contradiction. Hence

$$z_n \leq z_{n-1} \tag{2.8}$$

Therefore the sequence $\{z_n\}$ is decreasing. Thus there exists $r \geq 0$ such that $lim_{n\to\infty} z_n = r$.
Now we prove $r = 0$. Assume if $r > 0$, then from the condition (2.4), we have

$$\psi(z_n) \leq \psi(\mathscr{M}(\zeta_{n-1}, \zeta_n)) - \phi(\mathscr{M}(\zeta_{n-1}, \zeta_n)) + LN(\zeta_{n-1}, \zeta_n).$$

Now,

$$lim_{n\to\infty}\mathscr{M}(\zeta_{n-1}, \zeta_n) = lim_{n\to\infty} max\{z_{n-1}, 0, z_n, \frac{z_{n-1}}{1 + z_{n-1}}\} \leq r. \tag{2.9}$$

and

$$lim_{n\to\infty}N(\zeta_{n-1},\zeta_n) = lim_{n\to\infty}inf\{z_{n-2},0\} = 0.$$

Also, we know that

$$z_n \leq \mathscr{M}(\zeta_{n-1},\zeta_n),$$

taking limits as $n \to \infty$, the preceding inequality yields

$$r \leq lim_{n\to\infty}\mathscr{M}(\zeta_{n-1},\zeta_n). \tag{2.10}$$

Thus from (2.9) and (2.10), we can conclude that

$$limit_{n\to\infty}\mathscr{M}(\zeta_{n-1},\zeta_n) = r. \tag{2.11}$$

From (2.6) and (2.11), by taking the limit supremum as $n \to \infty$ in (2.4), we obtain

$$\psi(r) \leq \psi(r) - liminf_{n\to\infty}\phi(\mathscr{M}(\zeta_n,\zeta_{n-1}))$$

i.e., Using the ϕ property, we have the contradiction $\psi(r) < \psi(r)$. Thus, $r = 0$, i.e.,

$$lim_{n\to\infty}z_n = 0. \tag{2.12}$$

Next our purpose is to establish $\{\mathcal{F}\zeta_n\}$ is a Cauchy sequence in $\tilde{\mathtt{M}}$.
On the contrary, by applying Lemma 1.14, there exists $\epsilon > 0$ and two subsequences $\{\mathcal{F}\zeta_{m_p}\}$ and $\{\mathcal{F}\zeta_{n_p}\}$ with $\zeta \leq m_p \leq n_p$ so that

$$\eth(\mathcal{F}\zeta_{m_p},\mathcal{F}\zeta_{n_p-1}) < \epsilon < \eth(\mathcal{F}\zeta_{m_p},\mathcal{F}\zeta_{n_p}).$$

Moreover, if $\{\mathcal{F}\zeta_n\}$ is so that $\lim_{n\to\infty}\eth(\mathcal{F}\zeta_n,\mathcal{F}\zeta_{n+1}) = 0$ and

$$\lim_{p\to\infty}\eth(\mathcal{F}\zeta_{m_p},\mathcal{F}\zeta_{n_p}) = \lim_{p\to\infty}\eth(\mathcal{F}\zeta_{m_p-1},\mathcal{F}\zeta_{n_p}) = \lim_{p\to\infty}\eth(\mathcal{F}\zeta_{m_p},\mathcal{F}\zeta_{n_p-1}) = \epsilon. \tag{2.13}$$

As $\{\mathcal{F}\zeta_n\}$ is $\tilde{\mathtt{R}}$-preserving and $\{\mathcal{F}\zeta_n\} \subseteq S(\tilde{\mathtt{M}})$, on using local S-transitivity of $\tilde{\mathtt{R}}$, we have $\eth(\mathcal{F}\zeta_{m_p},\mathcal{F}\zeta_{n_p}) \in \tilde{\mathtt{R}}$.
Utilizing the axioms of metric, we obtain

$$\eth(\mathcal{F}\zeta_{m_p-1},\mathcal{F}\zeta_{n_p-1}) \leq \eth(\mathcal{F}\zeta_{m_p-1},\mathcal{F}\zeta_{m_p}) + \eth(\mathcal{F}\zeta_{m_p},\mathcal{F}\zeta_{n_p}) + \eth(\mathcal{F}\zeta_{m_p},\mathcal{F}\zeta_{n_p-1}).$$

On taking upper limit as $p \to \infty$, utilizing (2.12) and (2.13), we get

$$limsup_{p\to\infty}\eth(\mathcal{F}\zeta_{m_p-1},\mathcal{F}\zeta_{n_p-1}) \leq \epsilon. \tag{2.14}$$

Also,

$$\epsilon < \eth(\mathcal{F}\zeta_{m_p},\mathcal{F}\zeta_{n_p}) \leq \eth(\mathcal{F}\zeta_{m_p},\mathcal{F}\zeta_{m_p-1}) + \eth(\mathcal{F}\zeta_{m_p-1},\mathcal{F}\zeta_{n_p-1}) + \eth(\mathcal{F}\zeta_{n_p-1},\mathcal{F}\zeta_{n_p}).$$

Taking upper limits as $p \to \infty$, using (2.13) and (2.14), we get

$$limsup_{p\to\infty}\eth(\mathcal{F}\zeta_{m_p-1},\mathcal{F}\zeta_{n_p-1}) = \epsilon. \tag{2.15}$$

Similarly, we could prove

$$liminf_{p\to\infty}\eth(\mathcal{F}\zeta_{m_p-1},\mathcal{F}\zeta_{n_p-1}) = \epsilon. \tag{2.16}$$

Hence from (2.15) and (2.16), we get

$$lim_{p\to\infty}\eth(\mathcal{F}\zeta_{m_p-1},\mathcal{F}\zeta_{n_p-1}) = \epsilon. \tag{2.17}$$

