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Some results on complex Tricomi functions

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Abstract The current study explores certain extensions of the Tricomi functions analytically. The core objective of this research is to demonstrate how the generating function can be used effectively to produce specific results for special functions. The generating function introduces the complex version of Tricomi functions. In light of the generating function technique, several intriguing properties of complex Tricomi functions are examined. By evaluating the real and imaginary components of the complex Tricomi functions individually, the cosine-Tricomi functions and sine-Tricomi functions are obtained. Furthermore, the generating functions of degenerate Tricomi functions are examined, and a new identity for these functions is established. Finally, using a hypergeometric technique, the explicit forms of the real and imaginary parts of the complex Tricomi functions are also determined.

1 Introduction and preliminaries

The subject special functions is crucial to mathematical physics formalism. Among all the special functions, Bessel functions are among the most essential special functions, having many applications in engineering, physics and mathematical analysis spanning from theoretical astrophysics to quantum mechanics and abstract number theory. The theory of Bessel functions is vast and diverse, and it undoubtedly provides an endless area of study.

Tricomi functions, also known as confluent hypergeometric functions of the second kind, are mathematical functions that arise in a variety of physical and engineering problems. They are solutions to a second-order linear differential equation known as the Tricomi equation, which appears in fluid mechanics, plasma physics, and quantum mechanics. Tricomi functions have properties that make them useful in the study of asymptotic expansions and boundary value problems, and they can be expressed in terms of other special functions such as Bessel functions and Laguerre polynomials. Despite their utility, the computation of Tricomi functions can be challenging due to their complex nature and the need for numerical approximations in many cases.

Tricomi functions $C_n(x)$ are defined by means of the generating relation

$$\exp\left(t-\frac{x}{t}\right)=\sum_{n=-\infty}^{\infty}C_n(x)t^n,\ \ t\neq 0,\ \ \forall x. \tag{1.1}$$

Tricomi functions $C_n(x)$ are also defined by

$$C_n(x)\ =\ \sum_{k=0}^{\infty}\frac{(-1)^k\,x^k}{k!\,(n+k)!},\quad n=0,1,2,\ldots. \tag{1.2}$$

Further, Tricomi functions are characterized by the following link with the ordinary Bessel functions $J_n(x)$:

$$C_n(x)\ =\ x^{-\frac{n}{2}}J_n(2\sqrt{x}). \tag{1.3}$$

Tricomi functions satisfy the following pure and differential recurrence relations

$$x\,C_{n+1}(x)-nC_n(x)+C_{n-1}(x)\ =\ 0 \tag{1.4}$$

and

$$\frac{d}{dx}\,C_n(x) \;=\; -C_{n+1}(x), \tag{1.5}$$

$$x\,\frac{d}{dx}C_n(x) \;=\; -nC_n(x) + C_{n-1}(x) \tag{1.6}$$

respectively.

Tricomi functions are solutions of the differential equation

$$\left(x\,\frac{d^2}{dx^2} + (n+1)\frac{d}{dx} + 1\right) C_n(x) \;=\; 0. \tag{1.7}$$

We note that

$$C_n(x) \;=\; \frac{1}{\Gamma(1+n)}\,{}_0F_1[-;\,n+1;\,-x]. \tag{1.8}$$

The investigation of the properties of multivariable generalised special functions has provided new analytical tools for deriving solutions to broad classes of partial differential equations that are frequently encountered in physical concerns. Certain properties of the extended generalized Bessel function were studied in [11]. Also, the authors in [13] investigated a certain subclass of analytic functions defined by Gegenbauer polynomials.

To emphasise the significance of generalised special functions, two variable Tricomi functions are defined by the generating function [1]:

$$\exp\left(t - \frac{x}{t} + t^2 - \frac{y}{t^2}\right) = \sum_{n=-\infty}^{\infty} C_n(x,y)t^n. \tag{1.9}$$

The properties of mixed families of special functions related to the Tricomi functions have been studied in [8, 12]. The complex form of certain special polynomials and functions have been investigated by many authors as an extension of real-valued polynomials (functions), see [6, 5, 10, 4]. By considering the real and imaginary parts of the complex special functions separately, the new special functions have been obtained as cosine and sine versions. Several novel identities for the cosine and sine functions are established. Moreover, the generating function of the degenerate forms are investigated and certain novel identities for these functions are obtained. Inspired by the cited publications, in this article we introduce the complex version of Tricomi functions.

