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The S-vertices of the Annihilation Graphs of Commutator Posets and Lattices

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Keywords and phrases: Commutator Poset; commutator lattice; reduced poset with respect to an element; annihilation graph; complete bipartite graph; weakly perfect graph; maximal clique; diameter and girth of a graph; Smarandache vertex (= S-vertex) of a graph; Smarandache zero-divisor.

Abstract A vertex a in a simple graph G is said to be a Smarandache vertex (or S-vertex for short) provided that there exist three distinct vertices x, y, and b (all different from a) in G such that x—a, a—b, and b—y are edges in G, but there is no edge between x and y. In this interdisciplinary subject, we investigate the interplay between the algebraic properties of the commutator posets and lattices and their associated annihilation graphs using the notion of the Smarandache vertices. Actually, $\mathbb{AG}_z(L)$ (the annihilation graph of the commutator poset [lattice] L with respect to an element $z \in L$) is a widely generalized context for the study of the zero-divisor type (annihilating-ideal) graphs, where the vertices of the graphs are not elements/ideals of a commutative ring, but elements of an abstract ordered set [lattice] (imitating the lattice of ideals of a ring), equipped with a commutative (not necessarily associative) binary operation (imitating the product of ideals of a ring). We discuss when $\mathbb{AG}_z(L)$ is a complete bipartite graph together with some of its other graph-theoretic properties.

1 Introduction

In this work, we will extend the notion of the *Smarandache vertices* of a simple graph (Definition 1.1), which was discussed for the *zero-divisor* and *comaximal graphs* of a *commutative ring* in [33] and [27], respectively, to the *annihilation graphs* of a *commutator poset [commutator lattice]* L with respect to an element $z \in L$, denoted $\mathbb{AG}_z(L)$ (Definition 4.1). The notion of the annihilation graphs of a commutator poset and a commutator lattice L with respect to an element [25] and an ideal [26] of L was first introduced by Mehdi-Nezhad and Rahimi. The main purpose of their study in this interdisciplinary subject was an investigation between the algebraic properties of the *bounded* commutator lattices (Definition 3.1) and commutator posets (Definition 3.3) L and their associated annihilation graphs with respect to an element and an ideal of L.

The motivation of this study is mainly based on providing a bridge between the structures of commutator posets and lattices and their associated annihilation graphs (Definition 4.1) using the notion of the Smarandache vertices in a natural way by comparing some results similar to the cases in commutative rings.

• In this *introductory section*, we *recall* the definitions of the zero-divisor graph of a commutative ring, and the notion of a Smarandache vertex in a simple graph (Definition 1.1) from [6] and [33], respectively, for the sake of completeness and reference and end the section with a brief description related to the *organization* of the following sections.

Through out this paper (unless otherwise indicated) all posets [lattices] are bounded. A poset [lattice] L is said to be bounded if it contains a *least element 0* and *greatest element 1*. To avoid trivialities when $\mathbb{AG}_z(L)$ is empty, we will implicitly assume when necessary that z is not a *prime element* of L (see Remark 4.2(5)). We assume that the reader is familiar with the basic notion and definitions of poset, lattice, and graph theory. For the notation and definitions regarding lattice theory [resp. graph theory], the reader is referred to [32] or [12] [resp. [13], [14], and

[19] or any standard text on graph theory]. For commutative ring theory, see [20] and [7].

The familiar group-theoretic notion of a commutator has been generalized to various contexts of *universal algebra* and *category theory*. The universal-algebraic references to commutators usually begin with J. D. H. Smith [34], and then mention various further generalizations of Smith's definition (see for example [18] and references therein). For a general reference on commutator theory, see [18]. Also, for a detailed study of the annihilation graphs of commutator posets and lattices (with respect to an element and an ideal of the underlying posets and lattices), see [25] and [26], respectively. In addition in [29], there is a study related to an almost division algorithm of Euclidean commutator lattices.

The area of research on assigning a graph to an algebra (algebraic structure) has been very active (especially) since the last three decades and there are many papers which apply combinatorial methods (using graph-theoretic properties and parameters such as *planarity, clique number, chromatic number, independence number, domination number*, and so on) to obtain algebraic results and vice versa, for instance, there are many papers on this interdisciplinary subject and for a short list of them, see [1, 6, 5, 4, 10, 11, 15, 17, 22, 23, 28, 30, 3, 8, 9, 21, 31, 24].

• The concept of a Smarandache vertex (or S-vertex for short) in a *simple graph* (Definition 1.1), which is *independent* of any *algebraic structure* was first introduced by Rahimi [33] in order to study the *Smarandache zero-divisors* of a commutative ring which was introduced by Vasantha Kandasamy in [35] for semigroups and rings (not necessarily commutative). A nonzero element a in a commutative ring R is said to be a *Smarandache zero-divisor* if there exist *three different nonzero elements* x, y, and b (all different from a) in R such that $ax = ab = by = 0$ but $xy \neq 0$. This definition of a Smarandache zero-divisor (which was given in [33]) is slightly different from the definition of Vasantha Kandasamy in [35], where in her definition b could also be equal to a.
We now recall the definition of a Smarandache vertex for a simple graph (i.e., an undirected graph with no loops and no multiple edges) from [33].

Definition 1.1. A vertex a in a simple graph G is said to be a *Smarandache vertex* (or S-vertex for short) provided that there exist three distinct vertices x, y, and b (all different from a) in G such that x—a, a—b, and b—y are edges in G but there is no edge between x and y.