Now as $\{\mathcal{F}\zeta_n\}$ is $\tilde{\mathtt{R}}$ -preserving sequence, utilizing (2.1), we have

$$\psi(\eth(\mathcal{F}\zeta_{m_p},\mathcal{F}\zeta_{n_p})) \leq \psi(\mathscr{M}(\zeta_{m_p-1},\zeta_{n_p-1}) - \phi(\mathscr{M}(\zeta_{m_p-1},\zeta_{n_p-1})) + LN(\zeta_{m_p-1},\zeta_{n_p-1}), \tag{2.18}$$

where

$$\begin{aligned}\mathscr{M}(\zeta_{m_p-1},\zeta_{n_p-1}) &= max\{\eth(\digamma\zeta_{m_p-1},\digamma\zeta_{n_p-1}),\eth(\digamma\zeta_{n_p-1},S\zeta_{m_p-1}),\eth(\digamma\zeta_{n_p-1},S\zeta_{n_p-1}),\\ &\quad \frac{\eth(\digamma\zeta_{n_p-1},S\zeta_{m_p-1})[1+\eth(\digamma\zeta_{m_p-1},S\zeta_{m_p-1})]}{1+\eth(\digamma\zeta_{m_p-1},\digamma\zeta_{n_p-1})},\\ &\quad \frac{\eth(\digamma\zeta_{m_p-1},S\zeta_{m_p-1})[1+\eth(\digamma\zeta_{n_p-1},S\zeta_{m_p-1})]}{1+\eth(\digamma\zeta_{m_p-1},\digamma\zeta_{n_p-1})}\}\\ &= max\{\eth(\digamma\zeta_{m_p-1},\digamma\zeta_{n_p-1}),\eth(\digamma\zeta_{n_p-1},\digamma\zeta_{m_p}),\eth(\digamma\zeta_{n_p-1},\digamma\zeta_{n_p}),\\ &\quad \frac{\eth(\digamma\zeta_{n_p-1},\digamma\zeta_{m_p})[1+\eth(\digamma\zeta_{m_p-1},\digamma\zeta_{m_p})]}{1+\eth(\digamma\zeta_{m_p-1},\digamma\zeta_{n_p-1})},\\ &\quad \frac{\eth(\digamma\zeta_{m_p-1},\digamma\zeta_{m_p})[1+\eth(\digamma\zeta_{n_p-1},\digamma\zeta_{m_p})]}{1+\eth(\digamma\zeta_{m_p-1},\digamma\zeta_{n_p-1})}\}\end{aligned}$$

letting $p\to\infty$, on availing conditions (2.12), (2.13) and (2.17), we get

$$lim_{p\to\infty}\mathscr{M}(\zeta_{m_p-1},\zeta_{n_p-1}) = \{0,\epsilon,\frac{\epsilon}{1+\epsilon}\} = \epsilon. \tag{2.19}$$

and

$$\begin{aligned}N(\zeta_{m_p-1},\zeta_{n_p-1}) &= min\{\eth(S\zeta_{m_p-1},\digamma\zeta_{m_p-1}),\eth(\digamma\zeta_{n_p-1},S\zeta_{m_p-1})\}\\ &= min\{\eth(\digamma\zeta_{m_p},\digamma\zeta_{m_p-1}),\eth(\digamma\zeta_{n_p-1},\digamma\zeta_{m_p})\}\end{aligned}$$

which on taking limits as $p\to\infty$, using (2.12) and (2.13), we get

$$lim_{p\to\infty}N(\zeta_{m_p-1},\zeta_{n_p-1}) = 0. \tag{2.20}$$

Taking the limitsup as $p\to\infty$ in (2.18), using (2.19), (2.20) and property of ϕ, leads to a contradiction. Therefore, $\{\digamma\zeta_n\}$ is a $\tilde{\mathtt{R}}$-preserving Cauchy sequence in $\tilde{\mathtt{M}}$. As, $\{\digamma\zeta_n\}\subseteq\digamma\tilde{\mathtt{M}}$ and $\{\digamma\zeta_n\}\subseteq Q$, therefore $\{\digamma\zeta_n\}$ is a $\tilde{\mathtt{R}}$-preserving Cauchy sequence in Q. In virtue of $(Q,\eth)$ is $\tilde{\mathtt{R}}$-complete there exists $\ell\in Q$ such that $\lim_{n\to\infty}\digamma\zeta_n=\ell$. Also, $Q\subseteq\digamma(\tilde{\mathtt{M}})$ there exists $\imath\in\tilde{\mathtt{M}}$ such that

$$lim_{n\to\infty}\digamma\zeta_n = \ell = \digamma\imath. \tag{2.21}$$

To prove $\imath\in Coin(S,\digamma)$, we consider two different conditions of (d).
Subcase(i): If S is $(\digamma,\tilde{\mathtt{R}})$ continuous, then

$$lim_{n\to\infty}\digamma\zeta_{n+1} = lim_{n\to\infty}S\zeta_n = S\imath, \tag{2.22}$$

which leads to $S\imath=\digamma\imath$. Hence $\imath\in Coin(S,\digamma)$.
(ii): Suppose that $\tilde{\mathtt{R}}|_Q$ is $\eth$-self closed. Then for any $\tilde{\mathtt{R}}$ preserving sequence $\{\digamma\zeta_n\}$ with $\digamma\zeta_n\to\digamma\imath$, there is a subsequence $\{\digamma\zeta_{n_p}\}$ of $\{\digamma\zeta_n\}$ such that $[\digamma\zeta_{n_p},\digamma\imath]\in\tilde{\mathtt{R}}|_Q\subseteq\tilde{\mathtt{R}}$ for all $p\in\mathscr{N}$ and $\digamma\zeta_{n_p}\to\digamma\imath$. Now define $\delta=\eth(S\imath,\digamma\imath)$ and suppose that $\delta>0$.
By Proposition 2.2, and the fact that $[\digamma\zeta_{n_p},\digamma\imath]\in\tilde{\mathtt{R}}$ for all $p\in\mathscr{N}$, we have

$$\begin{aligned}\psi(\eth(\digamma\zeta_{n_p+1},S\imath)) &= \psi(\eth(S\zeta_{n_p},S\imath))\\ &\le \psi(\mathscr{M}(\zeta_{n_p},\imath)) - \phi(\mathscr{M}(\zeta_{n_p},\imath)) + LN(\zeta_{n_p},\imath)\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}\mathscr{M}(\zeta_{n_p},\imath) = max\{&\eth(\digamma\zeta_{n_p},\digamma\imath),\eth(\digamma\imath,S\imath),\eth(\digamma\imath,S\zeta_{n_p}),\frac{\eth(\digamma\imath,S\zeta_{n_p})[1+\eth(\digamma\zeta_{n_p},S\zeta_{n_p}]}{1+\eth(\digamma\zeta_{n_p},\digamma\imath)},\\ &\frac{\eth(\digamma\zeta_{n_p},S\zeta_{n_p})[1+\eth(\digamma\imath,S\zeta_{n_p})]}{1+\eth(\digamma\zeta_{n_p},\digamma\imath)}\}.\end{aligned}$$