Section 2 introduces the complex Tricomi functions and investigates their properties. In addition, the generating functions of cosine-Tricomi functions and sine-Tricomi functions are introduced. In Section 3, the generating equation is used to introduce degenerate Tricomi functions, and some of their features are studied. Section 4 obtains explicit representations of the real and complex components of complex Tricomi functions using a hypergeometric technique.

2 Complex Tricomi Functions

D. Kim [6, 5] introduced complex Appell and Sheffer polynomials and studied their properties. Further, Nahid and Ali studied complex Bessel functions [10]. Recently, Khan et.al. [4] proposed a family of complex Appell-Bessel functions and certain properties are explored. We introduce the complex Tricomi functions by means of the following generating function:

$$\exp\left(t - \frac{z}{t}\right) = \sum_{n=-\infty}^{\infty} C_n(z)t^n, \qquad z \in \mathbb{C}. \tag{2.1}$$

Since $z = x + iy$, $x, y \in \mathbb{R}$, we have

$$\exp\left(t - \frac{x}{t}\right)\left(\cos\left(\frac{y}{t}\right) - i\sin\left(\frac{y}{t}\right)\right) = \sum_{n=-\infty}^{\infty} C_n(z)t^n \tag{2.2}$$

$$\exp\left(t-\frac{x}{t}\right)\left(\cos\left(\frac{y}{t}\right)+i\sin\left(\frac{y}{t}\right)\right)=\sum_{n=-\infty}^{\infty}C_n(\bar{z})t^n. \tag{2.3}$$

In view of equations (2.2) and (2.3), it follows that

$$\exp\left(t-\frac{x}{t}\right)\cos\left(\frac{y}{t}\right)=\sum_{n=-\infty}^{\infty}\left(\frac{C_n(z)+C_n(\bar{z})}{2}\right)t^n \tag{2.4}$$

$$-\exp\left(t-\frac{x}{t}\right)\sin\left(\frac{y}{t}\right)=\sum_{n=-\infty}^{\infty}\left(\frac{C_n(z)-C_n(\bar{z})}{2i}\right)t^n. \tag{2.5}$$

Therefore, we define the cosine-Tricomi functions, denoted by $C_n^{(c)}(x,y)$ and sine-Tricomi functions, denoted by $C_n^{(s)}(x,y)$ by the following generating functions:

$$\exp\left(t-\frac{x}{t}\right)\cos\left(\frac{y}{t}\right)=\sum_{n=-\infty}^{\infty}C_n^{(c)}(x,y)t^n \tag{2.6}$$

and

$$-\exp\left(t-\frac{x}{t}\right)\sin\left(\frac{y}{t}\right)=\sum_{n=-\infty}^{\infty}C_n^{(s)}(x,y)t^n, \tag{2.7}$$

respectively.

These cosine-Tricomi functions $C_n^{(c)}(x,y)$ and sine-Tricomi functions $C_n^{(s)}(u,v)$ satisfy the following properties:

(**i**) $C_n^{(c)}(x,-y)=C_n^{(c)}(x,y)$,

(**ii**) $C_n^{(s)}(x,-y)=-C_n^{(s)}(x,y)$,

(**iii**) $C_n^{(c)}(x,0)=C_n^{(c)}(x)$,

(**iv**) $C_n^{(s)}(x,0)=0$.

Differentiating generating function (2.6) partially w.r.t. x and y respectively, the following recurrence relations for the cosine-Tricomi functions $C_n^{(c)}(x,y)$ are obtained:

$$\frac{\partial}{\partial x}C_n^{(c)}(x,y)=-C_{n+1}^{(c)}(x,y) \tag{2.8}$$

$$\frac{\partial}{\partial y}C_n^{(c)}(x,y)=C_{n+1}^{(s)}(x,y). \tag{2.9}$$

Similarly, we have the following recurrence relations for the sine-Tricomi functions $C_n^{(s)}(x,y)$:

$$\frac{\partial}{\partial x}C_n^{(s)}(x,y)=C_{n+1}^{(s)}(x,y) \tag{2.10}$$

$$\frac{\partial}{\partial y}C_n^{(s)}(x,y)=-C_{n+1}^{(c)}(x,y). \tag{2.11}$$

In view of equations (2.4) and (2.6), we get the real part of complex Tricomi functions $C_n(z)$ as:

$$C_n^{(c)}(x,y)=\frac{C_n(z)+C_n(\bar{z})}{2}=\Re(C_n(z)). \tag{2.12}$$

Similarly, we have the imaginary part of complex Tricomi functions $C_n(z)$:

$$C_n^{(s)}(x,y)=\frac{C_n(z)-C_n(\bar{z})}{2i}=\Im(C_n(z)). \tag{2.13}$$

Therefore, the complex Tricomi functions $C_n(z)$ is defined as:

$$C_n(z) = C_n^{(c)}(x,y) + iC_n^{(s)}(x,y) \tag{2.14}$$

and its conjugate is given by:

$$C_n(\bar{z}) = C_n^{(c)}(x,y) - iC_n^{(s)}(x,y). \tag{2.15}$$

Theorem 2.1. *For the complex Tricomi functions $C_n(z)$, the following differential recurrence relation holds true:*

$$\frac{d}{dz}C_n(z) = -2C_{n+1}^{(c)}(x,y), \quad z = x+iy. \tag{2.16}$$

Proof. Differentiating relation (2.14) w.r.t. z, we get

$$\frac{d}{dz}C_n(z) = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(C_n^{(c)}(x,y) + iC_n^{(s)}(x,y)\right). \tag{2.17}$$

Using relations (2.8)-(2.11) in equation (2.17) and after some simplification, we get assertion (2.16). □

Theorem 2.2. *For $n \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$, the following identity for the complex Tricomi functions $C_n(z)$ holds true:*

$$C_n(z)C_n(\bar{z}) = \left(C_n^{(c)}(x,y)\right)^2 + \left(C_n^{(s)}(x,y)\right)^2. \tag{2.18}$$

Proof. From equations (2.14) and (2.15), we get assertion (2.18).

□

In the next section, we give the generating function of the degenerate Tricomi functions and established certain properties satisfied by these functions.

3 Degenerate Tricomi functions

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined as:

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{3.1}$$

It should be noted that $\lim_{\lambda\to 0} e_\lambda^x(t)$ reduce to the classical exponential function e^{xt}.

The degenerate versions of the familiar polynomials like Bernoulli, Euler, falling factorial and Bell polynomials are studied by many researchers using generating functions, umbral calculus, and p-adic integrals, see for example [2, 3, 7, 9]. Fascinated by investigations concerning degenerate forms of special polynomials, we define the 1-variable degenerate Tricomi function $\mathcal{C}_n(x;\lambda)$ as follows:

$$(1+\lambda t)^{\frac{1}{\lambda}}\left(1+\frac{\lambda}{t}\right)^{-\frac{x}{\lambda}} = \sum_{n=-\infty}^{\infty} \mathcal{C}_n(x;\lambda)t^n. \tag{3.2}$$

Similarly, the 2-variable degenerate Tricomi functions $\mathcal{C}_n(x,y;\lambda)$ are defined by using the following generating function:

$$(1+\lambda t)^{\frac{1}{\lambda}}(1+\lambda t^2)^{\frac{1}{\lambda}}\left(1+\frac{\lambda}{t}\right)^{-\frac{x}{\lambda}}\left(1+\frac{\lambda}{t^2}\right)^{-\frac{y}{\lambda}} = \sum_{n=-\infty}^{\infty} \mathcal{C}_n(x,y;\lambda)t^n. \tag{3.3}$$

Remark 3.1. Taking $\lambda \to 0$ in equation (3.2) and (3.3), we get the generating functions of 1-variable Tricomi functions $C_n(x)$ and 2-variable Tricomi functions $C_n(x,y)$ defined by (1.1) and (1.9), respectively.

Theorem 3.2. *For the 2-variable degenerate Tricomi functions $\mathcal{C}_n(x,y;\lambda)$, the following relation holds true:*

$$\mathcal{C}_n(x,y;\lambda) = \sum_{l=-\infty}^{\infty} \mathcal{C}_{n-2l}(x;\lambda)\mathcal{C}_l(y;\lambda). \tag{3.4}$$

Proof. Using generating function (3.2) in equation (3.3), we get

$$\sum_{n=-\infty}^{\infty} \mathcal{C}_n(x,y;\lambda)t^n = \sum_{n=-\infty}^{\infty} \mathcal{C}_n(x;\lambda)t^n \sum_{l=-\infty}^{\infty} \mathcal{C}_l(y;\lambda)t^{2l}, \tag{3.5}$$

which on equating the coefficients of the identical powers of t in both sides, yields assertion (3.4).