Consequently, by this generalization, study of S-vertices of any simple graph can be done directly in a pure graph-theoretic sense, and specially, discussing the S-vertices of any simple graph associated to an algebra (algebraic structure) is possible and can lead to the study of the interplay between some graph-theoretic properties and algebraic properties of the related algebra. For instance, S. Visweswaran and Hiren D. Patel in [36] studied the S-vertices of *the complement of the annihilating-ideal graph* in connection to some ring-theoretic properties in Sections 2 (Lemma 2.5), 4 (Lemma 4.2(v)), and 5 (Proposition 5.1(iv)) of their paper.

Note that a graph containing a Smarandache vertex should have at least four vertices and three edges, and also the *degree* of each S-vertex must be at least 2.
We now recall the definition of the zero-divisor graph of a commutative ring from [6].

The *zero-divisor graph* of a commutative ring R, denoted $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is the empty graph if and only if R is an integral domain. Moreover, a nonempty graph $\Gamma(R)$ is finite if and only if R is finite and not a field [6, Theorem 2.2].

In [10] (1988), Beck introduced the concept of a *zero-divisor graph* of a commutative ring, but this work was mostly concerned with *colorings* of rings. The above definition first appeared in the work of Anderson and Livingston [6] (1999), which contains several fundamental results concerning $\Gamma(R)$. This definition, unlike the earlier work of Anderson and Naseer [4] and Beck [10], does not take zero to be a vertex of $\Gamma(R)$. The zero-divisor graph of a commutative ring has also been generalized to a k-zero-divisor hypergraph by Eslahchi and Rahimi in [16].

• The organization of this paper is as follows: In Section 2, we give several examples and study the S-vertices of a simple graph in general (Example 2.1, Lemmas 2.2, 2.3, 2.4, and Proposition 2.5 which provides a relation between a weakly perfect graph and its S-vertices). In Lemma 2.2, we show the existence or nonexistence of S-vertices of some known graphs and in Lemmas 2.3 and 2.4 show how to construct the S-vertices from the cliques of a graph. We end this section with an example that shows the converse of Proposition 2.5 need not be true in general (Example 2.7), see also Remark 2.6. Section 3 is devoted to the definitions of a commutator lattice and commutator poset (Definitions 3.1 and 3.3) together with some of their properties (Remarks 3.2, 3.4, Definition 3.5, and Proposition 3.6). In Section 4, besides Examples 4.3 and 4.4, we define and discuss some basic properties of $\mathbb{AG}_z(L)$ (Definition 4.1, Remark 4.2) and close this section by constructing two isomorphic graphs with no S-vertices in a commutator poset L with respect to two different elements of L (Example 4.5). In Section 5, we study the cases for which $\mathbb{AG}_z(L)$ has S-vertices exploiting a clique of $\mathbb{AG}_z(L)$ (Propositions 5.1 and 5.4). We also show that $\mathbb{AG}_z(L)$ contains $n \geq 3$ S-vertices when z is the meet of n prime elements of L (Theorem 5.6). Section 6 is related to the girth of $\mathbb{AG}_z(L)$ and cases when $\mathbb{AG}_z(L)$ has no S-vertices regarding a relationship between the complete bipartiteness of $\mathbb{AG}_z(L)$ and prime elements of the commutator poset [lattice] L (Theorems 6.5, 6.6, and 6.7). Finally, we end the paper by defining a *reduced commutator poset* L with respect to an element of L (Definition 6.9) and a result (Theorem 6.10) following a conjecture related to a universal vertex (i.e., a vertex which is adjacent to all other vertices of the graph) of $\mathbb{AG}_z(L)$, which implies nonexistence of S-vertices in $\mathbb{AG}_z(L)$.

2 Some Examples and Results Related to S-vertices of Some Known Simple Graphs

We provide several (in particular, graph-theoretic) examples (Lemmas 2.2, 2.3, 2.4, and Proposition 2.5 which provide a relation between a weakly perfect graph and its S-vertices). In Lemma 2.2, we show the existence or nonexistence of S-vertices of some known graphs and in Lemmas 2.3 and 2.4 show how to construct the S-vertices from the cliques of a graph. Finally, we end the section with an example that shows the converse of Proposition 2.5 need not be true in general (Example 2.7).

We begin this section by defining the join of two graphs to show the existence of the S-vertices in the join of two special graphs (Example 2.1, see also Lemma 2.4).

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets V_i and edge sets E_i $(1 \leq i \leq 2)$. The *join* of G_1 and G_2 is denoted by $G = G_1 \vee G_2$ with the vertex set $V_1 \cup V_2$ and the set of edges is

$$E_1 \cup E_2 \cup \{xy | x \in V_1 \text{and} y \in V_2\},$$

where xy is the edge between x and y.

Example 2.1. Let for any graph G, $\overline{G}$ denote the complement of G. The graph $K_n \vee \overline{K_m}$ contains exactly n S-vertices provided that $n \geq 2$ and $m \geq 2$ (see also Lemma 2.4 below). Moreover, for any $n \geq 2$, $K_n \vee \overline{K_2}$ has exactly n S-vertices.

The proof of the next three lemmas are not difficult and can be followed directly from the definition of S-vertex and we leave them to the reader. In these lemmas, we show the existence (or nonexistence) of S-vertices of some known graphs (Lemma 2.2) and in Lemmas 2.3 and 2.4 show how to construct the S-vertices from the cliques of a graph.

Lemma 2.2. *For any simple graph, the following results are true:*

(1) *A complete graph does not have any S-vertices.*

(2) *A star graph does not have any S-vertices.*

(3) *A complete bipartite graph has no S-vertices.*

(4) *Let G be a complete r-partite graph ($r \geq 3$) with parts $V_1, V_2, \ldots, V_r$. If at least one part, say V_1, has at least two elements, then every element not in V_1 is an S-vertex. Further, if*

there exist at least two parts of G such that each of which has at least two elements, then every element of G is an S-vertex.