Letting $p\to\infty$, we get

$$\lim_{p\to\infty}\mathscr{M}(\zeta_{n_p},\imath) = max\{0,\eth(\digamma\imath,S\imath),0,0\} = \eth(\digamma\imath,S\imath). \tag{2.24}$$

Also, $N(\zeta_{n_p}, \imath) = min\{\eth(\digamma\zeta_{n_p}, S\zeta_{n_p}), \eth(\digamma\imath, S\zeta_{n_p})\}$,
which on taking limits as $p \to \infty$, we get

$$lim_{p\to\infty}N(\zeta_{n_p}, \imath) = 0. \tag{2.25}$$

Now, taking limitsuperior as $p \to \infty$ in (2.23), we get

$$limsup_{p\to\infty}\psi(\eth(\digamma\zeta_{n_p+1}, S\imath)) \leq limsup_{p\to\infty}\psi(\mathscr{M}(\zeta_{n_p}, \imath)) - liminf_{p\to\infty}\phi(\mathscr{M}(\zeta_{n_p}, \imath)) + limsup_{\imath\to\infty}LN(\zeta_{n_p}, \imath).$$

By the property of ϕ, using (2.24) and (2.25) in the above inequality, we get $\psi(\delta) < \psi(\delta)$, a contradiction. Hence $\delta = 0$. Thus $S\imath = \digamma\imath$. i.e, $\imath \in Coin(S, \digamma)$.
Alternately, suppose the conditions of (e) hold.
Taking in account of condition (2.2), we have $\{\digamma\zeta_n\} \subseteq S\tilde{\mathtt{M}} \subseteq Q$, and hence $\{\digamma\zeta_n\}$ is $\tilde{\mathtt{R}}$-preserving Cauchy sequence in Q. Since Q is $\tilde{\mathtt{R}}$- complete, there exists $z \in Q$ such that

$$lim_{n\to\infty}\digamma\zeta_n = z \; and \; lim_{n\to\infty}S\zeta_n = z. \tag{2.26}$$

Now we demonstrate that $z \in Coin(S, \digamma)$. Based on the condition (2.26) and $\tilde{\mathtt{R}}$-continuity of $\digamma$ and S, it follows that

$$lim_{n\to\infty}\digamma(S\zeta_n) = \digamma z. \tag{2.27}$$

$$lim_{n\to\infty}S(\digamma\zeta_n) = Sz. \tag{2.28}$$

Since $\{S\zeta_n\}$ and $\{\digamma\zeta_n\}$ are $\tilde{\mathtt{R}}$-preserving sequences and using (2.26) together with condition $(e)(i)$ of our assumption, we obtain

$$lim_{n\to\infty}\eth(S\digamma\zeta_n, \digamma S\zeta_n) = 0. \tag{2.29}$$

Thus on using conditions (2.27), (2.28) and (2.29), we get

$$\eth(Sz, \digamma z) = \eth(lim_{n\to\infty}S\digamma\zeta_n, lim_{n\to\infty}\digamma S\zeta_n) = lim_{n\to\infty}\eth(S\digamma\zeta_n, \digamma S\zeta_n) = 0.$$

Hence $z \in Coin(S, \digamma)$. □

Theorem 2.4. *Suppose that all the premises of Theorem* 2.3 *is satisfied and assume that*
(i) $\digamma\hat{\mathfrak{a}}$ *and* $\digamma\hat{\mathfrak{b}}$ *are* $\tilde{\mathtt{R}}$ *comparable for all* $\hat{\mathfrak{a}}, \hat{\mathfrak{b}} \in Coin(S, \digamma)$ *with* $\hat{\mathfrak{a}} \neq \hat{\mathfrak{b}}$.
(ii) *One of* $\digamma$ *or* S *is one-one.*
Then $Coin(S, \digamma)$ *is unique. Further, the pair* $(S, \digamma)$ *is weakly compatible, then* $C(S, \digamma)$ *is singleton set.*

Proof. From Theorem 2.3, $Coin(S, \digamma) \neq \phi$. Let $\hat{\mathfrak{a}}, \hat{\mathfrak{b}} \in Coin(S, \digamma)$, we have $S\hat{\mathfrak{a}} = \digamma\hat{\mathfrak{a}}, S\hat{\mathfrak{b}} = \digamma\hat{\mathfrak{b}}$ and $[\digamma\hat{\mathfrak{a}}, \digamma\hat{\mathfrak{b}}] \in \tilde{\mathtt{R}}$. If $\digamma\hat{\mathfrak{a}} = \digamma\hat{\mathfrak{b}}$, then by utilizing the property of one of S and $\digamma$ is one-one, we get $\hat{\mathfrak{a}} = \hat{\mathfrak{b}}$. Hence, $Coin(S, \digamma)$ unique.
If $\digamma\hat{\mathfrak{a}} \neq \digamma\hat{\mathfrak{b}}$, then by condition (2.1) and Proposition 2.2, we have

$$\psi(\eth(\digamma\hat{\mathfrak{a}}, \digamma\hat{\mathfrak{b}})) = \psi(\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})) \leq \psi(\mathscr{M}(\hat{\mathfrak{a}}, \hat{\mathfrak{b}})) - \phi(\mathscr{M}(\hat{\mathfrak{a}}, \hat{\mathfrak{b}})) + LN(\hat{\mathfrak{a}}, \hat{\mathfrak{b}}), \tag{2.30}$$

where

$$\begin{aligned}\mathscr{M}(\hat{\mathfrak{a}}, \hat{\mathfrak{b}}) &= max\{\eth(\digamma\hat{\mathfrak{a}}, \digamma\hat{\mathfrak{b}}), \eth(\digamma\hat{\mathfrak{b}}, S\hat{\mathfrak{a}}), \eth(\digamma\hat{\mathfrak{b}}, S\hat{\mathfrak{b}}), \frac{\eth(\digamma\hat{\mathfrak{b}}, S\hat{\mathfrak{a}})[1 + \eth(\digamma\hat{\mathfrak{a}}, S\hat{\mathfrak{a}})]}{1 + \eth(\digamma\hat{\mathfrak{a}}, \digamma\hat{\mathfrak{b}})}, \\ &\qquad \frac{\eth(\digamma\hat{\mathfrak{a}}, S\hat{\mathfrak{a}}[1 + \eth(\digamma\hat{\mathfrak{b}}, S\hat{\mathfrak{a}})]}{1 + \eth(\digamma\hat{\mathfrak{a}}, \digamma\hat{\mathfrak{b}})}\} \\ &= max\{\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}}), \eth(S\hat{\mathfrak{b}}, S\hat{\mathfrak{a}}), \frac{\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})}{1 + \eth(S\hat{\mathfrak{b}}, S\hat{\mathfrak{a}})}\} \\ &= \eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})\end{aligned} \tag{2.31}$$

and

$$N(\hat{\mathfrak{a}}, \hat{\mathfrak{b}}) = min\{\eth(F\hat{\mathfrak{a}}, S\hat{\mathfrak{a}}), \eth(F\hat{\mathfrak{b}}, S\hat{\mathfrak{a}})\} = 0. \tag{2.32}$$

