Palestine Journal of Mathematics

Vol.15(3), (2026) , 1259–1271 © Palestine Polytechnic University-PPU 2026

Commuting Graph of Affine Group Aff$(1,\ \mathbb{Z}_p)$

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Communicated by Ayman Badawi

MSC 2020 Classifications: Primary 05C25; Secondary 05C07, 05C12, 05C50.

Keywords and phrases: Affine Group, Commuting Graph, Detour Distance, Metric Dimension, Resolving Polynomial, Signless Laplacian.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper. The second author (corresponding author) is thankful to the CSIR, India, for the Senior Research Fellowship.

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Abstract. The commuting graph $\Gamma = \mathcal{C}(G,\ \Omega)$ on a non-abelian group G consist of a vertex set $\Omega \subseteq G$ and any two vertices f and g in Ω are adjacent if and only if $fg = gf$ in G. In this article, we have studied certain distant and detour distant properties of commuting graph Γ on one dimensional affine group $\mathrm{Aff}(1, \mathbb{Z}_p)$, p an odd prime. We have also investigated the clique number, chromatic number and independence number of Γ. Further, we determine the metric dimension, resolving polynomial and spectral properties of Γ.

1 Introduction

A graph $\Gamma = (V(\Gamma),\ E(\Gamma))$ in this paper is formed up of a set of vertices $V(\Gamma)$ and an edge set $E(\Gamma)$. We further take into consideration basic undirected graphs without loops and numerous edges. The order of Γ is the cardinality of the set $V(\Gamma)$. If two vertices f and g are adjacent, then we denote it by $f \sim g$ and if they are non-adjacent, then we denote it by $f \not\sim g$. For more definitions and notations on graph theory one can see ([8], [9]).

The investigation of detour distance, metric dimension, resolving polynomial etc., in commuting graphs, becomes interesting as of their valuable applications. Chartrand *et al.* [6] developed the concept of detour distance, which has applications in frequency modulation channel assignment. The concept of metric dimension was first presented by Slatter [28] and subsequently examined independently by Harary *et al.* in [18]. The metric dimension of a graph has various applications in pharmaceutical chemistry ([5], [7]), robot navigation [22], sonar and Coast Guard LORAN [28], to name a few. The distant and detour distant properties of the commuting graph on the dihedral group D_{2n} were studied by Faisal *et al.* [3]. Further, they determined the metric dimension of the commuting graph on D_{2n} and its resolving polynomial. Siddharth *et al.* [24] studied the detour distance properties, resolving polynomial and spectral properties of the commuting graph of non-abelian groups of order p^4 with center having p elements. Fasfous and Nath [17] and Jannat and Nath [20] studied spectral properties and energies of commuting graphs using common neighborhood matrices.

Motivated with the work of ([3], [21], [23]) in this paper we have investigated some graph invariants for the commuting graph Γ of one dimensional affine group $\mathrm{Aff}(1, \mathbb{Z}_p)$, p an odd prime.

In this paper, we organize our work in the following manner. In the Section 2, we developed a necessary material and obtained the commuting graph Γ of one dimensional affine group $\mathrm{Aff}(1, \mathbb{Z}_p)$, p an odd prime. In Section 3, we have studied the detour distant properties of Γ. In Section 4, we determine interior, eccentric subgraph, closure, clique number, chromatic number, and independence number of Γ etc. In Section 5, we study the metric dimension and determine the resolving polynomial of commuting graph on $\mathrm{Aff}(1, \mathbb{Z}_p)$, p an odd prime. In section 6, we study the spectral properties commuting graph on $\mathrm{Aff}(1, \mathbb{Z}_p)$, p an odd prime. Throughout the article G will denote the one dimensional affine group $\mathrm{Aff}(1,\ \mathbb{Z}_p)$ of $\mathbb{Z}_p$, p an odd prime.

2 Commuting Graphs on Aff$(1, \mathbb{Z}_p)$

Let $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ be the group of integers modulo p, p an odd prime. Let G = Aff$(1,\ \mathbb{Z}_p)$ = $\{f_{a,b} : f_{a,b}(x) = ax + b, a \in U_p, b \in \mathbb{Z}_p\}$ denote the one dimensional affine group and U_p be the group of units of integers modulo p. Since $|\mathbb{Z}_p| = p$ and $|U_p| = p - 1$, it is clear that $|G| = p(p-1)$.

Lemma 2.1. *The center $Z(G)$ of the group G =Aff$(1,\ \mathbb{Z}_p)$ is trivial.*

Proof. Let $f_{a,b} \in Z(G)$. Then $f_{a,b}f_{c,d} = f_{c,d}f_{a,b}$, for every $c \in U_p$ and $d \in \mathbb{Z}_p$. Thus for $x \in Z_p$, we have

$$\begin{aligned} cax + (cb + d) &= acx + (ad + b) \pmod p, \\ cb + d &= ad + b \pmod p, \\ b(c-1) &= d(a-1) \pmod p, \end{aligned} \tag{2.1}$$

We observe that equation (2.1) is true for all $c \in U_p$ and $d \in \mathbb{Z}_p$ only if $a = 1$ and $b = 0$. Thus $Z(G) = \{f_{1,0}\}$. One can clearly note that $f_{1,0}$ is the identity map on $\mathbb{Z}_p$. □

Lemma 2.2. *Let $f_{1,i} \in G$, where $i \in \mathbb{Z}_p \setminus \{0\}$. Then the centralizer $C_G(f_{1,i}) = \{f_{1,b} :\ b \in \mathbb{Z}_p\}$. Further, $C_G(f_{1,i})$ is a cyclic subgroup of G of order p.*

Proof. Let $f_{a,b} \in C_G(f_{1,i})$, where $a \in U_p$ and $b \in \mathbb{Z}_p$. Let $x \in \mathbb{Z}_p$. Then $f_{a,b}f_{1,i}(x) = ax + (ia + b)$ and $f_{1,i}f_{a,b}(x) = ax + (b + i)$. Thus, $f_{a,b} \in C_G(f_{1,i})$ if and only if

$$\begin{aligned} & f_{a,b}f_{1,i} = f_{1,i}f_{a,b}, \\ \Leftrightarrow\quad & ia + b = b + i \pmod p, \\ \Leftrightarrow\quad & i(a-1) = 0 \pmod p. \end{aligned} \tag{2.2}$$

Note that the equation 2.2, holds if and only if p divides $i(a-1)$. Since $i \in \mathbb{Z}_p \setminus \{0\}$, p does not divide i. Hence, p divides $i(a-1)$ if and only if p divides $a-1$. Since $a \in U_p$, p divides $a-1$ if and only if $a = 1$.

