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G-2-frames in 2-Hilbert spaces

S. Tatouti and N. Bounader

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Corresponding Author: S. Tatouti

Abstract. G-2-frames are generalized 2-frames that include ordinary 2-frames in 2-Hilbert space. In this article, we prove a version of the Riesz representation theorem in 2-Hilbert space to introduce a definition of g-2-frames, a generalization of 2-frames in 2-Hilbert spaces that is compatible with g-frames already known in Hilbert spaces, and provide some results concerning this notion. Finally, we will define g-2-frame operators and establish some of their characterizations in 2-Hilbert spaces, similar to those in Hilbert spaces.

1 Introduction

Frames in Hilbert spaces were introduced by Duffin and Schaeffer to study non-harmonic Fourier series [6], and were later reintroduced by Daubechies et al. [4] in 1986, with ongoing research in frame theory ever since. Frame theory has significantly impacted applied mathematics and engineering; indeed, certain unique properties of frames have made them a crucial aspect of functional analysis [7, 8, 12].

Over the years, various extensions of frame theory have been explored. In the elegant theory of g-frames, Sun [13] introduced g-frames as generalized frames, demonstrating that oblique frames, pseudo-frames, and fusion frames are all special cases of this framework

The concept of 2-inner product spaces was first introduced by Diminnie et al. [5], and frames in 2-Hilbert spaces were subsequently introduced by Arefijamaal and Ghadir Sadeghi [1].

To achieve this, it is first necessary to prove the Riesz representation theorem in 2-Hilbert spaces. Finally, we will establish the pre-frame and frame operators for g-2-frames.

Our work is organized as follows: In the preliminaries, we recall the foundational concepts of frames, g-frames, 2-Hilbert spaces, and 2-frames. In the main results section, we first establish a Riesz representation theorem in 2-Hilbert spaces. This theorem allows us to define g-2-frames as a natural generalization of 2-frames, analogous to how g-frames generalize classical frames in Hilbert spaces, and to study the essential properties of these sequences. Finally, the latter part of this section is devoted to characterizing these generalized 2-frames through their analysis and frame operators, establishing key results, and outlining their most important structural properties.

2 Notation and Preliminaries

Throughout this paper,we will use the following notations:

- $\mathbb{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$.
- $(\cdot,\cdot)$ is the notation used for inner product.
- $(H,(\cdot,\cdot))$ is the notation used for Hilbert space.

- $\ell^2(\mathbb{N})$ refers to the Hilbert space , defined by:

$$\ell^2(\mathbb{N}) = \{(u_i)_{i\in\mathbb{N}} \mid u_i \in \mathbb{C}, \sum_{i\in\mathbb{N}} |u_i|^2 < \infty\}$$

- $\{H_i\}_{i\in\mathbb{N}}$ denotes a family of separable Hilbert spaces.
- $\left(\bigoplus_{i\in\mathbb{N}} H_i\right)_{\ell^2}$ denotes the external Hilbert direct sum of the Hilbert spaces H_i, defined by:

$$\left(\bigoplus_{i\in\mathbb{N}} H_i\right)_{\ell^2} = \left\{(h_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} H_i \;\middle|\; \sum_{i\in\mathbb{N}} \|h_i\|^2 < \infty\right\}$$

- $B(H, H_i)$ is the collection of all bounded linear operators from H to H_i.
- $\mathcal{X}$ is the notation used for linear space of dimension greater than 1 over the field $\mathbb{K}$.
- $\langle\cdot,\cdot|\cdot\rangle$ is the notation used for 2-inner product on $\mathcal{X}$.
- $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ is the notation used for 2-Hilbert space.
- $\mathcal{L}_\eta$ is the notation used for the subspace generated by a fixed element $\eta \in \mathcal{H}$.
- $\mathcal{M}_\eta$ is the notation used for the algebraic complement of $\mathcal{L}_\eta$ in $\mathcal{H}$.
- $\mathcal{H}_\eta$ is the notation used for the completion of the inner product space $\mathcal{M}_\eta$.
- $\tilde{\mathcal{H}}$ is the notation used for the quotient space $\mathcal{H}/\mathcal{L}_\eta$.
- $\tilde{g}$ is the notation for the class $g + \mathcal{L}_\eta \in \tilde{\mathcal{H}}$ represented by $g \in \mathcal{H}$.
- $B_\eta(\mathcal{H}, H_i)$ denotes the space of all η-bounded linear operators from $\mathcal{H}$ to H_i, equipped with the norm defined for all $\tau \in B_\eta(\mathcal{H}, H_i)$ by:

$$\|\tau\| = \sup\{\|\tau(f)\|_{H_i} \mid f \in \mathcal{H},\ \|f, \eta\| = 1\}$$

In this section, we review some basic facts about frames and g-frames in a Hilbert space H, as well as 2-frames in a 2-Hilbert space $\mathcal{H}$ from [9, 10, 11], which are necessary for our discussion.