(5) *A bistar graph has two S-vertices; namely, the center of each star is an S-vertex. Recall that a bistar graph is a graph generated by two star graphs when their centers are joined.*

(6) *Every vertex in a cycle of size greater than or equal to five in a graph is an S-vertex provided that there is no edge between the nonneighbouring vertices. In particular, every vertex in the cycle C_n of size larger than or equal to 5 is an S-vertex. Note that for odd integers $n \geq 5$, $\chi(C_n) = 3$ and $\omega(C_n) = 2$. Moreover, for even integers $n \geq 5$, $\chi(C_n) = \omega(C_n) = 2$.*

(7) *Let G be a simple graph containing two distinct vertices x and y such that $d(x, y) = 3$. Then G has an S-vertex. But the converse is not true in general. Suppose G is the graph x—a, a—b, b—y, and a—y; where obviously, a is an S-vertex and $d(x, y) = 2$. Note that if the diameter of G is 3, then it has an S-vertex since there exist two distinct vertices x and y in G such that $d(x, y) = 3$.*

Lemma 2.3. *Let C be a clique in a graph G such that $|C| \geq 3$. Suppose that x is a vertex in $G \setminus C$ and x makes a link with at least one vertex or at most $|C| - 2$ vertices of C. Then every vertex of C is an S-vertex. In other case, if x makes links with $|C| - 1$ vertices of C, then all those $|C| - 1$ vertices are S-vertices.*

Lemma 2.4. *Let C be a clique in a graph G such that $|C| = n \geq 3$. Then by removing any edge e from C, the remaining subgraph $C \setminus \{e\}$ is isomorphic to $K_{n-2} \vee \overline{K_2}$. In this case, $C \setminus \{e\}$ has exactly $n - 2$ S-vertices provided that $n \geq 4$ (see Example 2.1).*

The following proposition provides a relation between a weakly perfect graph and its S-vertices.

Proposition 2.5. *Let G be a connected simple graph whose clique number is strictly larger than 2. If $\omega(G) \neq \chi(G)$, then G has an S-vertex. In other words, for any connected simple graph G with $\omega(G) \geq 3$ and no S-vertices, then $\omega(G) = \chi(G)$ (i.e., G is weakly perfect).*

Proof. Let C be a (largest) clique in G with $|C| \geq 3$. Since $\omega(G) \neq \chi(G)$, then G is not a complete graph. Thus, there exists a vertex x in $G \setminus C$ which makes edge(s) with at least one or at most $\omega(G) - 1$ element(s) of C. Now the proof is immediate from Lemma 2.3. □

Remark 2.6. In the next example we show that the converse of the above proposition need not be true in general (Example 2.7). Also none of the graphs in Parts (1), (2), and (3) of Lemma 2.2, has an S-vertex where $\omega(G) = \chi(G)$. Note that each of the graphs in Parts (2) and (3) has $\omega(G) = \chi(G) = 2$. The graph in Part (5) has two S-vertices and $\omega(G) = \chi(G) = 2$. See also Part (6) of Lemma 2.2.

We end this section with an example that shows the converse of Proposition 2.5 is not true in general.

Example 2.7. Suppose G is the graph x—a, a—b, b—y, and a—y, where obviously, a is an S-vertex of G. Clearly, this graph shows that the converse of Proposition 2.5 is not true in general since it has an S-vertex and $\omega(G) = \chi(G) = 3$.

3 Definitions of the Commutator Poset and Commutator Lattice with Some of Their Basic Properties

In this section we recall the definitions of the commutator lattice and commutator poset L (Definitions 3.1 and 3.3) and study some of their (trivial) properties (Remarks 3.2 and 3.4). Finally, we close this section with some definitions and trivial properties relating to (commutator) posets and lattices (Definition 3.5 and Proposition 3.6). Moreover, we will define and discuss some (basic) properties of $\mathbb{AG}_z(L)$, the annihilation graph of L with respect to an element $z \in L$ (Definition 4.1), in the following sections.

Definition 3.1. ([25, Definition 2.1]) A commutator lattice is a (bounded) lattice L equipped with a binary operation $[-, -]$, also written as $[x, y] = xy$, that satisfies the following relations:

L1 $xy \leq x \wedge y$;

L2 $xy = yx$;

L3 $x(y \vee z) = (xy) \vee (xz)$;

for all x, y, z in L.

Remark 3.2. From **L1** of the above definition, it is clear that $xy \leq x$ since $x \wedge y \leq x$ for all $x, y \in L$. Also from **L3** above, we conclude $x \leq y$ implies $xz \leq yz$ for all z in L since $x \leq y$ implies $x \vee y = y$, which by **L3** it implies $zx \vee zy = zy$.

We now (more generally) from the above remark, define the following:

Definition 3.3. ([25, Definition 2.3]) A commutator poset is a (bounded) poset (= partially ordered set) L equipped with a binary operation $[-,-]$, also written as $[x, y] = xy$, that satisfies the following relations:

P1 $xy \leq x$;

P2 $xy = yx$;

P3 $x \leq y \Rightarrow xz \leq yz$;

for all x, y, z in L.