Next, since $f_{1,i} \in C_G(f_{1,i})$, $f_{1,i}^2 = f_{1,i}f_{1,i} = f_{1,2i}$. One can inductively note that $f_{1,i}^k = f_{1,ki}$. Now, since $\mathbb{Z}_p$ is a cyclic group of order p and i is a generator of $\mathbb{Z}_p$, $ki \in \mathbb{Z}_p$ for each k. Thus, if $g \in C_G(f_{1,i})$, then $g = f_{1,ki}$ for some $k \in \{0,\ 1,\ 2, \ldots, p-1\}$. Thus $C_G(f_{1,i}) = \langle f_{1,i} \rangle$. This completes the proof. □

Lemma 2.3. *Let $f_{2,i} \in G$, where $i \in \mathbb{Z}_p$. Then the centralizer $C_G(f_{2,i}) = \{f_{a,i(a-1)} :\ a \in U_p\}$. Further, $C_G(f_{2,i})$ is a cyclic subgroup of G of order $p-1$.*

Proof. Let $f_{a,b} \in C_G(f_{2,i})$, where $a \in U_p$ and $b \in \mathbb{Z}_p$. Let $x \in \mathbb{Z}_p$. Then $f_{a,b}f_{2,i}(x) = 2ax + (ia + b)$ and $f_{2,i}f_{a,b}(x) = 2ax + (2b + i)$. Thus, $f_{a,b} \in C_G(f_{2,i})$ if and only if

$$\begin{aligned} & f_{a,b}f_{2,i} = f_{2,i}f_{a,b}, \\ \Leftrightarrow\quad & ia + b = 2b + i \pmod p, \\ \Leftrightarrow\quad & b = i(a-1) \pmod p. \end{aligned} \tag{2.3}$$

Thus, $C_G(f_{2,i}) = \{f_{a,i(a-1)} :\ a \in U_p\}$.

Next, since $f_{2,i} \in C_G(f_{2,i})$, $f_{2,i}^2 = f_{2,i}f_{2,i} = f_{2^2,i(2^2-1)}$. One can inductively observe that $f_{2,i}^k = f_{2^k,i(2^k-1)}$. Now, since U_p is a cyclic group of order $p-1$ and 2 is a generator of U_p, $2^k \in U_p$ for each k. Thus, if $g \in C_G(f_{2,i})$, then $g = f_{2^k,i(2^k-1)}$ for some $k \in \{1,\ 2, \ldots, p-1\}$. Thus $C_G(f_{2,i}) = \langle f_{2,i} \rangle$. This completes the proof. □

Lemma 2.4. *Let $f_{2,i},\ f_{2,j} \in G$, where $i,\ j \in \mathbb{Z}_p$. Then $f_{2,j} \in C_G(f_{2,i})$ if and only if $i = j$ (mod p).*

Proof. Let $f_{2,j} \in C_G(f_{2,i})$. Let $x \in \mathbb{Z}_p$. Then $f_{2,j}f_{2,i}(x) = 4x + (2i + j)$ and $f_{2,i}f_{2,j}(x) = 4x + (2j + i)$. Thus, $f_{2,j} \in C_G(f_{2,i})$ if and only if

$$\begin{aligned} & f_{2,j}f_{2,i} = f_{2,i}f_{2,j}, \\ \Leftrightarrow \quad & 2i + j = 2j + i \pmod p, \\ \Leftrightarrow \quad & i = j \pmod p. \end{aligned} \tag{2.4}$$

This completes the proof. □

Remark 2.5. By Lemma 2.3 and Lemma 2.4, we observe that $C_G(f_{2,i}) \cap C_G(f_{2,j}) = \{f_{1,0}\}$ if $i \neq j \pmod p$.

Next, let $\Omega_1 = \{f_{1,0}\}$, $\Omega_2 = \{f_{1,u} : u \in \mathbb{Z}_p \setminus \{0\}\}$ and $\Omega_3 = \{f_{\lambda,u} : \lambda \in U_p \setminus \{1\},\ u \in \mathbb{Z}_p\}$. Then G = Aff$(1,\ p) = \Omega_1 \cup \Omega_2 \cup \Omega_3$. By Lemma 2.1, Lemma 2.2 and Lemma 2.3, we observe that $\Omega_1 = Z(G)$, $\Omega_2 = C_G(f_{1,i}) \setminus \Omega_1$ and $\Omega_3 = \bigcup_{k=0}^{p-1} \Omega_3^k$, where $\Omega_3^k = C_G(f_{2,k}) \setminus \Omega_1$. It is clear that $fg = gf$ for every $f,\ g \in \Omega_2$. By Remark 2.5, we observe that for different k_1, $k_2 \in \{0,\ 1, \ldots,\ p-1\}$, $\Omega_3^{k_1}$ and $\Omega_3^{k_2}$ are disjoint. By Lemma 2.3, $|C_G(f_{2,k})| = p - 1$. Therefore $|\Omega_3^k| = p - 2$ for each k and any two elements of Ω_3^k commute with each other. Thus Ω_3 is the union of p disjoint sets each having $p - 2$ elements. Thus $|G| = |\Omega_1| + |\Omega_2| + |\Omega_3| = 1 + p - 1 + p(p - 2) = p(p - 1)$.

Now, we define the commuting graph $\Gamma = \mathcal{C}(G,\ \Omega)$ in G = Aff$(1,\ p)$ on the vertex set $\Omega \subseteq G$ and two distinct vertices f and g in Ω are adjacent if and only if $fg = gf$. We will denote K_n for a complete graph of order n and $\overline{K}_n$ for the complement of K_n. The following proposition follows by the discussion made in the above paragraph.

Proposition 2.6. *Let $\Gamma = \mathcal{C}(G,\ \Omega)$ be the commuting graph on G.*

(1) If $\Omega = \Omega_1$, then $\Gamma \cong K_1$

(2) If $\Omega = \Omega_2$, then $\Gamma \cong K_{p-1}$

(3) If $\Omega = \Omega_3$, then for $p \geq 5$, $\Gamma \cong pK_{p-2}$ and for $p = 3$, $\Gamma \cong \overline{K}_p$.

The following proposition follows by the above proposition.

Proposition 2.7. *If $\Gamma = \mathcal{C}(G,\ G)$ is the commuting graph on G, then*

$$\Gamma = \begin{cases} K_1 \vee (K_{p-1} \cup \overline{K}_p) & \text{if } p = 3, \\ K_1 \vee (K_{p-1} \cup pK_{p-2}) & \text{if } p \geq 5\ . \end{cases} \tag{2.5}$$

3 Some Detour Distant Properties of Commuting Graph on Aff$(1, \mathbb{Z}_p)$

Let Γ be a graph. Then the length of a longest f - g path in Γ is called the detour distance $d_D(f, g)$ between two vertices f and g in Γ. A f - g path of length $d_D(f, g)$ is called a f - g detour geodesic. The maximum detour distance between a vertex f and any other vertices in Γ is known as the detour eccentricity (ecc$_D(f)$) of that vertex. The minimum detour eccentricity among the vertices of Γ is the detour radius of Γ, represented by rad$_D(\Gamma)$. The maximum detour eccentricity in Γ is called the detour diameter diam$_D(\Gamma)$ of a graph Γ. For more definitions on detour distance and detour degree one can see ([3], [8]). Next, we have the following theorem.