Now utilizing (2.31) and (2.32) in (2.30), we get

$$\psi(\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})) \leq \psi(\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})) - \phi(\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})). \tag{2.33}$$

Suppose, we define a sequence $\{r_n\}$ by $r_n = \eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})$ for all $n \in \mathscr{N}$ then $r_n \to \eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}}) > 0$ as $n \to \infty$. Hence $liminf_{n\to\infty}\phi(r_n) > 0$ that is

$$\phi(r_n) > 0 \implies \phi(\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})) > 0. \tag{2.34}$$

Thus from (2.33) and (2.34), it follows that

$$\psi(\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})) < \psi(\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})),$$

a contradiction. Hence $\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}}) = 0 \implies S\hat{\mathfrak{a}} = S\hat{\mathfrak{b}}$, since S or F is one-one, it follows that $\hat{\mathfrak{a}} = \hat{\mathfrak{b}}$. Hence $Coin(S, F)$ unique.
Next suppose that (S, F) is weakly compatible. Since $\hat{\mathfrak{a}} \in Coin(S, F)$ which implies $S\hat{\mathfrak{a}} = F\hat{\mathfrak{a}} = v$(say). As the pair (S, F) is weakly compatible, we have $SF\hat{\mathfrak{a}} = FS\hat{\mathfrak{a}} \implies Sv = Fv$. Hence $v \in Coin(S, F)$. In view of uniqueness of the coincidence point, we have $v = \hat{\mathfrak{a}}$ and $Sv = Fv = v$. Hence $v \in C(S, F)$.
In order to prove $Coin(S, F)$ is a singleton set, let w be the another common fixed point of (S, F) then $w \in C(S, F)$. Therefore, by the above argument, we have $w = v$.
Hence $Fw = w = Sw = Sv = wv = v$.
Therefore (S, F) is a singleton set. □

Theorem 2.5. *In addition to the conditions of Theorem* 2.3, *if* $\tilde{\mathfrak{R}}|S\tilde{\mathfrak{M}}$ *is complete, then* $Coin(S, F)$ *is a singleton set. Furthermore, if the pair* (S, F) *is weakly compatible, there exists at least one* $\nu \in \tilde{\mathfrak{M}}$ *such that* $\nu \in C(S, F)$.

Proof. On the account of proof of Theorem 2.3, $Coin(S, F) \neq \emptyset$.
If $Coin(S, F)$ is a singleton set, then the proof is completed. Otherwise, choose $\hat{\mathfrak{a}}, \hat{\mathfrak{b}} \in C(S, F)$, so that

$$S\hat{\mathfrak{a}} = F\hat{\mathfrak{a}} = \nu \; and \; S\hat{\mathfrak{b}} = F\hat{\mathfrak{b}} = \omega. \tag{2.35}$$

Suppose that $\tilde{\mathfrak{R}}|SM$ is complete, $[\nu, \omega] \in \tilde{\mathfrak{R}}$. Using Proposition 2.2, and condition (a), we get

$$\psi(\eth(\nu, \omega)) = \psi(\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})) \leq \psi(\mathscr{M}(\hat{\mathfrak{a}}, \hat{\mathfrak{b}})) - \phi(\mathscr{M}(\hat{\mathfrak{a}}, \hat{\mathfrak{b}})) + LN(\hat{\mathfrak{a}}, \hat{\mathfrak{b}}), \tag{2.36}$$

where

$$\begin{aligned}\mathscr{M}(\hat{\mathfrak{a}}, \hat{\mathfrak{b}}) &= max\{\eth(F\hat{\mathfrak{a}}, F\hat{\mathfrak{b}}), \eth(F\hat{\mathfrak{b}}, S\hat{\mathfrak{a}}), \eth(F\hat{\mathfrak{b}}, S\hat{\mathfrak{b}}), \frac{\eth(F\hat{\mathfrak{b}}, S\hat{\mathfrak{a}})[1 + \eth(F\hat{\mathfrak{a}}, S\hat{\mathfrak{a}})]}{1 + \eth(F\hat{\mathfrak{a}}, F\hat{\mathfrak{b}})}, \\ &\qquad \frac{\eth(F\hat{\mathfrak{a}}, S\hat{\mathfrak{a}})[1 + \eth(F\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})]}{1 + \eth(F\hat{\mathfrak{a}}, F\hat{\mathfrak{b}})}\} \\ &= max\{\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}}), \eth(S\hat{\mathfrak{b}}, S\hat{\mathfrak{a}}), \frac{\eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}})}{1 + \eth(S\hat{\mathfrak{b}}, S\hat{\mathfrak{a}})}\} \\ &= \eth(S\hat{\mathfrak{a}}, S\hat{\mathfrak{b}}).\end{aligned} \tag{2.37}$$

and

$$N(\hat{\mathfrak{a}}, \hat{\mathfrak{b}}) = min\{\eth(F\hat{\mathfrak{a}}, S\hat{\mathfrak{a}}), \eth(F\hat{\mathfrak{b}}, S\hat{\mathfrak{a}})\} = 0. \tag{2.38}$$

Now utilizing (2.37) and (2.38) in (2.36), we get

$$\psi(\eth(\nu, \omega)) \leq \psi(\eth(\nu, \omega)) - \phi(\eth(\nu, \omega)),$$

inview of property ϕ, it follows that

$$\psi(\eth(\nu, \omega)) < \psi(\eth(\nu, \omega)),$$

which encounters a contradiction. Consequently, $\nu = \omega$. □

3 Consequences and examples

Corollary 3.1. *Consider selfmaps S and F on $(\tilde{\mathtt{M}}, \eth)$ associated with $\tilde{\mathtt{R}}$. Suppose that conditions $(a),(b),(c),(d)$ or (e) of Theorem 2.3 and (i), (ii) of Theorem 2.4 together with the following assumptions :*
(i) there exists $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)), \tag{3.1}$$

where

$$\mathscr{M}(\zeta, \int) = max\{\eth(F\zeta, F\int), \eth(F\int, S\zeta), \eth(F\int, S\int), \frac{\eth(F\int, S\zeta)[1+\eth(F\zeta, S\zeta)]}{1+\eth(F\zeta, F\int)}, \frac{\eth(F\zeta, S\zeta)[1+\eth(F\int, S\zeta)]}{1+\eth(F\zeta, F\int)}\}$$

for all $\zeta, \int \in \tilde{\mathtt{M}}$ with $(F\zeta, F\int) \in \tilde{\mathtt{R}}$.
(ii) the pair (S, F) is weakly compatible, under the above condition, S and F posses a unique common fixed point within $\tilde{\mathtt{M}}$.