□

4 Explicit expressions

A strong mathematical technique for approximating complex functions as a sum of simpler functions is series expansion. Bessel functions, hypergeometric functions, and Legendre functions are fundamental mathematical tools utilised in many sectors of science and engineering. These functions frequently exhibit varied and complicated behaviour, making it difficult to compute their values for various input parameters. Series expansion is a practical way for approximating these functions as a series of simpler functions that may be easily evaluated, such as polynomials or trigonometric functions. The resulting series can then be used to compute the function values to the necessary level of accuracy, making series expansion a useful tool for practical applications of special functions.

The generalized hypergeometric functions and their extensions in the form of basic (or $q-$) hypergeometric functions, elliptic hypergeometric functions and multiple hypergeometric functions are ubiquitous. One characteristic of these functions is a tendency to appear in a variety of mathematical and physical circumstances. A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$${}_pF_q\left[\begin{array}{ll}(\alpha_p);& \\ & z\\ (\beta_q);&\end{array}\right] = {}_pF_q\left[\begin{array}{ll}\alpha_1,\alpha_2,\ldots,\alpha_p;& \\ & z\\ \beta_1,\beta_2,\ldots,\beta_q;&\end{array}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\,\frac{z^n}{n!} \tag{4.1}$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$\beta_j \neq 0,-1,-2,\ldots\ ;\ \ j=1,2,\ldots,q.$$

In contracted notation, the sequence of p numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ is denoted by (α_p). The Pochammer symbol $(\alpha)_n$, where α denotes any number, real or complex and n any integer positive, negative or zero, is defined by

$$(\alpha)_n = \begin{cases} 1, & \text{if } n=0,\\ \alpha(\alpha+1)\cdots(\alpha+n-1), & \text{if } n\geq 1,\end{cases}$$

and

$$(\alpha)_{-n} = \frac{(-1)^n}{(1-\alpha)_n},\ \ n=1,2,3,\ldots;\ \ \alpha\neq 0,\pm1,\pm2,\cdots.$$

The hypergeometric function ${}_pF_q$ form the core of the special functions and include as special cases most of the commonly used functions. Some of the important functions in terms of ${}_pF_q$ are as follows:

$$\exp(z) = {}_0F_0\left[\begin{matrix} - & ; \\ & z \\ - & ; \end{matrix}\right], \quad \cos z = {}_0F_1\left[\begin{matrix} - & ; \\ & -\frac{z^2}{4} \\ \frac{1}{2} & ; \end{matrix}\right], \quad \sin z = z\,{}_0F_1\left[\begin{matrix} - & ; \\ & -\frac{z^2}{4} \\ \frac{3}{2} & ; \end{matrix}\right]$$

$$(1-z)^{-\alpha} = {}_1F_0\left[\begin{matrix} \alpha & ; \\ & z \\ - & ; \end{matrix}\right], \quad |z| < 1.$$

In 2021, Qureshi et al. obtained certain explicit expressions of hybrid special functions related to the Tricomi functions. The following result has been proved [12]:
For $S \leq S'+1$, $T \leq T'+1$, $U \leq U'+1$, $W \leq W'+1$ and $t \neq 0$, the following generating function (in terms of the product of four generalized hypergeometric functions of one variable) for the Srivastava-Daoust double hypergeometric function holds true:

$${}_SF_{S'}\left[\begin{matrix} (s_S) & ; \\ & \frac{\alpha x}{t^m} \\ (s'_{S'}) & ; \end{matrix}\right] {}_TF_{T'}\left[\begin{matrix} (t_T) & ; \\ & \frac{\beta y}{t} \\ (t'_{T'}) & ; \end{matrix}\right] {}_UF_{U'}\left[\begin{matrix} (u_{U'}) & ; \\ & \mu u t \\ (u'_{U'}) & ; \end{matrix}\right] {}_WF_{W'}\left[\begin{matrix} (w_W) & ; \\ & \frac{\lambda z}{t^r} \\ (w'_{W'}) & ; \end{matrix}\right]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{(\beta y)^{j+n^\star} (\mu u)^{j+n^\star+n} \prod_{i=1}^{T} (t_i)_{j+n^\star} \prod_{i=1}^{U} (u_i)_{j+n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)! \prod_{i=1}^{T'} (t'_i)_{j+n^\star} \prod_{i=1}^{U'} (u'_i)_{j+n^\star+n}}$$