Theorem 3.1. *Let $\Gamma = \mathcal{C}(G,\ G)$ be the commuting graph on G. Then,*

$$ecc_D(f) = \begin{cases} p - 1 & \text{if } f \in \Omega_1 \\ 2p - 3 & \text{if } f \in \Omega_2 \cup \Omega_3\ . \end{cases} \tag{3.1}$$

Proof. We note that $ff' = f'f$ for $f \in Z(G)$ and for each $f' \in \Omega_2 \cup \Omega_3$. Also for all $h,\ h' \in \Omega_2$, $hh' = h'h$ $(h \neq h')$. By Lemma 2.3, it is clear that for each $g' \in \Omega_3$ there exists exactly $p - 3$

vertices in Ω_3 different from g' which are adjacent to g'. By Lemma 2.2 and Lemma 2.3, if $f \in \Omega_2$ and $g \in \Omega_3$, then $fg \neq gf$. Thus, it is easy to observe the following,
(i) For $f \in \Omega_1$, there is always a $f - f'$ detour of length $p-1$ for every $f' \in \Omega_2$ and a $f - f''$ detour of length $p-2$ for every $f'' \in \Omega_3$,
(ii) For $h \in \Omega_2$, one has always a $h - h'$ detour of length $p-1$ for all $h' \in \Omega_2 \setminus \{h\}$ and a $h - h''$ detour of length $2p-3$ for all $h'' \in \Omega_3$ and
(iii) For every $g \in \Omega_3$, there is always a $g - g'$ detour of length $p-2$ for any $g' \in \Omega_3 \setminus \{g\}$. □

Following proposition follows by Theorem 3.1.

Proposition 3.2. *Let $\Gamma = \mathcal{C}(G,\ G)$ be the commuting graph on G. Then, $rad_D(G) = p-1$ and $diam_D(G) = 2p-3$.*

Theorem 3.3. *If $\Gamma = \mathcal{C}(G,\ G)$ be the commuting graph on G, then*

$$dds_D(\Gamma) = \Big(\big(1, 0^{p-3}, p(p-2), p-1\big),\ \big(1, 0^{p-2}, p-1, 0^{p-3}, p(p-2)\big)^{p-1},$$
$$\big(1, 0^{p-3}, p-2, 0^{p-3}, (p-1)(p-2), p-1\big)^{p(p-2)} \Big).$$

Proof. In view of Proposition 2.7, and Theorem 3.1, it follows that if $f \in V(K_1)$, then $ecc_D(f) = p-1$ and

$$dds_D(f) = \big(1,\ \underbrace{0,\ 0, \ldots, 0}_{p-3\ times},\ p(p-2),\ p-1\big).$$

Again, by Theorem 3.1, if $f \in V(K_{p-1}) \cup V(pK_{p-2}))$, then $ecc_D(f) = 2p-3$ and

$$\mathrm{dds}_D(f) = \begin{cases} \big(1,\ \underbrace{0,\ 0, \ldots, 0}_{p-2\ times},\ p-1,\ \underbrace{0,\ 0, \ldots, 0}_{p-3\ times},\ p(p-2)\big) & \text{if } f \in \Omega_2 \\ \big(1,\ \underbrace{0,\ 0, \ldots, 0}_{p-3\ times},\ p-2,\ \underbrace{0,\ 0, \ldots, 0}_{p-3\ times},\ (p-1)(p-2),\ p-1\big) & \text{if } f \in \Omega_3\ . \end{cases} \tag{3.2}$$

□

Corollary 3.4. *If $\Gamma = \mathcal{C}(G,\ G)$ be the commuting graph on G, then*

$$D(\Gamma) = \Big(\big(p(p-2)\big)^{p-1},\ (p-1)^{p(p-2)},\ p-1 \Big) \text{ and } D_{av}(\Gamma) = \tfrac{1+2p(p-2)}{p}.$$

Proof. By Theorem 3.3, we have

$$d_D(f) = \begin{cases} p-1 & \text{if } f \in \Omega_1, \\ p(p-2) & \text{if } f \in \Omega_2, \\ p-1 & \text{if } f \in \Omega_3. \end{cases} \tag{3.3}$$

Therefore, $D(\Gamma) = \Big(\big(p(p-2)\big)^{p-1},\ (p-1)^{p(p-2)},\ p-1 \Big)$ and $D_{av}(\Gamma) = \tfrac{1+2p(p-2)}{p}$. □

4 Some Distant Properties of Commuting Graph of Aff$(1, \mathbb{Z}_p)$

Consider a graph Γ. In a graph Γ, the length of a shortest f - g path is $d(f,g)$ between two vertices f and g. The maximum distance between a vertex f and any other vertices of Γ is its eccentricity, represented by the ecc(f). The radius of Γ is the least eccentricity between its vertices, and it is represented by rad (Γ). The diameter of Γ, dim(Γ), is the largest eccentricity among the vertices of Γ. A vertex f in Γ is defined as central vertex if and only if ecc(f) = rad(f) and the center Cen(Γ) of Γ is the subgraph induced by the central vertices of Γ. When ecc(f) = diam(Γ), we say that a vertex f in Γ is a peripheral vertex, and a subgraph of Γ induced by the peripheral vertices is called the periphery of Γ, denoted by Per(Γ). For the definitions of $Int(\Gamma)$ and $Cl(\Gamma)$ one can see ([3], [9]).

Lemma 4.1. *Let* $\Gamma = \mathcal{C}(G, G)$ *be the commuting graph on* $G = Aff(1, \mathbb{Z}_p)$.*Then*
(i) $rad(\Gamma) = 1$.
(ii) $diam(\Gamma) = 2$.

Proof. (i) Since $Z(G) = \Omega_1 = \{f_{1,0}\}$, $ecc(f_{1,0}) = 1 \leq ecc(g)$ for any $g \in V(\Gamma)$.
(ii) For any $f,\ g \in V(\Gamma)$ with $f \not\sim g$, there is always a path $f - f_{1,0} - g$ of shortest length in Γ. Consequently, $ecc(f) = ecc(g) = 2 \geq ecc(h)$ for all $h \in V(\Gamma)$. □

Remark 4.2. If $\Gamma = \mathcal{C}(G, G)$ be the commuting graph on $G = \text{Aff}(1, \mathbb{Z}_p)$, then
(i) By ([3], Lemma 1.2, p. 2391) it is easy to observe that $Cen(\Gamma) = K_1$.
(ii) By ([3], Corollary 1.4, p.2391) it follows that $Per(\Gamma)$ is induced by $\Omega_2 \cup \Omega_3$.