Proof. The proof follows immediately by setting $L = 0$ in Theorem 2.4. □

Corollary 3.2. *Let S be a self map on $(\tilde{\mathtt{M}}, \eth)$ with $\tilde{\mathtt{R}}$. Assume that*

(a) S is almost $\tilde{\mathtt{R}} - (\psi, \phi)$ weak contractive map on $\tilde{\mathtt{M}}$ that is

there exists $\psi \in \Psi$, $\phi \in \Phi$ and $L \geq 0$ such that

$$\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int) \tag{3.2}$$

where

$$\mathscr{M}(\zeta, \int) = max\{\eth(\zeta, \int), \eth(\int, S\zeta), \eth(\int, S\int), \frac{\eth(\int, S\zeta)[1+\eth(\zeta, S\zeta)]}{1+\eth(\zeta, \int)}, \frac{\eth(\zeta, S\zeta)[1+\eth(\int, S\zeta)]}{1+\eth(\zeta, \int)}\}$$

and

$$N(\zeta, \int) = min\{\eth(\zeta, S\zeta), \eth(\int, S\zeta)\}$$

for all $\zeta, \int \in \tilde{\mathtt{M}}$ with $(\zeta, F) \in \tilde{\mathtt{R}}$.
(b) There exists $\zeta_o \in \tilde{\mathtt{M}}$ such that $(\zeta_o, S\zeta_o) \in \tilde{\mathtt{R}}$.
(c) There is a complete subset Q of $\tilde{\mathtt{M}}$ such that $S\tilde{\mathtt{M}} \subseteq Q \bigcap F\tilde{\mathtt{M}}$.
$(\eth)$ $\tilde{\mathtt{R}}$ is S closed.
(e) One of the succeeding conditions is satisfied:
(i) S is $\tilde{\mathtt{R}}$ continuous
(ii) $\tilde{\mathtt{R}}|_Q$ is $\eth$-self closed,
then S has a fixed point. Further
(f) if $[\zeta, q] \in Fix(S) \implies [\zeta, \int] \in \tilde{\mathtt{R}}$,
then S has a unique fixed point in $\tilde{\mathtt{M}}$.

Proof. The proof follows immediately by setting $F = I$ in Theorem 2.4. □

Corollary 3.3. *Corollary 3.2 remains correct even if (f^*) takes the place of (f).*

(f^*) $S\tilde{\mathtt{M}}$ is $\tilde{\mathtt{R}}^s$ is connected,

then S has a unique fixed point in $\tilde{\mathtt{M}}$.

Proof. We can assume that by above Corollary 3.2, S has a fixed point, say ν. Suppose that S has another fixed point ω such that $\nu \neq \omega$. Then there exists a path $R^s \subseteq Fix(S)$. Let us denote this path by $\{\vartheta_0, \vartheta_1, ..., \vartheta_k\}$ such that $\nu = \vartheta_0,\ \omega = \vartheta_k$, such that $\vartheta_i \neq \vartheta_{i+1}$ for all i, where $0 \leq i \leq k-1$ and

$$[\vartheta_i, \vartheta_{i+1}] \in \tilde{\mathtt{R}}. \tag{3.3}$$

As $\vartheta_i \in Fix(S)$, so that $S(\vartheta_i) = \vartheta_i$ for all $i \in \{1, 2, ..., k-1\}$.
Consequently, on using proposition 2.2, we have

$$\psi(\eth(S\vartheta_i, S\vartheta_{i+1})) \leq \psi(\mathscr{M}(\vartheta_i, \vartheta_{i+1})) - \phi(\mathscr{M}(S\vartheta_i, S\vartheta_{i+1})) + LN(\vartheta_i, \vartheta_{i+1}), \tag{3.4}$$

where

$$\begin{aligned}\mathscr{M}(\vartheta_i, \vartheta_{i+1}) &= max\{\eth(\vartheta_i, \vartheta_{i+1}), \eth(\vartheta_{i+1}, S\vartheta_i), \eth(\vartheta_{i+1}, S\vartheta_{i+1}), \frac{\eth(\vartheta_{i+1}, S\vartheta_i)[1+\eth(\vartheta_i, \vartheta_i)}{1+\eth(\vartheta_i, \vartheta_{i+1})},\\ &\qquad \frac{\eth(\vartheta_i, \vartheta_i)[1+\eth(\vartheta_{i+1}, \vartheta_i)]}{1+\eth(\vartheta_i, \vartheta_{i+1})}\}\\ &= max\{\eth(\vartheta_i, \vartheta_{i+1}), \frac{\eth(\vartheta_i, \vartheta_{i+1})}{1+\eth(\vartheta_i, \vartheta_{i+1})}\}\\ &= \eth(\vartheta_i, \vartheta_{i+1})\end{aligned} \tag{3.5}$$

and

$$N(\vartheta_i, \vartheta_{i+1}) = min\{\eth(\vartheta_i, \vartheta_{i+1}), \eth(\vartheta_{i+1}, \vartheta_{i+1})\} = 0. \tag{3.6}$$

Thus, on making use of inequalities (3.4), (3.5) and (3.6), we attain

$$\psi(\eth(\vartheta_i, \vartheta_{i+1})) \leq \psi(\eth(\vartheta_i, \vartheta_{i+1})) - \phi(\eth(\vartheta_i, \vartheta_{i+1})),$$

taking account of properties of ϕ, we get $\psi(\eth(\vartheta_i, \vartheta_{i+1})) < \psi(\eth(\vartheta_i, \vartheta_{i+1}))$, a contradiction. Hence $\vartheta_i = \vartheta_{i+1}$. Hence S has a unique fixed point in $\tilde{\mathtt{M}}$. □