Remark 3.4. Let 0 and 1 be the least and greatest elements of a commutator poset L, respectively. Clearly $0 \leq 0 \cdot x \leq 0$ implies $0 \cdot x = 0$ for each $x \in L$. Now suppose L contains a multiplicative identity element u. Then $u = 1$ since $1 = u \cdot 1 \leq u \leq 1$, which implies $u = 1$. For example, $R \in L = \text{ideal}(R)$, the commutator poset [lattice] of ideals (under multiplication of ideals) of a commutative ring R with identity, is the largest and multiplicative identity of L. Note that $1 \in L$ need not be the multiplicative identity of L in general. That is, $1 \cdot x$ need not be equal to $x \in L$ (e.g. $1 \cdot 1$ need not be equal to 1).

Finally, we close this section with some definitions and trivial properties relating to (commutator) posets and lattices.

Definition 3.5. Let L be a commutator poset [lattice].

(a) A nonempty subset I of L is said to be an ideal of L when $rL \subseteq I$ for each $r \in I$. Further, $I \neq L$ is a prime ideal of L if $xy \in I$, then either $x \in I$ or $y \in I$.

(b) Let $z \neq 1$ be an element of a commutator poset (lattice) L. Let $C_z(L)$ [resp., $N_z(L)$] be the set of all elements of L that are less than or equal to z [resp., that are not less than or equal to z]. We also simply write C_z [resp., N_z] for $C_z(L)$ [resp., $N_z(L)$] whenever there is no confusion in the context. It is clear that $C_z(L)$ is an ideal of L.

(c) An element x in L is said to be a zero-divisor with respect to $1 \neq z \in L$ if there exists an element $y \in N_z(L)$ such that $xy \leq z$. In particular, x is said to be a zero divisor in L when $z = 0$. That is, x is a zero divisor in L if there exists a nonzero $y \in L$ such that $xy = 0$. Clearly, $x \leq z$ implies $xy \leq zy \leq z$ for any $y \in L$, which means that each $x \in C_z(L)$ is a zero divisor with respect to z and we call it a trivial zero divisor with respect to z. But when we say that x is a nontrivial zero divisor with respect to z, it means that $x \in N_z(L)$ and is a zero divisor with respect to z.

(d) An element $z \neq 1$ in L is said to be *prime* when $xy \leq z$ implies either $x \leq z$ or $y \leq z$ for all $x, y \in L$.

Proposition 3.6. *Let $p \neq 1$ be an element of a commutator poset L. Then p is a prime element of L if and only if C_p is a prime ideal of L.*

Proof. The proof follows directly from the definition. □

4 Definition of the $\mathbb{AG}_z(L)$ with Some of its Basic Properties

We begin this section by recalling the definition of the annihilation graphs of commutator posets and lattices with respect to an element $z \neq 1$ of the poset [lattice] L, denoted $\mathbb{AG}_z(L)$, (Definition 4.1) from [25] and some basic facts about it (Remark 4.2), respectively, which will be used in the sequel. Besides Examples 4.3 and 4.4, we close this section by constructing two isomorphic graphs with no S-vertices in a commutator poset L with respect to two different elements of L (Example 4.5).

The definition of an annihilating-ideal graph $\mathbb{AG}_I(R)$ [2] of a commutative ring R with respect to an ideal I of R immediately extends to the context of a commutator poset [lattice] as follows.

Definition 4.1. Let $z \neq 1$ be an element of a commutator poset [lattice] L. Let $N_z(L)$ be the set of all elements of L that are not less than or equal to z. The *annihilation graph of L with respect to z*, denoted by $\mathbb{AG}_z(L)$, is the graph whose vertex set is $\{a \in N_z(L) | \ ab \leq z \text{ for some } b \in N_z(L)\}$ with distinct vertices a and b adjacent if and only if $ab \leq z$. We shall also write $\mathbb{AG}(L) = \mathbb{AG}_0(L)$ and call this graph the *annihilation graph of L*.

We now recall some basic facts related to $\mathbb{AG}_z(L)$ from [25], which will be used in the sequel, for the sake of completeness.

Remark 4.2. The following are true for a commutator poset [lattice] L.

(1) $\mathbb{AG}_z(L)$ is connected with $\text{diam}(\mathbb{AG}_z(L)) \leq 3$. Thus, $\text{diam}(\mathbb{AG}_z(L)) = 0, 1, 2$ or 3 ([25, Theorem 4.9]).

(2) If $\mathbb{AG}_z(L)$ contains a cycle, then $\text{gr}(\mathbb{AG}_z(L)) \leq 4$. Thus, $\text{gr}(\mathbb{AG}_z(L)) = 3, 4$ or ∞ ([25, Theorem 4.9]).

(3) Let $z \neq 1$ be a nonzero element of a commutator poset L. Let p_1 and p_2 be two prime elements of L such that $z = inf\{p_1, p_2\}$. Then $\mathbb{AG}_z(L)$ is a complete bipartite graph ([25, Theorem 6.1]).

(4) Let $z \neq 1$ be a nonzero element of a commutator lattice L. Let p_1 and p_2 be two prime elements of L such that $z = p_1 \wedge p_2$. Then $\mathbb{AG}_z(L)$ is a complete bipartite graph ([25, Theorem 6.2]).

(5) Let $z \neq 1$ be an element of a commutator poset [lattice] L. Then $\mathbb{AG}_z(L)$ is the empty graph if and only if z is a prime element of L ([25, Proposition 3.3]).

We now define a simple example (a property) of commutator posets [lattices], which will provide new examples of the annihilation graphs [with respect to an element].