Theorem 4.3. *If* $\Gamma = \mathcal{C}(G, G)$ *is the commuting graph on* G, *then Int*$(\Gamma) =$ *Cen*(Γ).

Proof. By [[3], Theorem 2.9, p.2396], the interior vertices of Γ are precisely those vertices of Γ that are not boundary vertices. Thus by [[3], Theorem 2.10, p.2396], It is adequate to locate those vertices of Γ which are complete. For this, we will find out $N(f)$ for all $f \in V(\Gamma)$, where $N(f)$ is the neighborhood of f in Γ .
Case 1. For $p = 3$. By Proposition 2.7, $\Gamma = K_1 \vee (K_{p-1} \cup \overline{K}_p)$. Then

$$\mathrm{N}(f) = \begin{cases} \Omega_2 \cup \Omega_3 & \text{if } f \in V(K_1), \\ \Omega_1 \cup \Omega_2 \setminus \{f\} & \text{if } f \in V(K_{p-1}) , \\ \Omega_1 & \text{if } f \in V(\overline{K}_p). \end{cases} \tag{4.1}$$

One can observe that the subgraph induced by $\Omega_2 \cup \Omega_3$ is disconnected, subgraph induced by $\Omega_2 \setminus \{f\} \cup \Omega_1$ and Ω_1 are complete. Thus, each $f \in V(K_{p-1} \cup \overline{K}_p)$ is a boundary vertex. Thus $f \in V(K_1)$ is the only interior vertex of Γ. Therefore, $\text{Int}(\Gamma) = K_1 = Cen(\Gamma)$ (see Remark 4.2).
Case 2. For $p \geq 5$. By Proposition 2.7, $\Gamma = K_1 \vee (K_{p-1} \cup pK_{p-2})$. Then

$$\mathrm{N}(f) = \begin{cases} \Omega_2 \cup \Omega_3 & \text{if } f \in V(K_1), \\ \Omega_2 \setminus \{f\} \cup \Omega_1 & \text{if } f \in V(K_{p-1}) , \\ \Omega_1 \cup (\Omega_3^i \setminus \{f\}) & \text{if } f \in V(K_{p-2}^i), \end{cases} \tag{4.2}$$

where $1 \leq i \leq p$ and K_{p-2}^i is the i^{th} component of the graph pK_{p-2} corresponding to the vertex set Ω_3^i. We observe that the subgraph induced by $\Omega_2 \cup \Omega_3$ is disconnected and the subgraphs induced by $(\Omega_2 \setminus \{f\}) \cup \Omega_1$ and $\Omega_1 \cup (\Omega_3^i \setminus \{f\})$ are complete for each $1 \leq i \leq p$. Consequently, every $f \in V(K_{p-1} \cup pK_{p-2})$ is a boundary vertex. Thus $f \in V(K_1)$ is the only interior vertex of Γ. Hence, $\text{Int}(\Gamma) = K_1 = Cen(\Gamma)$ (see Remark 4.2). □

Suppose that f is a vertex in the graph Γ. A vertex at distance $ecc(f)$ from f is therefore an eccentric vertex for f. If f is also an eccentric vertex of some other vertex of Γ, then f is an eccentric vertex of the graph Γ. The eccentric subgraph, denoted by $\text{Ecc}(\Gamma)$, of a graph Γ is the subgraph of Γ induced by the eccentric vertices of Γ (see [8]).

Theorem 4.4. *If* $\Gamma = \mathcal{C}(G, G)$ *is the commuting graph on* $G =$*Aff*$(1, \mathbb{Z}_p)$, *then Ecc*(Γ) = *Per*(Γ).

Proof. By Proposition 2.7, we have

$$\Gamma = \begin{cases} K_1 \vee (K_{p-1} \cup \overline{K}_p) & \text{if } p = 3, \\ K_1 \vee (K_{p-1} \cup pK_{p-2}) & \text{if } p \geq 5 . \end{cases} \tag{4.3}$$

By [[3], Corollary 1.4, p.2391] and ([8], p.337), each $f \in V(\Gamma) \setminus V(K_1)$ is an eccentric vertex. It is clear that $f \in V(K_1)$ is not an eccentric vertex. Since $d(f, g) = 1$ and $ecc(g) = 2$ for every $g \in V(K_{p-1} \cup pK_{p-2})$, f is not an eccentric vertex for any $g \in V(K_{p-1} \cup pK_{p-2})$. Hence, $\text{Ecc}(\Gamma) = \text{Per}(\Gamma)$. □

Corollary 4.5. *If* $\Gamma = \mathcal{C}(G, G)$ *is the commuting graph on* G, *then* Γ *is not an eccentric graph.*

Proof. By Theorem 4.4, it is clear that $f \in V(K_1)$ is not an eccentric vertex of Γ. □

Lemma 4.6. *Let $G = Aff(1, \mathbb{Z}_p)$. Let $\Gamma = \mathcal{C}(G, \Omega)$, $\Omega \subseteq G$. Let $f \in \Omega$. Then $deg_G(f) = |C_G(f)| - 1$.*

Proof. Let $f \in \Omega \subseteq G$. Then a vertex $g \in \Omega$ is adjacent to f if and only if $g \in C_G(f)$. Since $f \in C_G(f)$, therefore $deg_G(f) = |C_G(f)| - 1$. □

Theorem 4.7. *If $\Gamma = \mathcal{C}(G, G)$ is the commuting graph on $G = Aff(1, \mathbb{Z}_p)$, then*

$$deg(f) = \begin{cases} p(p-1) - 1 & \text{if } f \in \Omega_1 \\ p - 1 & \text{if } f \in \Omega_2 \\ p - 2 & \text{if } f \in \Omega_3 \,. \end{cases} \tag{4.4}$$

Proof. Let $f \in \Omega_1 = Z(G)$. Therefore, $C_G(f) = G$. Thus by Lemma 4.6, $\deg(f) = |G| - 1$. Let $f \in \Omega_2$. Then by Lemma 2.2, $C_G(f) = \{f_{1,b} : b \in \mathbb{Z}_p\}$. Thus $\deg(f) = |\{f_{1,b} : b \in \mathbb{Z}_p\}| - 1$. Let $f \in \Omega_3$. Then $f \in \Omega_3^k$ for some $k \in \{0, 1, 2, \ldots, p-1\}$. Since $\Omega_3^k = C_G(f_{2,k}) \setminus \Omega_1$, by Lemma 2.3 $f = f_{2^l, k(a^l-1)}$ for some $l \in \{1, 2, \ldots, p\}$. By Lemma 2.3, $C_G(f) = C_G(f_{2,k})$. Thus by Lemma 4.6, $\deg(f) = |C_G(f_{2,k})| - 1$. This completes the proof. □