Example 3.4. Let $\tilde{\mathtt{M}} = (-1, 1]$, we define $\eth : \tilde{\mathtt{M}} \times \tilde{\mathtt{M}} \to \tilde{\mathtt{M}}$ by

$$\eth(\zeta, \int) = \begin{cases} max\{\zeta, \int\} & if\ \zeta \neq \int \\ \\ 0 & if\ \zeta = \int \end{cases}$$

Define $\tilde{\mathtt{R}}$ on $\tilde{\mathtt{M}}$ by $\tilde{\mathtt{R}} = P \times P$, where $P = [0, 1]$. We define $S, \digamma$ on $\tilde{\mathtt{M}}$ by

$$S\zeta = \begin{cases} \frac{2}{3} & if\ \zeta \in (-1, \frac{2}{3}] \\ \\ \frac{4}{3} - \zeta & if\ \zeta \in (\frac{2}{3}, 1] \end{cases} \text{ and } \digamma\zeta = \begin{cases} \zeta & if\ \zeta \in (-1, \frac{2}{3}] \\ \\ \frac{5}{6} & if\ \zeta \in (\frac{2}{3}, 1] \end{cases}$$

We now verify the conditions of the Theorem 2.3. There exists $\zeta_o = \frac{2}{3}$ such that $(S\zeta_o, \digamma\zeta_o) \in \tilde{\mathtt{R}}$. Let $Q = [\frac{1}{3}, \frac{2}{3}]$. Clearly, $Q \subseteq \digamma\tilde{\mathtt{M}} = [0, \frac{2}{3}] \cup \{\frac{5}{6}\}$ and $[\frac{1}{3}, \frac{2}{3}] = S\tilde{\mathtt{M}} \subseteq Q \cap \digamma\tilde{\mathtt{M}}$. Clearly, Q is complete subset of $\tilde{\mathtt{M}}$. The relation $\tilde{\mathtt{R}}$ satisfies the property of locally S transitive and $\tilde{\mathtt{R}}$ is $(S, \digamma)$ closed.
We now define ψ and $\varphi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \frac{t^2}{2} \text{ and } \varphi(t) = \begin{cases} \frac{t}{16} & if\ t \leq 1 \\ \\ \frac{11}{10} & if\ t > 1. \end{cases}$$

Then obviously, $\psi \in \Psi$ and $\varphi \in \Phi$.
We now show that S and $\digamma$ are $\tilde{\mathtt{R}}$- continuous maps, for let $\{\zeta_n\}$ be an $\tilde{\mathtt{R}}$ preserving sequence converging to ζ in $[0, \frac{2}{3}]$ then $S\zeta_n \to \frac{2}{3} = S\zeta$ and $\digamma\zeta_n \to \zeta = \digamma\zeta$ as $n \to \infty$ and $\{\zeta_n\}$ be an $\tilde{\mathtt{R}}$ preserving sequence converging to ζ in $(\frac{2}{3}, 1]$ then $S\zeta_n \to \frac{4}{3} - \zeta = S\zeta$ and $\digamma\zeta_n \to \frac{5}{6} = \digamma\zeta$ as $n \to \infty$, hence S and $\digamma$ are $\tilde{\mathtt{R}}$- continuous maps. Also, the maps S and $\digamma$ are $\tilde{\mathtt{R}}-$ compatible. Indeed, let $\zeta_n = \frac{2}{3} - \frac{1}{n}$, then $S\zeta_n = \frac{2}{3}$ and $\digamma\zeta_n = \frac{2}{3} - \frac{1}{n}$ are $\tilde{\mathtt{R}}$ preserving sequences and

converging to $\frac{2}{3}$ as $n \to \infty$, then $lim_{n\to\infty}\eth(S\digamma\zeta_n, \digamma S\zeta_n) = 0$.
We now verify the contraction condition (2.1) when $(\digamma\zeta, \digamma\int) \in \tilde{\mathtt{R}}$, for any $L \geq \frac{1}{2}$.

Case-1: When $\zeta, \int \in [0, \frac{2}{3}]$, then we have $\psi(\eth(S\zeta, S\int)) = 0$. Hence

$$\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int).$$

Case-2: When $\zeta, \int \in (\frac{2}{3}, 1]$, then without loss of generality $\zeta > \int$. We have

$$\psi(\eth(S\zeta, S\int)) = \frac{(\frac{4}{3} - \int)^2}{2},\ \psi(\mathscr{M}(\zeta, \int)) = \frac{3025}{2592},\ \phi(\mathscr{M}(\zeta, \int)) = \frac{11}{10},\ N(\zeta, \int) = \frac{5}{6}.$$

The max value of LHS attains at $\int = 1$, therefore

$$\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int).$$

Case-3: When $\zeta \in (\frac{2}{3}, 1]$, and $\int \in [0, \frac{2}{3}]$, then we have

$$\psi(\eth(S\zeta, S\int)) = \frac{2}{9},\ \psi(\mathscr{M}(\zeta, \int)) = \frac{25}{72},\ \phi(\mathscr{M}(\zeta, \int)) = \frac{5}{96},\ N(\zeta, \int) = \frac{2}{3}.$$

Hence

$$\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int).$$

Case-4: When $\zeta \in [0, \frac{2}{3}]$, and $\int \in (\frac{2}{3}, 1]$. We have

$$\psi(\eth(S\zeta, S\int)) = \frac{2}{9},\ \psi(\mathscr{M}(\zeta, \int)) = \frac{25}{72},\ \phi(\mathscr{M}(\zeta, \int)) = \frac{5}{96},\ N(\zeta, \int) = 0.$$

The max value of RHS attains at $\zeta = \frac{2}{3}$, therefore

$$\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int).$$

Noticeably, S and $\digamma$ commute at their coincidence point, satisfying the hypotheses of Theorem 2.3. Additionally, $\frac{2}{3}$ is the unique common fixed point of S and $\digamma$.