$$\times \sum_{\ell,s=0}^{m\ell+rs \leq j+n^\star} (-j-n^\star)_{m\ell+rs} \frac{\prod_{i=1}^{T'} (1-t'_i-j-n^\star)_{m\ell+rs} \prod_{i=1}^{S} (s_i)_\ell \prod_{i=1}^{W} (w_i)_s}{\prod_{i=1}^{T} (1-t_i-j-n^\star)_{m\ell+rs} \prod_{i=1}^{S'} (s'_i)_\ell \prod_{i=1}^{W'} (w'_i)_s}$$

$$\times \frac{\left(\frac{\alpha x}{(\beta y)^m}\right)^\ell}{\ell!} \frac{\left(\frac{\lambda z}{(\beta y)^r}\right)^s}{s!}\, t^n, \tag{4.2}$$

where m, r are positive integers and $n^\star$ is defined as:

$$n^\star = \max\{0, -n\} = \begin{cases} -n, & \text{when } n = \cdots, -3, -2, -1 \\ 0, & \text{when } n = 0, 1, 2, \cdots. \end{cases} \tag{4.3}$$

Special functions are usually shown to be expressible as series, because this is frequently the most practical way to obtain numerical values for the functions. By using the Laurent type hypergeometric generating relation (4.2), the explicit representations of the real and imaginary parts of the complex Tricomi functions are derived as follows:

I. Taking $U = U' = 0$, $\mu = u = 1$, $T = T' = 0$, $\beta = -1$, $y \to x$, $S = 0$, $S' = 1$, $s'_1 = \frac{1}{2}$, $x \to y^2$, $m = 2$, $\alpha = -\frac{1}{4}$ and $z = 0$ in equation (4.2), we get

$${}_0F_0\left[\begin{matrix} - & ; \\ & t \\ - & ; \end{matrix}\right] {}_0F_0\left[\begin{matrix} - & ; \\ & -\frac{x}{t} \\ - & ; \end{matrix}\right] {}_0F_1\left[\begin{matrix} - & ; \\ & -\frac{1}{4}\left(\frac{y}{t}\right)^2 \\ \frac{1}{2} & ; \end{matrix}\right]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{(-x)^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!} \sum_{\ell=0}^{2\ell \leq j+n^\star} \frac{(-j-n^\star)_{2\ell}}{\left(\frac{1}{2}\right)_\ell} \frac{\left(-\frac{y^2}{4x^2}\right)^\ell}{\ell!} t^n. \tag{4.4}$$

Using Legendre's duplication formula in r.h.s. of above equation, we get:

$$\exp\left(t-\frac{x}{t}\right)\cos\left(\frac{y}{t}\right) = \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-x)^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!} \times {}_2F_1\left[\begin{matrix}\frac{-j-n^\star}{2}, & \frac{-j-n^\star+1}{2} \\ \frac{1}{2} & \end{matrix}\;;\; -\frac{y^2}{x^2}\right] t^n. \tag{4.5}$$

Comparing equation (4.5) with equation (2.6), the following explicit representation of the cosine-Tricomi functions $C_n^{(c)}(x,y)$ is obtained:

$$C_n^{(c)}(x,y) = \sum_{j=0}^{\infty}\frac{(-x)^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!}\,{}_2F_1\left[\begin{matrix}\frac{-j-n^\star}{2}, & \frac{-j-n^\star+1}{2} \\ \frac{1}{2} & \end{matrix}\;;\; -\frac{y^2}{x^2}\right]. \tag{4.6}$$

II. Multiplying by $\frac{1}{2i}$ and taking $U = U' = 0$, $\mu = u = 1$, $T = T' = 0$, $\beta = -1$, $y \to x$, $S = S' = 0$, $x \to y^2$, $m = 1$, $\alpha = i$ $(i^2 = -1)$ and $z = 0$ in equation (4.2), we get

$$\frac{1}{2i}\exp\left(t-\frac{x}{t}\right)\exp\left(\frac{iy}{2}\right) = \frac{1}{2i}\sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-x)^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!} \times {}_1F_0\left[\begin{matrix}-j-n^\star & \\ — & \end{matrix}\;;\; -\frac{iy}{x}\right] t^n. \tag{4.7}$$

Replacing y by $-y$ in equation (4.7), we get

$$\frac{1}{2i}\exp\left(t-\frac{x}{t}\right)\exp\left(-\frac{iy}{2}\right) = \frac{1}{2i}\sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-x)^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!} \times {}_1F_0\left[\begin{matrix}-j-n^\star & \\ — & \end{matrix}\;;\; \frac{iy}{x}\right] t^n. \tag{4.8}$$