Theorem 4.8. *Let $\Gamma = \mathcal{C}(G, \Omega)$ be the commuting graph on $G = Aff(1, \mathbb{Z}_p)$ and $\Omega \subseteq G$. Then $\Gamma = K_p$ if and only if $\Omega = \Omega_2 \cup \Omega_1$.*

Proof. Let $\Omega = \Omega_2 \cup \Omega_1$. Then Ω is a cyclic subgroup of G. Therefore $\Gamma = \mathcal{C}(G, \Omega)$ is a complete graph of order p $(= |\Omega|)$. Hence $\Gamma = K_p$. Conversely, if $\Gamma = K_p$, then degree of any vertex in Γ is $p - 1$. Thus by Theorem 4.7, $\Omega = \Omega_2 \cup \Omega_1$. This completes the proof. □

Theorem 4.9. *If $\Gamma = \mathcal{C}(G, G)$ is the commuting graph on $G = Aff(1, \mathbb{Z}_p)$, then the number of edges in Γ is $\mathcal{E}(\Gamma) = \frac{p(p-1)^2}{2}$.*

Proof. The graph induced by $\Omega = \Omega_1 \cup \Omega_2$ is K_p . Further, the graph induced by Ω_3 is pK_{p-2}. Therefore the number of edges in Γ is the sum of number of edges in subgraph induced by Ω, the number of edges in subgraph induced by Ω_3 and the number of edges from Ω_1 to the vertices in Ω_3. Therefore, $\mathcal{E}(\Gamma) = \frac{p(p-1)}{2} + \frac{p(p-2)(p-3)}{2} + p(p-2) = \frac{p(p-1)^2}{2}$. □

Theorem 4.10. *If $\Gamma = \mathcal{C}(G, G)$ is the commuting graph on $G = Aff(1, \mathbb{Z}_p)$, then $Cl(\Gamma) = G$.*

Proof. Case 1. For $p = 3$, by Proposition 2.7, we have $\Gamma = K_1 \vee (K_{p-1} \cup \overline{K}_p)$. In view of Theorem 4.7, we have

$$\deg(f) = \begin{cases} 2 & \text{if } f \in V(K_{p-1}) \\ 1 & \text{if } f \in V(\overline{K}_p) \,. \end{cases} \tag{4.5}$$

Now, we observe that for any two non-adjacent vertices f and g in Γ, we have $d(f) + d(g) < p(p-1) = 6$. Thus by [[8], Lemma 2.15, p.2398], $Cl(\Gamma) = \Gamma$.
Case 2. For $p \geq 5$, by Proposition 2.7, we have $\Gamma = K_1 \vee (K_{p-1} \cup pK_{p-2})$. Further by Theorem 4.7, we have

$$\deg(f) = \begin{cases} p - 1 & \text{if } f \in V(K_{p-1}) \\ p - 2 & \text{if } f \in V(pK_{p-2}) \,. \end{cases} \tag{4.6}$$

Now, it is easy to observed that for any two non-adjacent vertices f and g in Γ, we have $d(f) + d(g) < p(p-1) = |\Gamma|$. Therefore by [[8], Lemma 2.15, p.2398], $Cl(\Gamma) = \Gamma$. □

Corollary 4.11. *If $\Gamma = \mathcal{C}(G, G)$ is the commuting graph on $G = Aff(1, \mathbb{Z}_p)$, then Γ is a closed graph.*

Proof. Follows by Theorem 4.10. □

For the definitions of the clique number, chromatic number and independence number one can see ([8], [21]).

Theorem 4.12. *If $\Gamma = \mathcal{C}(G,G)$ is the commuting graph on $G = Aff(1,\ \mathbb{Z}_p)$, then the clique number $\omega(\Gamma)$ of Γ is p.*

Proof. By Theorem 4.8, it is clear that $\Gamma = \mathcal{C}(G,\Omega)$ is a maximal complete subgraph of Γ if $\Omega = \Omega_2 \cup \Omega_1$. Thus, $\omega(\Gamma) = p$. □

Theorem 4.13. *Let $\Gamma = \mathcal{C}(G,G)$ be the commuting graph on $G = Aff(1,\mathbb{Z}_p)$. Then the chromatic number of Γ is $\psi(\Gamma) = p$.*

Proof. Let $\Omega = \Omega_2 \cup \Omega_1$. By Theorem 4.8, it is clear that to color the subgraph induced by Ω, one needs p different color. Since $f_{1,0}$ is adjacent to all vertices of Γ, the color assigned to this vertex cannot be assigned to any other vertex of Γ. By Lemma 2.2 and Lemma 2.3, it is clear that the remaining $p-1$ vertices in Ω are not adjacent to any vertex in $\Omega_3 = G \setminus \Omega = \bigcup_{k=0}^{p-1} \Omega_3^k$,. By Lemma 2.3, it is clear that for a fixed k all the vertices in Ω_3^k are adjacent to each other but for different k_1 and k_2 no vertex of $\Omega_3^{k_1}$ is adjacent to any vertex of $\Omega_3^{k_2}$. Therefore, the vertices in Ω_3 can be colored by remaining $p-1$ colors. Thus $\psi(\Gamma) = p$. This completes the proof. □

Theorem 4.14. *Let $\Gamma = \mathcal{C}(G,G)$ be the commuting graph on $G = Aff(1,\mathbb{Z}_p)$. Then the independence number of Γ is $\alpha(\Gamma) = p+1$.*

Proof. By Proposition 2.7, it follows that $f_{1,0} \in \Omega_1$ is adjacent to each vertex of Γ, therefore it cannot be an element of any independent set of Γ. Next, since any two vertices in Ω_2 are adjacent, any independent set of Γ can contain at most one vertex of Ω_2. Next, since $\Omega_3 = \bigcup_{k=0}^{p-1} \Omega_3^k$ and for a fixed k any two vertices of Ω_3^k are adjacent, in view of Remark 2.5, it easy to observe that any independent set of Γ can contain at most one vertex of Ω_3^k for each $k \in \{0,1,\ldots,p-1\}$. Thus the maximum size of an independent set is $p+1$. Hence $\alpha(\Gamma) = p+1$. □

5 The Resolving Polynomial of Commuting Graph on Aff$(1,\mathbb{Z}_p)$

For an ordered set $W = \{f_1, f_2, f_3, \ldots, f_l\} \subset V(\Gamma)$ and a vertex f of Γ, the l-vector $r(f|W) = \big(d(f,f_1), d(f,f_2), \ldots, d(f,f_l)\big)$ is said to be the representation of f with respect to W. We say that W is a resolving set for Γ if different vertices of Γ have separate representations with regard to $W \subset V(\Gamma)$. If W is a resolving set of Γ, then we say that W resolves Γ. The metric dimension of Γ, denoted by $\beta(\Gamma)$, is the number of vertices in a minimum resolving set for Γ. We denote the family of resolving sets of Γ by $\mathcal{R}(\Gamma,k)$ each having cardinality k. Further, let $l_k = |\mathcal{R}(\Gamma,k)|$. Then the resolving polynomial of a graph Γ of order n is defined as $\beta(\Gamma,x) = \sum_{k=\beta(\Gamma)}^{n} l_k x^k$.