Example 3.5. Let $\tilde{\mathtt{M}} = [0, 1)$ with the usual metric. Let us define a binary relation on $\tilde{\mathtt{M}}$ by $\tilde{\mathtt{R}} = \{(0,0), (\frac{1}{2}, 0), (\frac{1}{3}, 0), (\frac{1}{4}, 0), (\frac{1}{4}, \frac{1}{4}), (\frac{1}{3}, \frac{1}{3}), (\frac{1}{5}, 0)\}$. We define selfmaps S and $\digamma$ on $\tilde{\mathtt{M}}$ by

$$S\zeta = \begin{cases} 0 & if\ \zeta \in [0, \frac{1}{4}] \\ \\ \frac{1}{4} & if\ \zeta \in (\frac{1}{4}, 1) \end{cases} \text{ and } \digamma\zeta = \begin{cases} \zeta & if\ \zeta \in \{0, \frac{1}{2}\} \\ \\ \frac{\zeta}{2} & otherwise. \end{cases}$$

We now verify the conditions of the Theorem 2.3. There exists $\zeta_o = 0$ such that $(S\zeta_o, \digamma\zeta_o) \in \tilde{\mathtt{R}}$. Let $Q = \{0, \frac{1}{4}\}$. Obviously, $Q \subseteq \digamma\tilde{\mathtt{M}} = [0, \frac{1}{2}]$ and $\{0, \frac{1}{4}\} = S\tilde{\mathtt{M}} \subseteq Q \cap \digamma\tilde{\mathtt{M}}$. Clearly, Q is complete subset of $\tilde{\mathtt{M}}$. The relation $\tilde{\mathtt{R}}$ exhibits the property of being locally S transitive and $(S, \digamma)$ closed.

We now show that S and $\digamma$ are $\tilde{\mathtt{R}}$- continuous maps, for let $\{\zeta_n\}$ be an $\tilde{\mathtt{R}}$ preserving sequence in Q converging to ζ then there exists $m \in \mathscr{N}$ such that $\zeta_n = \zeta \in \{0, \frac{1}{4}\}$ for all $n \geq m$. We can choose a subsequence $\zeta_{n_p} = \zeta$ for all $p \in \mathscr{N} \implies [\zeta_{n_p}, \zeta] \in R|Q$ for all p. Therefore $\tilde{\mathtt{R}}|Q$ is $\eth$-self closed. Moreover, it is readily apparent that $\tilde{\mathtt{R}}|S\tilde{\mathtt{M}}$ is complete.

The functions ψ and $\phi : [0, \infty) \to [0, \infty)$ are defined as follows:

$$\psi(t) = ln(1 + t) \text{ and } \phi(t) = \begin{cases} \frac{t}{2} & if\ t \leq 1 \\ \\ 1 & if\ t > 1. \end{cases}$$

Then obviously, $\psi \in \Psi$ and $\phi \in \Phi$. To authenticate the contraction condition (a) of Theorem 2.3, we have to verify when $(\digamma\zeta, \digamma\int) \in \{(\frac{2}{3}, 0), (\frac{1}{2}, 0), (\frac{2}{5}, 0)\}$, or else $\eth(S\zeta, S\int) = 0$.
Case-1: When $\zeta = \frac{2}{3}$, $\int = 0$, then

$$\psi(\eth(S\zeta, S\int)) = ln\frac{5}{4},\ \psi(\mathscr{M}(\zeta, \int)) = ln\frac{4}{3},\ \phi(\mathscr{M}(\zeta, \int)) = \frac{1}{6},\ N(\zeta, \int) = \frac{1}{12}.$$

Hence

$$\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int),$$

holds for any $L \geq 2$.

Case-2: When $\zeta = \frac{1}{2}$, $\int = 0$, then

$$\psi(\eth(S\zeta, S\int)) = ln\frac{5}{4},\ \psi(\mathscr{M}(\zeta, \int)) = ln\frac{3}{2},\ \phi(\mathscr{M}(\zeta, \int)) = \frac{1}{4},\ N(\zeta, \int) = \frac{1}{4}.$$

Hence

$$\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int),$$

holds for any $L \geq 1$.

Case-3: When $\zeta = \frac{2}{5}$, $\int = 0$, then

$$\psi(\eth(S\zeta, S\int)) = ln\frac{5}{4},\ \psi(\mathscr{M}(\zeta, \int)) = ln\frac{5}{4},\ \phi(\mathscr{M}(\zeta, \int)) = \frac{1}{8},\ N(\zeta, \int) = \frac{1}{20}.$$

Hence

$$\psi(\eth(S\zeta, S\int)) \leq \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)) + LN(\zeta, \int),$$

holds for any $L \geq 3$.

Intelligibly, S and $\digamma$ commute at their coincidence point, satisfying prerequisites of Theorem 2.5 and 0 is the unique common fixed point of S and $\digamma$. We note that Condition (3.1) of Corollary 3.1 fails to hold when $\zeta = \frac{2}{3}$, $\int = 0$, indeed

$$\psi(\eth(S\zeta, S\int)) = ln\frac{5}{4}, \psi(\mathscr{M}(\zeta, \int)) = ln\frac{4}{3}, \phi(\mathscr{M}(\zeta, \int)) = \frac{1}{6}.$$

We have

$$\psi(\eth(S\zeta, S\int)) = ln\frac{5}{4} \nleq ln\frac{4}{3} - \frac{1}{6} = \psi(\mathscr{M}(\zeta, \int)) - \phi(\mathscr{M}(\zeta, \int)).$$

Thus, Theorem 2.3 generalizes Corollary 3.1.

Here it is worth noting that the mapping $\digamma$ is not a generalized weakly contractive map on $\tilde{\mathbb{R}}$. This is evident that when considering $\zeta = \frac{2}{3}$, $\int = 0$, and $\phi(\tau) = \frac{\tau}{2}$, we have

$$\psi(\digamma\zeta, \digamma\int)) = log(\frac{4}{3}) \nleq log(\frac{5}{3}) - \frac{1}{3} = \phi(\Lambda_1(\zeta, \int)) - \phi(\Lambda_2(\zeta, \int)).$$

Hence, condition (2.1) is more general than condition (1.1).

4 Applications

Boundary value issues are an important branch of mathematics that find extensive use in engineering, chemistry, biology, physics, and other fileds where real-world phenomena are modeled. Motivated the applications of boundary value problems we solve the following differential equation taken from Samet et. al., [31].

$$-\frac{\eth^2 u}{\eth\kappa^2} = f(k, u(k)),\ k \in [0, 1], u(0) = u(1) = 0, \tag{4.1}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Green's function associated with (4.1) is

$$\digamma(k, s) = \begin{cases} k(1-s) & if\ 0 \leq k \leq s \leq 1 \\ \\ s(1-k) & if\ 0 \leq s \leq k \leq 1. \end{cases}$$

Let $\tilde{\mathtt{M}} = C[0, 1]$ be set of all continuous functions defined on $[0, 1]$ and we define $\eth : \tilde{\mathtt{M}} \times \tilde{\mathtt{M}} \to \mathbb{R}$ by $\eth(\zeta, \int) = sup_{k\in[0,1]}|\zeta(k) - \int(k)|$. We are aware that $(C[0, 1], \eth)$ forms a complete metric space.