Example 4.3. Let $L = L_1 \times L_2 \times \cdots \times L_n$ $(n \geq 2)$ where L_i is a (commutator) poset [lattice] for each $1 \leq i \leq n$. Clearly, L is a (commutator) poset [lattice] by defining its operations *(multiplication), order $\leq$, meet, and join*, componentwise, respectively. Let $z_1 \neq 1$ and $z_2 \neq 1$ be two fixed elements of L_1 and L_2, respectively. Let $a = (a_1, 1, \ldots, 1)$, $b = (1, a_2, \ldots, 1)$, and $z = (z_1, z_2, 1, \ldots, 1)$, where $a_1 \leq z_1$ and $a_2 \leq z_2$. Clearly $ab \leq z$, but neither a nor b is less than or equal to z.

In the following simple example, we show that $\mathbb{AG}_z(L)$ is nonempty (Part (a)) and define a commutator poset L whose annihilation graph is complete bipartite (Part (b)) and consequently has no S-vertices (Lemma 2.2(3)).

Example 4.4. Let $L = L_1 \times L_2$, where each of L_1 and L_2 is a commutator poset with at least two elements 0 and 1. Clearly, L is a commutator poset by defining its multiplicative operation and its order $\leq$ componentwise (see also Example 4.3 and Remark 3.4).

(a) Let $a_1 \neq 1$ and $a_2 \neq 1$ be two elements of L_1 and L_2, respectively. Suppose $a = (a_1, 1)$, $b = (1, a_2)$, and $z = (a_1, a_2)$. Clearly $ab \leq z$, but neither a nor b is less than or equal to z.

(b) If each of L_1 and L_2 contains no nonzero zero-divisors, then $\mathbb{AG}(L)$ is a complete bipartite graph with parts $\{(a,0)|0 \neq a \in L_1\}$ and $\{(0,b)|0 \neq b \in L_2\}$ and hence has no S-vertices (Lemma 2.2(3)).

Finally, we close this section by constructing two isomorphic graphs with no S-vertices in a commutator poset L with respect to two different elements of L.

Example 4.5. Let $X = \{1,2,3\}$ and $L = (P(X), \cap, \subseteq)$, be a commutator poset, where $P(X)$ is the power set of X, which obviously contains $2^3 = 8$ elements included the empty set. Let $z = \{1\}$ and $w = \{2\}$ be two elements of L. It is not difficult to show that each of $\mathbb{AG}_z(L)$ and $\mathbb{AG}_w(L)$ is a square and so $\mathbb{AG}_z(L) \cong \mathbb{AG}_w(L)$. That is, $\mathbb{AG}_z(L)$ is the square

$$(\{1,2\},\{1,3\},\{2\},\{3\},\{1,2\})$$

and $\mathbb{AG}_w(L)$ is the square

$$(\{1,2\},\{2,3\},\{1\},\{3\},\{1,2\}).$$

Thus, there is no S-vertices in these two isomorphic graphs.

5 Constructing S-vertices in $\mathbb{AG}_z(L)$

In this section, we study the cases for which $\mathbb{AG}_z(L)$ has S-vertices using the clique of $\mathbb{AG}_z(L)$ (Propositions 5.1 and 5.4). We will end the section by showing the existence of $n \geq 3$ S-vertices in $\mathbb{AG}_z(L)$ when z is the meet of n prime elements of L (Theorem 5.6).

Proposition 5.1. *Let $n \geq 3$ be a fixed integer, $z \neq 1$ an element of the commutator latticeL, and $\{x_1, x_2, \ldots, x_n\}$ be a clique in $\mathbb{AG}_z(L)$ such that $x_1^2, x_2^2 \in N_z$. Then $\mathbb{AG}_z(L)$ contains n S-vertices.*

Proof. Let $y = x_1 \vee x_2$. First, we show that $y \neq x_i$ for each $1 \leq i \leq n$. Suppose, for instance, $y = x_1$. Thus, $x_2 \leq x1$ and hence $x_2^2 \leq x_2 x_1 \leq z$, yielding a contradiction. Clearly, $yx_i \leq z$ only for $3 \leq i \leq n$ and y doesn't make link with x_1 and x_2. Otherwise, for $i = 1, 2$, $x_i y \leq z$ implies $x_i^2 \leq z$ since $x_i \leq x_1 \vee x_2$, yielding a contradiction. Now, the proof follows directly from the definition and Lemma 2.3. □

Corollary 5.2. *Let $z \neq 1$ be an element of a commutator lattice L with $\mathrm{gr}(\mathbb{AG}_z(L)) = 3$. Suppose $\mathbb{AG}_z(L)$ contains a triangle with vertices a, b, and c such that $a^2, b^2 \in N_z$. Then $\mathbb{AG}_z(L)$ contains three S-vertices; namely a, b, and c are three S-vertices of $\mathbb{AG}_z(L)$.*

Corollary 5.3. *Let $n \geq 3$ be a fixed integer and L_i a commutator lattice with $1_{L_i} \neq z_i \in L_i$ for each $i = 1, 2, \ldots, n$. Let $1 \neq z = (z_1, z_2, \ldots, z_n)$ be an element of the commutator lattice $L = L_1 \times L_2 \times \cdots \times L_n$ with $1_{L_i}^2 \notin C_{z_i}(L_i)$ for $i = 1, 2$. Then $\mathbb{AG}_z(L)$ contains at least n S-vertices.*

Proof. Let $x_i = (0, 0, \ldots, 1, 0, \ldots, 0)$ be an n-tuple whose ith component is 1 and the other components are zero for each $1 \leq i \leq n$. Clearly, N_z is the set of all n-tuples whose at least one of its components; say the ith component is not less than or equal z_i for at least one $1 \leq i \leq n$. Now, the result follows directly from Proposition 5.1. □

In the following proposition, we provide some conditions for the existence of a S-vertex in the direct product of $n \geq 2$ commutator posets. We will also, in the next section, show the nonexistence of S-vertices in the product of two commutator posets (Theorem 6.1) in contrast to Parts (b) and (c) of the following proposition.