Let $f \in V(\Gamma)$. Then the open and closed neighborhood of f in Γ are defined as $N(f) = \{f \in V(\Gamma) : f \sim g \text{ in } \Gamma\}$ and $N[f] = N(f) \cup \{f\}$, respectively. We say that vertices f and g, $(f \neq g)$ of Γ are true (or adjacent) twins if $N[f] = N[g]$ and false (or non-adjacent) twins if $N(f) = N(g)$. Let $U \subset V(\Gamma)$. If any two distinct vertices in U are twins in Γ, then we say that U is a twin set in Γ.

Theorem 5.1. *Let $\Gamma = \mathcal{C}(G,\ G)$ be the commuting graph on G. Then*

$$\beta(\Gamma) = \begin{cases} 3 & \text{if } p = 3 \\ p^2 - 2p - 2 & \text{if } p \geq 5\,. \end{cases} \tag{5.1}$$

Proof. Case 1. For $p = 3$, one can easily observe that Ω_2 and Ω_3 are twin-sets in Γ of cardinality 2 and 3 respectively. Thus, by [[3], Remark 3.3, p.2398], we have $\beta(\Gamma) \geq 1 + 2 = 3$. Further, we observe that the set $W = \{f_{1,1}, f_{2,1}, f_{2,0}\}$ is a resolving set for Γ. Thus, $\beta(\Gamma) = 3$.
Case 2. For $p \geq 5$, we observe that Ω_2 is a twin-set with cardinality $p-1$ and Ω_3 is the union of p disjoint twin-sets each of cardinality $p-2$. Thus by [[3], Remark 3.3, p.2398], we have $\beta(\Gamma) \geq p(p-3) + p - 2 = p^2 - 2p - 2$. Next, we observe that the set $W = \{f_{1,1}^l,\ f_{1,1}^m f_{2,0}^n : 1 \leq l \leq p-2,\ 0 \leq m \leq p-1,\ 1 \leq n \leq p-3\}$ of cardinality $p^2 - 2p - 2$ is a resolving set for Γ. Thus, $\beta(\Gamma) = p^2 - 2p - 2$. □

Theorem 5.2. *Let $p = 3$. Let $\Gamma = \mathcal{C}(G,\ G)$ be the commuting graph on $G = Aff(1, \mathbb{Z}_p)$. Then $\beta(\Gamma,\ x) = x^3(x^3 + 6x^2 + 11x + 6)$.*

Proof. By Theorem 5.1, $\beta(\Gamma) = 3$. Therefore, we need to find out the resolving sequence $(l_3,\ l_4,\ l_5,\ l_6)$. By Corollary 3.2, and by the principal of multiplication, we have $l_3 = 3 \times 2 = 6$. Next to evaluate l_4, we need to calculate the number of resolving set for Γ of cardinality 4. Let f and g be two to vertices of g that do not belong to any resolving set W having cardinality 4. Then, by [[3], Remark 3.3, p.2398], we observe that either $f \in \Omega_2$ and $g \in \Omega_3$, or $f \in \Omega_1$ and $g \in \Omega_3$, or $f \in \Omega_1$ and $g \in \Omega_2$. By principal of multiplication there are $2 \cdot 3 = 6$, $1 \cdot 3 = 3$ and $1 \cdot 2 = 2$ possible resolving sets having cardinality 4, in the first, second and third case respectively. Thus, by principal of addition $l_4 = 6 + 3 + 2 = 11$. Also by [[3], Proposition 3.5 p.2399], we have $l_5 = 6$, $l_6 = 1$. Thus, $\beta(\Gamma,\ x) = x^3(x^3 + 6x^2 + 11x + 6)$. □

Theorem 5.3. *For all prime $p \geq 5$, let $\Gamma = \mathcal{C}(G, G)$ be the commuting graph on $G = Aff(1, \mathbb{Z}_p)$. Then*

$$\beta(\Gamma, x) = (p-2)^p |\Omega_2| x^{(p-2)^p |\Omega_2|} \ +\ |G| x^{|G|-1} + \ x^{|G|} \ + \sum_{j=p^2-2p-1}^{p^2-p-2} l_j x^j \ ,$$

where $l_j = 2\ {}^pC_{m-j-1}\ (p-2)^{m-j-1} \ +\ {}^pC_{m-j-2}\ (p-2)^{m-j-2}$ and ${}^nC_r = \frac{n!}{r!(n-r)!}$.

Proof. Since $p \geq 5$, by Theorem 5.1, the metric dimension $\beta(\Gamma) = p^2 - 2p - 2$. To get the resolving polynomial of the graph Γ, we need to calculate the resolving sequence $(l_{\beta(\Gamma)},\ l_{\beta(\Gamma)+1},\ l_{\beta(\Gamma)+2}, \ldots, l_{p(p-1)})$, of length $p+3$. By [[3], Proposition 3.5, p.2399], $l_{p(p-1)} = 1$ and $l_{p(p-1)-1} = p(p-1)$.

For $l_{\beta(\Gamma)}$, since Ω_1, Ω_2 and Ω_3 are mutually disjoint sets out of which Ω_2 is a twin-set and Ω_3 is the disjoint union of p twin sets each of cardinality $p-2$, by [[3], Corollary 3.2, p.2398], and the principal of multiplication, we have

$$l_{\beta(\Gamma)} = \underbrace{{}^{p-2}C_{p-3} \cdot\ {}^{p-2}C_{p-3}\ \cdots\ {}^{p-2}C_{p-3}}_{p \text{ times}} \cdot\ {}^{p-1}C_{p-2} = (p-2)^p \times (p-1).$$

Next, we shall calculate l_j, for fixed j, where $p^2 - 2p - 1 \leq j \leq p^2 - p - 2$. Let W be a resolving set of cardinality j. Let $f_1,\ f_2,\ \ldots, f_{m-j}$ be the vertices of graph Γ that do not belong to any resolving set of cardinality j, where $m = |\Gamma|$. Then the three cases are possible:
(i) One of $f_1,\ f_2,\ \ldots, f_{m-j}$, say $f_1 \in \Omega_1$ and rest are in Ω_3;
(ii) One of $f_1,\ f_2,\ \ldots, f_{m-j}$, say $f_1 \in \Omega_2$ and rest are in Ω_3;
(iii) One of $f_1,\ f_2,\ \ldots, f_{m-j}$, say $f_1 \in \Omega_1$; One of $f_2, \ldots, f_{m-j}$, say $f_2 \in \Omega_2$, and rest are in Ω_3.