Theorem 4.1. *Let S be a self maps of a complete metric space $\tilde{\mathtt{M}} = C[0,1]$ satisfying:*

1. A function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is satisfying:

$$|f(k,\zeta(k)) - f(k,\int(k))| \leq 8(\mathscr{M}(\zeta(k),\int(k)) - \varphi(\mathscr{M}(\zeta(k),\int(k)))$$

such that for all $\zeta(k), \int(k) \in \tilde{\mathtt{R}}$ with $\zeta(k) \leq \int(k)$, where $\varphi \in \Phi$ and

$$\mathscr{M}(\zeta(k),\int(k)) = max\{|\zeta(k) - \int(k)|, |\int(k) - S\zeta(k)|, |\int(k) - S\int(k)|, \frac{|\int(k) - S\zeta(k)|[1 + |\zeta(k) - S\zeta(k)|]}{1 + |\zeta(k) - \int(k)|}, \frac{|\zeta(k) - S\zeta(k)|[1 + |\int(k) - S\zeta(k)|]}{1 + |\zeta(k) - \int(k)|}\}.$$

2. There exists $\zeta_o \in C([0,1])$ such that

$$S\zeta(k) = \int_0^1 F(k,s)f(s,\zeta(s))ds. \tag{4.2}$$

Under the above assumptions the equation (4.1) has a unique solution.

Proof. Finding solution of (4.1) is equal to finding fixed point in $\tilde{\mathtt{M}}$. We define a binary relation

$$\tilde{\mathtt{R}} = \{(\zeta,\int) \in C[0,1] \times C[0,1] : \zeta(k) \leq \int(k),\ for\ all\ k \in [0,1]\}.$$

Let us consider the sequence $\{\zeta_n\}$ which $\tilde{\mathtt{R}}$ preserving and converges to z. For every k in the range $[0,1]$, it holds that

$$\zeta_0(k) \leq \zeta_1(k) \leq \zeta_2(k) \leq ..... \leq \zeta_n(k) \leq .....$$

and converges to $\zeta(k)$ which implies $\zeta_n(k) \leq \zeta(k)$ for all $k \in [0,1]$ we can choose subsequence $\{\zeta_{n_p}(k)\}$ of $\{\zeta_n(k)\}$ such that $[\zeta_{n_p},\zeta(k)] \in \tilde{\mathtt{R}}$ for all $n \in \mathscr{N}$, which proves $\tilde{\mathtt{R}}$ is $\eth$-self closed.
For $(\zeta,\int) \in \tilde{\mathtt{R}}$, *i.e.*, $\zeta(s) \leq \int(s)$ and $F(k,s) > 0$ for all $k,s \in [0,1]$, we have

$$S\zeta(k) = \int_0^1 F(k,s)f(s,\zeta(s))ds \leq \int_0^1 F(k,s)f(s,\int(s))ds = S\int(k)$$

which implies $(S\zeta(k), S\int(k)) \in \tilde{\mathtt{R}}$, thus $\tilde{\mathtt{R}}$ is S- closed.
Now for $(\zeta,\int) \in \tilde{\mathtt{R}}$, we have

$$\begin{aligned}
\eth(S\zeta, S\int) &= sup_{k\in[0,1]}|S\zeta(k) - S\int(k)| \\
&= sup_{k\in[0,1]}|\int_0^1 F(k,s)f(s,\zeta(s))ds - \int_0^1 F(k,s)f(s,\int(s))ds| \\
&\leq sup_{k\in[0,1]}\int_0^1 |F(k,s)||f(s,\zeta(s)) - f(s,\int(s))ds| \\
&\leq 8(\mathscr{M}(\zeta,\int) - \varphi(\mathscr{M}(\zeta,\int))sup_{k\in[0,1]}\int_0^1 |F(k,s)|ds \\
&= 8(\mathscr{M}(\zeta,\int) - \varphi(\mathscr{M}(\zeta,\int))sup_{k\in[0,1]}(\frac{k}{2} - \frac{k^2}{2}) \\
&= (\mathscr{M}(\zeta,\int) - \varphi(\mathscr{M}(\zeta,\int))(since\ sup_{k\in[0,1]}(\frac{k}{2} - \frac{k^2}{2}) = \frac{1}{8}) \\
&\leq \mathscr{M}(\zeta,\int) - \varphi(\mathscr{M}(\zeta,\int)) + LN(\zeta,\int).
\end{aligned}$$

This shows that S satisfies condition (a) of Corollary 3.2. Let $x \in C([0,1],\mathbb{R})$ be a solution of (a), that is,

$$\begin{aligned}
x(k) &\leq \int_0^s k(1-s)f(k,x(k))dk + \int_s^1 t(1-s)f(k,x(k))dk \\
&= \int_0^1 F(k,s)f(k,x(k))dk = Sx(k),
\end{aligned}$$

for all $k \in [0,1]$, $(x(k), Sx(k)) \in \tilde{\mathtt{R}}$ i.e., $\tilde{\mathtt{M}}(S, \tilde{\mathtt{R}}) \neq \phi$.
Let us define r as $r = max\{\beta, \vartheta\}$. Consequently, for any t, $\beta(k) \leq r(k)$ and $\vartheta(k) \leq r(k)$, $(\beta, r) \in \tilde{\mathtt{R}}$ and $(\vartheta, r) \in \tilde{\mathtt{R}}$. This establishes the sequence $\{\beta, r, \vartheta\}$ forms a path connecting β and ϑ in $\tilde{\mathtt{R}}$. Consequently, all the hypotheses of Corollary 3.2 is satisfied, leading to the conclusion that S has a unique fixed point, which is the solution of the problem (4.1).

□

5 Conclusions

Utilizing a relation-theoretic technique, we studied the existence of common fixed points for almost weak contraction in complete metric spaces. Recent studies in the literature have employed relational contraction conditions, which are less restrictive compared to normal contraction conditions. Our theorems not only extend but also generalize the well-known results in the literature. It is worth mentioning that in our results, we often use weaker notions like $\tilde{\mathtt{R}}$-completeness on a subspace rather than completeness on the whole space and $\tilde{\mathtt{R}}$-continuous, not necessarily continuous and $\tilde{\mathtt{R}}|_Q$ is $\eth$-self closed. Notably, these contraction conditions reduce to their classical forms when the underlying relation is taken to be universal.

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