Proposition 5.4. *Let $z \neq 1$ be an element of the commutator poset $L \cong L_1 \times L_2 \times \cdots \times L_n$ be a direct product of $n \geq 2$ commutator posets. Suppose $1_{L_i} \neq z_i \in L_i$ for each $1 \leq i \leq n$ and $z = (z_1, z_2, \ldots, z_n)$. Then the following results are true:*

(a) *If $n \geq 3$, then $\mathbb{AG}_z(L)$ has at least n S-vertices provided that $1_{L_i}^2 \notin C_{z_i} L_i$ for each $1 \leq i \leq n-1$.*

(b) *For $n = 2$, suppose that one of L_1 or L_2; say L_2 has at least two nontrivial zero divisors $a \neq 1$ and $b \neq 1$ (not necessarily different) in L_2 with respect to z_2 such that $ab \leq z_2$. Then $(1,0)$ and $(0,a)$ are two S-vertices in $\mathbb{AG}_z(L)$ provided that $1 \cdot b \notin C_{z_2}(L_2)$.*

(c) *For $n = 2$, suppose that one of L_1 or L_2; say L_1 has at least two nontrivial zero divisors $a \neq 1$ and $b \neq 1$ (not necessarily different) in L_1 with respect to z_1 such that $ab \leq z_1$. Then $(0,1)$ and $(a,0)$ are two S-vertices in $\mathbb{AG}_z(L)$ provided that $1 \cdot b \notin C_{z_1}(L_1)$.*

Proof. Part (c) is left to the reader since its proof is similar to the proof of Part (b). (a) Let $S = \{x_1, x_2, \ldots, x_n\}$, where $x_i = (0, 0, \ldots, 1, 0, \ldots, 0)$ is an n-tuple whose ith component is 1 and the other components are zero for each $1 \leq i \leq n$ and let $x = (1, 1, 1, \ldots, 1, 0)$. Clearly, S is a clique of size n in $\mathbb{AG}_z(L)$. Now, by hypothesis, $(0, 0, \ldots, 0, 1)$ is the only vertex in S which makes a link with x and hence the proof follows directly from Lemma 2.3.

(b) By the hypothesis, we have the following path

$$(0,1) - (1,0) - (0,a) - (1,b)$$

which implies that $(1,0)$ and $(0,a)$ are S-vertices in $\mathbb{AG}_z(L)$. Note that, in this case, a can also be equal to b. □

We end this section with a remark and a theorem related to the (non)existence of S-vertices in $\mathbb{AG}_z(L)$ when z is the meet of a finite number of prime elements of a commutator lattice L.

Remark 5.5. In Theorem 6.7, it is shown that for any nonzero element z of a commutator lattice L with $z = p_1 \wedge p_2 \neq 0$, for some prime elements p_1 and p_2 of L, then $\mathbb{AG}_z(L)$ is a complete bipartite graph. Hence, $\mathbb{AG}_z(L)$ contains no S-vertices. But we next show that $\mathbb{AG}_z(L)$ contains $n \geq 3$ S-vertices when z is the meet of n prime elements of L.

Theorem 5.6. *Let $n \geq 3$ be a fixed integer and $z \neq 1$ an element of the commutator lattice L such that $z = \bigwedge_1^n p_i$, where p_i's are prime elements of L for each $1 \leq i \leq n$. Suppose for each $1 \leq j \leq n$, $\bigwedge_{1 \leq i \leq n,\ i \neq j} p_i \notin C_z$, $p_1 \notin C_{p_2}$, and $p_1 \notin C_{p_3}$. Then $\mathbb{AG}_z(L)$ contains n S-vertices.*

Proof. We assume $\mathbb{AG}_z(L)$ is nonempty and hence z is not a prime element of L by Remark 4.2(5) and hence $z \neq p_i$ for each $1 \leq i \leq n$. Actually, $p_i \notin C_z$ since by hypothesis, $z = \bigwedge_{i=1}^n p_i$ and hence $p_i \leq z \leq p_i$ implies $p_i = z$, yielding a contradiction. Consider $x_j = \bigwedge_{1 \leq i \leq n,\ i \neq j} p_i$. It is easy to see that

$$X = \{x_1, \ldots, x_n\}$$

is a clique in $\mathbb{AG}_z(L)$. Clearly, $p_1 \notin X$ since $p_1 = x_i$ implies $p_1 \leq p_2$ or p_3 for each $1 \leq i \leq n$, yielding a contradiction by hypothesis. We now show that p_1 is adjacent only to $x_1 \in X$. Clearly, p_1 is adjacent to x_1 since $p_1 x_1 \leq p_1 \wedge x_1 = z$. On the other hand, for any $2 \leq i \leq n$, $p_1 x_i \leq p_1 \wedge x_i = x_i \notin C_z$, by hypothesis, which implies that p_1 is not adjacent to x_i. Now the proof is immediate from Lemma 2.3. □

6 No S-vertices in $\mathbb{AG}_z(L)$ as a Complete Bipartite Graph

In this section we discuss a relationship between the girth, complete bipartiteness of $\mathbb{AG}_z(L)$ and prime elements of the commutator poset [lattice] L (Theorems 6.5, 6.6, and 6.7). Finally, we end the paper by defining a reduced commutator poset with respect to an element (Definition 6.9) and a result (Theorem 6.10) following a conjecture related to a universal vertex of $\mathbb{AG}_z(L)$, which implies nonexistence of S-vertices in $\mathbb{AG}_z(L)$.