In case (i), we have $f_1 \in \Omega_1$ and $f_2, f_3, \ldots, f_{m-j} \in \Omega_3$. Since, $\Omega_3 = \bigcup_{k=1}^{p} \Omega_3^k$, where Ω_3^k is a twin set of cardinality $p-2$ for each k, by [[3], Remark 3.3, p.2398], exactly one vertex among $f_2, f_3\ \ldots, f_{m-j}$ is into exactly one set of the family $\{\Omega_3^1,\ \Omega_3^2\ \ldots,\ \Omega_3^p\}$ of p twin set. Thus, by principal of addition and multiplication, there are

$${}^pC_{m-j-1} \cdot ({}^{p-2}C_{p-3})^{m-j-1} \cdot\ {}^{p-1}C_{p-1}$$

possible resolving sets of cardinality j such that $f_1,\ f_2,\ \ldots, f_{m-j}$ do not belong to any resolving sets of cardinality j.

In case (ii), one of $f_1,\ f_2,\ \ldots, f_{m-j}$, say $f_1 \in \Omega_2$ and rest are in Ω_3. Again, since $\Omega_3 = \bigcup_{k=1}^{p} \Omega_3^k$, by [[3], Remark 3.3, p.2398], exactly one vertex among $f_2, f_3\ \ldots,$
f_{m-j} is into exactly one set of the family $\{\Omega_3^1,\ \Omega_3^2\ \ldots,\ \Omega_3^p\}$ of p twin sets. Thus, by addition and multiplication principal, there are

$${}^1C_1 \cdot\ {}^{p-2}C_{p-2} \cdot\ {}^pC_{m-j-1} \cdot ({}^{p-2}C_{p-3})^{m-j-1}$$

possible resolving sets of cardinality j such that $f_1,\ f_2,\ \ldots, f_{m-j}$ do not belong to any resolving set of cardinality j.

Lastly, when one of $f_1,\ f_2,\ \ldots, f_{m-j}$, say $f_1 \in \Omega_1$; one of $f_2,\ \ldots, f_{m-j}$, say $f_2 \in \Omega_2$, and rest are in Ω_3. Again, as in the previous case, since $\Omega_3 = \bigcup_{k=1}^{p} \Omega_3^k$, by [[3], Remark 3.3, p.2398], exactly one vertex among $f_3,\ f_4\ \ldots, f_{m-j}$ is into exactly one set of the family $\{\Omega_3^1,\ \Omega_3^2,\ \ldots,\ \Omega_3^p\}$ of p twin set. Thus, by principal of addition and multiplication, there are

$$ {}^{p}C_{m-j-2} \cdot ({}^{p-2}C_{p-3})^{m-j-2} \cdot\ {}^{p-2}C_{p-2} $$

possible resolving sets of cardinality j such that $f_1,\ f_2,\ \ldots, f_{m-j}$ do not belong to any resolving set of cardinality j. Thus, by principal of addition

$$ l_j = 2\ {}^{p}C_{m-j-1}\ (p-2)^{m-j-1}\ +\ {}^{p}C_{m-j-2}\ (p-2)^{m-j-2}. $$

This completes the proof. □

6 Spectral Properties of Commuting Graph on Aff$(1, \mathbb{Z}_p)$

In this section, we will study the spectrum of normalized Laplacian and signless Laplacian matrices of the commuting graph of the affine group.

For any given graph Γ, let $A(\Gamma)$ represent its adjacency matrix. Then, with $D(\Gamma)$ being a diagonal matrix of degrees, the matrices $L(\Gamma) = D(\Gamma) - A(\Gamma)$ and $Q(\Gamma) = D(\Gamma) + A(\Gamma)$ are referred to as the Laplacian and signless Laplacian, respectively.

The normalized Laplacian matrix $\mathcal{L}$ (Γ)=(l_{ij}) where,

$$ l_{ij} = \begin{cases} 1, & i = j, deg(v_i) \neq 0, \\ \frac{-1}{\sqrt{deg(v_i)deg(v_j)}}, & v_i v_j \in E(\Gamma), \\ 0, & \text{otherwise.} \end{cases} $$

For the complete graph $K_n, Spec(K_n), Spec_Q(K_n)$, and $Spec_{\mathcal{L}}(K_n)$ denotes the spectrum of the adjacency, signless Laplacian and normalized Laplacian matrices of K_n respectively.

The signless Laplacian matrix is known to be positive semi-definite. The number of bipartite connected components in the graph is correlated with the multiplicity of 0 as an eigenvalue[12].

One can show that $Spec(K_n) = \{(-1)^{n-1}, (n-1)^1\}$ and $Spec(mK_n) = \{(-1)^{m(n-1)},\ (n-1)^m\}$, where $\{(-1)^{n-1}, (n-1)^1\}$ means (-1) eigenvalue repeats $n-1$ times, and $(n-1)$ eigenvalue came 1 time. Then, one can easily observe the following

Define $G_1 = K_1$, $G_2 = K_{p-1}$, and $G_3 = \overline{K_p}$ for $p = 3$, and $G_1 = K_1$, $G_2 = K_{p-1}$, and $G_3 = pK_{p-2}$ for $p = 5$. Let S_2 denote the star graph with vertex set $\{1, 2, 3\}$. Then S_2-join of the graphs G_1, G_2, and G_3 is the commuting graph $\mathcal{C}(G,\ G)$. For the definition of star graph and S_2-join see [8] and [29].

$$ Spec_Q(G_1) = \begin{cases} \{0^1\}, & \text{if } p = 3 \\ \{0^1\}, & \text{if } p \geq 5 \end{cases}, $$

$$ Spec_Q(G_2) = \begin{cases} \{(2(p-2))^1, (p-3)^{(p-2)}\}, & \text{if } p = 3 \\ \{(2(p-2))^1, (p-3)^{(p-2)}\}, & \text{if } p \geq 5 \end{cases}, $$

$$ Spec_Q(G_3) = \begin{cases} \{0^p\}, & \text{if } p = 3 \\ \{(2(p-3))^p, (p-4)^{p(p-3)}\}, & \text{if } p \geq 5 \end{cases}. $$

Let H be a graph with $V(H) = \{1, 2, , \ldots, k\}$, and G_i be a r_i -regular graph of order n_i, that is, $n_i = \mid V(G_i) \mid$, where $1 \leq i \leq k$. For the graph H, $N_H(i)$ denote the neighbors of i. Define $N_i = \sum_{j \in N_H(i)} n_j$ and $C_Q(H) = (c_{ij})$ [29], where

$$c_{ij} = \begin{cases} 2r_i + N_i, & i = j \\ \sqrt{n_i n_j}, & ij \in E(H) \\ 0, & \text{otherwise} \end{cases}.$$