Subtracting equation (4.8) from equation (4.7), the following generating relation is obtained:

$$\exp\left(t-\frac{x}{t}\right)\sin\left(\frac{y}{t}\right) = \frac{1}{2i}\sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-x)^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!} \times \left({}_1F_0\left[\begin{matrix}-j-n^\star & \\ — & \end{matrix}\;;\; -\frac{iy}{x}\right] - {}_1F_0\left[\begin{matrix}-j-n^\star & \\ — & \end{matrix}\;;\; \frac{iy}{x}\right]\right) t^n. \tag{4.9}$$

Comparing equation (4.9) with equation (2.7), the following explicit representation of the

sine-Tricomi functions $C_n^{(s)}(x,y)$ is obtained:

$$C_n^{(s)}(x,y) = \frac{1}{2i}\sum_{j=0}^{\infty}\frac{(-x)^{j+n^\star}}{(j+n^\star)!\,(j+n^\star+n)!} \times \left({}_1F_0\left[\begin{array}{ccc} -j-n^\star & ; & \\ & & -\frac{iy}{x} \\ — & ; & \end{array}\right] - {}_1F_0\left[\begin{array}{ccc} -j-n^\star & ; & \\ & & \frac{iy}{x} \\ — & ; & \end{array}\right]\right). \tag{4.10}$$

III. Taking $x=z=0$, $U=1$, $U'=0$, $u_1=-\frac{1}{\lambda}$, $\mu=-1$, $u=\lambda$, $T=1$, $T'=0$, $t_1=\frac{x}{\lambda}$, $\beta=-1$, $y=\lambda$ in equation (4.2), we get

$$\begin{aligned} &{}_1F_0\left[\begin{array}{ccc} -\frac{1}{\lambda} & ; & \\ & & -\lambda t \\ — & ; & \end{array}\right] {}_1F_0\left[\begin{array}{ccc} \frac{x}{\lambda} & ; & \\ & & -\frac{\lambda}{t} \\ — & ; & \end{array}\right] \\ &= \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-\lambda)^{2j+2n^\star+n}\left(\frac{x}{\lambda}\right)_{j+n^\star}\left(-\frac{1}{\lambda}\right)_{j+n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)!}\,t^n \end{aligned} \tag{4.11}$$

or

$$(1+\lambda t)^{\frac{1}{\lambda}}\left(1+\frac{\lambda}{t}\right)^{-\frac{x}{\lambda}} = \sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(-\lambda)^{2j+2n^\star+n}\left(\frac{x}{\lambda}\right)_{j+n^\star}\left(-\frac{1}{\lambda}\right)_{j+n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)!}\,t^n. \tag{4.12}$$

Comparing equation (4.12) with equation (3.2), the following explicit representation of the 1-variable degenerate Tricomi functions $\mathcal{C}_n(x;\lambda)$ is obtained:

$$\mathcal{C}_n(x;\lambda) = \sum_{j=0}^{\infty}\frac{(-\lambda)^{2j+2n^\star+n}\left(\frac{x}{\lambda}\right)_{j+n^\star}\left(-\frac{1}{\lambda}\right)_{j+n^\star+n}}{(j+n^\star)!\,(j+n^\star+n)!}. \tag{4.13}$$

5 Conclusions

In this work, we explore certain extensions of the Tricomi functions analytically. We have demonstrated how the generating function may be used effectively to produce specific results for special functions. By evaluating the real and imaginary components of the complex Tricomi functions individually, the cosine-Tricomi functions and sine-Tricomi functions are obtained. Furthermore, the generating functions of degenerate Tricomi functions are examined, and a new identity for these functions is established. Finally, using a hypergeometric technique, the explicit forms of the real and imaginary parts of the complex Tricomi have been determined. Tricomi function appears in the solution of the Schrödinger equation for a hydrogen atom in a strong magnetic field, as well as in the analysis of the flow of an ideal fluid past a wedge. Additionally, Tricomi functions have been used in the modeling of wave propagation in heterogeneous media, such as the atmosphere or the ocean, as well as in the design of antennas and other electromagnetic devices. The versatility and broad range of applications of Tricomi functions make them a valuable tool in many areas of science and engineering. The properties satisfied by the newly introduced relatives of Tricomi functions can be utilised to address present and emerging challenges in mathematical physics.

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