Theorem 6.1. *For $n = 2$, let $1 \neq z \in L$, where $L = L_1 \times L_2$ is the direct product of two commutator posets [lattices] and $z = (z_1, z_2)$ with $1 \neq z_1 \in L_1$ and $1 \neq z_2 \in L_2$. Then $\mathbb{AG}_z(L)$ has no S-vertices provided that both L_1 and L_2 have no nontrivial zero divisors with respect to z_1 and z_2 (see Definition 3.5(c)), respectively.*

Proof. If each of L_1 and L_2 contains no nontrivial zero divisors with respect to z_1 and z_2, respectively, then $\mathbb{AG}_z(L)$ is a complete bipartite graph with parts $\{(a,0) | a \notin C_{z_1}(L_1)\}$ and $\{(0,b) | b \notin C_{z_2}(L_2)\}$ and hence has no S-vertices (Lemma 2.2(3), see also Example 4.4(b) when $z = 0$). □

Lemma 6.2. *Let $n \geq 3$ be a fixed integer and $L = L_1 \times L_2 \times \cdots, \times L_n$ be the direct product of n commutator posets with $1 \neq z \in L$ and $z = (z_1, z_2, \ldots, z_n)$, where $1 \neq z_i \in L_i$ for each $1 \leq i \leq n$. Then the girth of $\mathbb{AG}_z(L)$ is three.*

Proof. Clearly, an n-tuple is in $N_z(L)$ when at least one of its components; say the ith component does not belong to $C_{z_i}(L_i)$ for some $1 \leq i \leq n$ (see Example 4.3). Without loss of generality, assume $n = 3$. Now the proof follows from the fact that the set consisting of $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ is a clique of size 3 in $\mathbb{AG}_z(L)$. □

Theorem 6.3. *Let $n \geq 2$ be a fixed integer and $L = L_1 \times L_2 \times \cdots, \times L_n$ a direct product of n commutator posets such that 1_{L_i} is not a zero divisor with respect to z_i in L_i for each $1 \leq i \leq n$. Let $z = (z_1, z_2, \ldots, z_n)$, where $1 \neq z_i \in L_i$ for each $1 \leq i \leq n$. Then $\mathbb{AG}_z(L)$ is a complete bipartite graph with each part of size greater than or equal to 2 if and only if it contains no S-vertices and its girth is 4.*

Proof. Actually, $1 \notin C_{z_i}(L_i)$ since it is not a zero divisor with respect to z_i for each $1 \leq i \leq n$ by hypothesis. The necessary part is clear. For the sufficient part, by the above lemma, $n = 2$ since $n \geq 3$ implies that the girth of the graph to be equal to three, which is a contradiction by hypothesis. That is, L must be isomorphic to the product of two commutator posets. Now, without loss of generality, assume $L = L_1 \times L_2$ and L_2 has two nontrivial zero divisors with respect to z_2. Consequently, there exist $a, b \in N_{z_2}(L_2)$ (not necessarily distinct) with $a \neq 1$ and $b \neq 1$, by hypothesis, such that $ab \leq z_2$ since 1 is not a zero divisor in L_2 with respect to z_2. Clearly, a and b are in $N_{z_2}(L_2)$. Otherwise, for instance, $b \leq z_2$ implies that 1 is a zero divisor with respect to z_2 since $1 \cdot b \leq b \leq z_2$, yielding a contradiction. Therefore,

$$(0,1) - (1,0) - (0,a) - (1,b)$$

is a path in $\mathbb{AG}_z(L)$, yielding a contradiction since $\mathbb{AG}_z(L)$ contains no S-vertices by hypothesis. Thus, each of L_1 and L_2 has no nontrivial zero divisors with respect to z_1 and z_2, respectively, which implies $\mathbb{AG}_z(L)$ is complete bipartite by Theorem 6.1 and the cardinality of each is greater than or equal to 3 since $\mathrm{gr}(\mathbb{AG}_z(L)) = 4$. □

Lemma 6.4. *Let a and z be two elements of a commutator poset [lattice] L. Let $(z : a)$ be the set of all elements $r \in L$ such that $ra \leq z$. Then $(z : a)$ is an ideal of L.*

Proof. Let $r \in L$ and $s \in (z : a)$. Clearly, $sa \leq z$ and $rs \leq s$. Hence $(rs)a \leq sa \leq z$ implies $rs \in (z : a)$. □

We now provide a relation between the girth and nonexistence of S-vertices of $\mathbb{AG}_z(L)$.

Theorem 6.5. *Let $z \neq 1$ be an element in a commutator poset L. Then the following statements are true for L.*

(1) *Let $\mathrm{gr}(\mathbb{AG}_z(L)) = 4$ and $a - b - c - d - a$ be a cycle in $\mathbb{AG}_z(L)$ such that $a^2 \notin C_z$. Then $\mathbb{AG}_z(L)$ is complete bipartite when it has no S-vertices.*

(2) *If $\mathbb{AG}_z(L)$ is complete bipartite, then $\mathbb{AG}_z(L)$ has no S-vertices with $\mathrm{gr}(\mathbb{AG}_z(L)) = 4$ or ∞.*