Therefore, we have the following

$$(n_1, n_2, n_3) = \begin{cases} (1, p-1, p), & \text{if } p = 3 \\ (1, p-1, p(p-2)), & \text{if } p \geq 5 \end{cases},$$

$$(r_1, r_2, r_3) = \begin{cases} (0, p-2, 0), & \text{if } p = 3 \\ (0, p-2, (p-2)-1), & \text{if } p \geq 5 \end{cases},$$

$$(N_1, N_2, N_3) = \begin{cases} (2p-1, 1, 1), & \text{if } p = 3 \\ (p^2-p-1, 1, 1), & \text{if } p \geq 5 \end{cases},$$

and

$$C_Q(S_2) = \begin{cases} \begin{pmatrix} 2p-1 & \sqrt{p-1} & \sqrt{p} \\ \sqrt{p-1} & 2p-3 & 0 \\ \sqrt{p} & 0 & 1 \end{pmatrix}, & \text{if } p = 3 \\ \begin{pmatrix} p^2-p-1 & \sqrt{p-1} & \sqrt{p(p-2)} \\ \sqrt{p-1} & 2p-3 & 0 \\ \sqrt{p(p-2)} & 0 & 2p-5 \end{pmatrix}, & \text{if } p \geq 5 \end{cases}.$$

Theorem 6.1. *Let* $\Gamma = C_(G, G)$. *Then,*

$$Spec_Q(\Gamma) = \begin{cases} Spec\,(C_Q(S_2)) \cup A & \text{if } p = 3 \\ Spec\,(C_Q(S_2)) \cup B, & \text{if } p \geq 5 \end{cases},$$

where
$A = \{2p-1, 1, 1\}$,
$B = \{p^2-p-1, (p-2)^{p-2}, (2p-5)^{p-1}, (p-3)^{p(p-3)}\}$.

Proof. By [29, Theorem 2.1, p.4],

$$Spec_Q(\Gamma) = \left(\bigcup_{i=1}^{3} (N_i + (Spec_Q(G_i) \backslash \{2r_i\})) \right) \bigcup Spec(C_Q(S_2)),$$

We first assume that $p = 3$. Then,

$$\begin{aligned} Spec_Q(\Gamma) &= \{2p-1 + (\{0\}\backslash\{0\})\} \\ &\quad \cup (\{1 + (\{0, p-1\}\backslash\{p-1\}) \\ &\quad \cup \{1 + (\{0\}\backslash\{0\}) \\ &\quad \cup Spec(C_Q(S_2)) \\ &= \{2p-1, 1, 1\} \cup Spec(C_Q(S_2)) \\ &= A \cup Spec\,(C_Q(S_2))\,. \end{aligned}$$

Now, let $p \geq 5$. Then,

$$\begin{aligned} Spec_Q(\Gamma) &= \{p^2-p-1+(\{0\}\backslash\{0\})\} \\ &\cup \{1+(\{2p-4,(p-3)^{p-2}\}\backslash\{2p-4\})\} \\ &\cup \{1+(\{(2(p-3))^p,(p-4)^{p(p-3)}\}\backslash\{2(p-3)\})\} \\ &\cup Spec(C_Q(S_2)) \\ &= \{p^2-p-1,(p-2)^{p-2},(2p-5)^{p-1},(p-3)^{p(p-3)}\} \\ &\cup Spec(C_Q(S_2)) \\ &= B \cup Spec\left(C_Q(S_2)\right). \end{aligned}$$

□

Let H be a graph with no isolated vertices and $V(H) = \{1,2,\ldots,k\}$, and let G_i be a r_i-regular graph of order n_i, that is, $n_i = \mid V(G_i) \mid$, where $1 \leq i \leq k$. For the graph H, $N_H(i)$ denote the neighbors of i. Define $N_i = \sum_{j \in N_H(i)} n_j$ and the matrix $C_{\mathcal{L}}(H) = (c_{ij})$ [29], where

$$c_{ij} = \begin{cases} \frac{N_i}{r_i+N_i}, & i=j, deg(v_i) \neq 0 \\ -\sqrt{\frac{n_i n_j}{(r_i+N_i)(r_j+N_j)}}, & ij \in E(H) \\ 0, & \text{otherwise.} \end{cases}.$$

Theorem 6.2. *Let* $\Gamma = C(G,G)$. *Then The normalized Laplacian spectrum of* Γ *is given as follows:-*

$$Spec_{\mathcal{L}}(\Gamma) = \bigcup_{i=1}^{3}\left(\frac{N_i}{r_i+N_i} + \frac{r_i}{r_i+N_i}(Spec_{\mathcal{L}}(G_i)\backslash\{0\}))\right)\bigcup Spec(C_{\mathcal{L}}(S_2)).$$

Proof. First, for the graph G_1, G_2, G_3, we compute their normalized Laplacian spectrum. We have

$$Spec_{\mathcal{L}}(G_1) = \begin{cases} Spec_{\mathcal{L}}(K_1) = \{0^1\}, & \text{if } p=3 \\ Spec_{\mathcal{L}}(K_1) = \{0^1\}, & \text{if } p \geq 5 \end{cases},$$

$$Spec_{\mathcal{L}}(G_2) = \begin{cases} Spec_{\mathcal{L}}(K_{p-1}) = \{(\frac{p-1}{p-2})^{p-2},0^1\}, & \text{if } p=3 \\ Spec_{\mathcal{L}}(K_{p-1}) = \{(\frac{p-1}{p-2})^{p-2},0^1\}, & \text{if } p \geq 5 \end{cases},$$

$$Spec_{\mathcal{L}}(G_3) = \begin{cases} Spec_{\mathcal{L}}(\overline{K}_p) = \{0^p\}, & \text{if } p=3 \\ Spec_{\mathcal{L}}(pK_{p-2}) = \{\left(\left(\frac{p-2}{p-3}\right)^{p-3},0^1\right)^p\}, & \text{if } p \geq 5 \end{cases}.$$

If λ is eigenvalue of K_n, then $1-\frac{\lambda}{n-1}$ is eigenvalue of normalized Laplacian matrix of K_n[11].

Therefore,

$$C_{\mathcal{L}}(S_2) = \begin{cases} \begin{pmatrix} 1 & -\sqrt{\frac{1}{2p-1}} & -\sqrt{\frac{p}{2p-1}} \\ -\sqrt{\frac{1}{2p-1}} & \frac{1}{p-1} & 0 \\ -\sqrt{\frac{p}{2p-1}} & 0 & 1 \end{pmatrix}, & \text{if } p=3 \\ \begin{pmatrix} 1 & -\sqrt{\frac{1}{p^2-p-1}} & -\sqrt{\frac{p}{p^2-p-1}} \\ -\sqrt{\frac{1}{p^2-p-1}} & \frac{1}{p-1} & 0 \\ -\sqrt{\frac{p}{p^2-p-1}} & 0 & \frac{1}{p-2} \end{pmatrix}, & \text{if } p \geq 5 \end{cases}.$$

The proof can now be obtained using [29, Theorem3.1, p.8] and a similar approach to proof of Theorem 2.5. □

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Received: 2025-01-03
Accepted: 2026-06-03.