Proof. We just give a proof for Part (1) since the other part is obvious. Clearly, $\mathrm{diam}(\mathbb{AG}_z(L)) \neq 3$ since $\mathbb{AG}_z(L)$ has no S-vertices. If $\mathrm{diam}(\mathbb{AG}_z(L)) = 0$ or 1, then $\mathbb{AG}_z(L)$ is a complete graph and so $\mathrm{gr}(\mathbb{AG}_z(L))$ is 3 or ∞, yielding a contradiction. Therefore, $\mathrm{diam}(\mathbb{AG}_z(L)) = 2$. We now show that $\mathbb{AG}_z(L)$ is a complete bipartite graph. Since $\mathrm{gr}(\mathbb{AG}_z(L)) = 4$, there exist $a, b, c, d \in \mathbb{AG}_z(L)$ such that $a-b-c-d-a$ is a cycle with $a^2 \notin C_z$ by hypothesis. We show that $\mathbb{AG}_z(L) \cong K_{|V_1|,|V_2|}$, where $V_1 = \{t \in N_z(L) : t \in (z : a)\}$ and $V_2 = \{s \in N_z(L) : s \notin (z : a)\}$. Let $t, t_1 \in V_1$ and $s, s_1 \in V_2$. Then $at \leq z$ and $as \notin C_z$. Assume that $ts \notin C_z$. Since $\mathrm{diam}(\mathbb{AG}_z(L)) = 2$, there exists $h \in N_z(L)$ such that $a - h - s$. Clearly, $ts \notin C_z$ implies that t is not less than or equal to h. If $th \leq z$, then $\mathrm{gr}(\mathbb{AG}_z(L)) = 3$, yielding a contradiction. Also t is not less than or equal to s since $th \notin C_z$. Thus a is an S-vertex in $\mathbb{AG}_z(L)$ which is a contradiction. Therefore $ts \leq z$. If $tt_1 \leq z$, then $a-t-t_1-a$ is a cycle, yielding a contradiction. So, $tt_1 \notin C_z$. Similarly $ss_1 \notin C_z$, otherwise, $t - s - s_1 - t$ is a cycle of length 3, yielding a contradiction. Also $V_1 \cap V_2 = \emptyset$. Therefore, $\mathbb{AG}_z(L) \cong K_{|V_1|,|V_2|}$ and so $\mathbb{AG}_z(L)$ is a complete bipartite graph. □

Theorem 6.6. (cf. [25, Theorem 6.1]) *Let $z \neq 1$ be a nonzero element of a commutator poset L. Let p_1 and p_2 be two prime elements of L such that $z = inf\{p_1, p_2\}$. Then $\mathbb{AG}_z(L)$ is a complete bipartite graph and hence has no S-vertices.*

Proof. See Remark 4.2(3) and Lemma 2.2(3). □

Theorem 6.7. (cf. [25, Theorem 6.2]) *Let $z \neq 1$ be a nonzero element of a commutator lattice L. Let p_1 and p_2 be two prime elements of L such that $z = p_1 \wedge p_2$. Then $\mathbb{AG}_z(L)$ is a complete bipartite graph and hence has no S-vertices.*

Proof. See Remark 4.2(4) and Lemma 2.2(3). □

Remark 6.8. In Example 6.3 of [25], it is shown that the primeness condition in the above theorem is not a superfluous assumption.

Definition 6.9. Let $z \neq 1$ be an element of a commutator poset L. A commutator poset L is said to be reduced with respect to z if and only if the square of no elements of $N_z(L)$ is less than or equal to z. In other words, the square of no elements of $N_z(L)$ is in $C_z(L)$.

Consequently, from the above definition, there is no loop in $\mathbb{AG}_z(L)$ when L is a reduced commutator poset with respect to z. Finally, we end the paper with a result and a conjecture related to a universal vertex of $\mathbb{AG}_z(L)$, which implies nonexistence of S-vertices in $\mathbb{AG}_z(L)$.

Theorem 6.10. *Let $z \neq 1$ be an element of a commutator poset L which is reduced with respect to z. Suppose that $\mathbb{AG}_z(L)$ is nonempty with $\mathrm{gr}(\mathbb{AG}_z(L)) = \infty$. Then there is a vertex in $\mathbb{AG}_z(L)$ which is adjacent to every vertex of $\mathbb{AG}_z(L)$.*

Proof. Note that $\mathbb{AG}_z(L)$ has more than one vertex since it is nonempty and L is reduced with respect to z by hypothesis and hence has no loops. Now, suppose to the contrary that there is not a vertex in $\mathbb{AG}_z(L)$ which is adjacent to every vertex of $\mathbb{AG}_z(L)$. Therefore, there exist distinct vertices a_1, a_2, a_3, a_4 such that $a_1 - a_2 - a_3 - a_4$, $a_1a_3 \notin C_z$, and $a_2a_4 \notin C_z$. If $a_1a_4 \leq z$, then $a_1 - a_2 - a_3 - a_4 - a_1$ is a cycle, contrary to $\mathrm{gr}(\mathbb{AG}_z(L)) = \infty$. So we may assume that $a_1a_4 \notin C_z$. Therefore, $(a_1a_4)a_2 \leq a_1a_2 \leq z$ and $(a_1a_4)a_3 \leq a_4a_3 \leq z$ implies $a_2 - a_3 - a_1a_4 - a_2$ is a cycle, contrary to $\mathrm{gr}(\mathbb{AG}_z(L)) = \infty$ (note that $a_1a_4 \neq a_2, a_3$ since L is reduced with respect to z). Thus, there exists a vertex in $\mathbb{AG}_z(L)$ which is adjacent to every vertex of $\mathbb{AG}_z(L)$. □

Conjecture: Let $z \neq 1$ be an element of a commutator poset L which is reduced with respect to z. Suppose that there exists a vertex in $\mathbb{AG}_z(L)$ which is adjacent to every vertex of $\mathbb{AG}_z(L)$. Then $\mathbb{AG}_z(L)$ is a star graph and hence has no S-vertices.

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