Definition 2.1. [10, 11] Let H be a Hilbert space with inner product $(\cdot,\cdot)$. A sequence $F = \{f_n\}_{n\in\mathbb{N}} \subset H$ is called a frame for H if there exist two constants A, B with $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{n\in\mathbb{N}} |(f, f_n)|^2 \leqslant B\|f\|^2, \quad \forall f \in H. \tag{2.1}$$

The constants A and B are referred to as the lower and upper frame bounds, respectively for the frame $\{f_n\}_{n\in\mathbb{N}}$

$F = \{f_n\}_{n\in\mathbb{N}}$ is called a tight frame if $A = B$.

$F = \{f_n\}_{n\in\mathbb{N}}$ is called a Bessel sequence if only if the right-hand side inequality is required in (2.1).

Given a frame $F = \{f_n\}_{n\in\mathbb{N}}$ for H, it is obviously a Bessel sequence; thus, we can define the frame operator S as follows:

$$Sf = \sum_{n\in\mathbb{N}} (f, f_n) f_n, \quad \forall f \in H. \tag{2.2}$$

Below we quote the Cauchy-Schwarz inequality in a Hilert space.

Lemma 2.2. *Let $(H, (\cdot,\cdot))$ be a Hilbert space. Then, we have*

$$|(f, g)|^2 \leq (f, f)(g, g) \quad \forall f, g \in H. \tag{2.3}$$

Definition 2.3. [9, 12] Let $(H,(\cdot,\cdot))$ be a Hilbert space, $\{H_i\}_{i\in\mathbb{N}}$ a sequence of separable Hilbert spaces, and $B(H,H_i)$ the set of all bounded linear operators from H to H_i. A sequence $\Lambda=\{\Lambda_i\in B(H,H_i):i\in\mathbb{N}\}$ is called a generalized frame, or simply a g-frame, for H with respect to $\{H_i\}_{i\in\mathbb{N}}$ if there exist two constants A,B with $0<A\leq B<\infty$ such that

$$A\|f\|^2\leqslant\sum_{i\in\mathbb{N}}\|\Lambda_i f\|^2\leqslant B\|f\|^2,\quad\forall f\in H. \tag{2.4}$$

The constants A and B are referred to as the lower and upper frame bounds, respectively for the g-frame $\{\Lambda_i\}_{i\in\mathbb{N}}$.

$\Lambda=\{\Lambda_i\}_{i\in\mathbb{N}}$ is called a tight g-frame if $A=B$.

$\Lambda=\{\Lambda_i\}_{i\in\mathbb{N}}$ is termed a g-Bessel sequence if only the upper bound the right-hand side inequality in (2.4) is required.

Definition 2.4. [1] The g-synthesis operator T_Λ and the g-frame operator S_Λ of a g-frame for H w.r.t $\{H_i\}_{i\in\mathbb{N}}$ $\Lambda=\{\Lambda_i\}_{i\in\mathbb{N}}$ are defined, respectively, by:

$$T_\Lambda:\left(\bigoplus_{i\in\mathbb{N}}H_i\right)_{\ell^2}\to H,\quad T_\Lambda\big((h_i)_{i\in\mathbb{N}}\big)=\sum_{i\in\mathbb{N}}\Lambda_i^*h_i. \tag{2.5}$$

and

$$S_\Lambda:H\to H,\quad S_\Lambda f=T_\Lambda T_\Lambda^*f=\sum_{i\in\mathbb{N}}\Lambda_i^*\Lambda_i f. \tag{2.6}$$

We recall here the basic definitions of 2-normed spaces and 2-inner product spaces as presented in [1, 10].

Definition 2.5. [1] Let $\mathcal{X}$ be a linear space of dimension greater than 1 over the field $\mathbb{K}$. A $\mathbb{K}$-valued function $\langle\cdot,\cdot\mid\cdot\rangle$ defined on $\mathcal{X}\times\mathcal{X}\times\mathcal{X}$ is called a 2-inner product on $\mathcal{X}$ if it satisfies the following conditions:

(1) $\langle f+f',g\mid h\rangle=\langle f,g\mid h\rangle+\langle f',g\mid h\rangle$ for all $f,f',g,h\in\mathcal{X}$;

(2) $\langle\alpha f,g\mid h\rangle=\alpha\langle f,g\mid h\rangle$ for all $f,g,h\in\mathcal{X}$ and $\alpha\in\mathbb{K}$;

(3) $\langle g,f\mid h\rangle=\overline{\langle f,g\mid h\rangle}$ for all $f,g,h\in\mathcal{X}$;

(4) $\langle f,f\mid g\rangle=\langle g,g\mid f\rangle$ for all $f,g\in\mathcal{X}$;

(5) $\langle f,f\mid g\rangle\geqslant 0$ for all $f,g\in\mathcal{X}$, and $\langle f,f\mid g\rangle=0$ if and only if f and g are linearly dependent.

$(\mathcal{X},\langle\cdot,\cdot\mid\cdot\rangle)$ is referred to as a 2-inner product space.

Definition 2.6. [1] Let $\mathcal{X}$ be a linear space of dimension greater than 1 over the field $\mathbb{K}$. The function $\|\cdot,\cdot\|$ defined on $\mathcal{X}\times\mathcal{X}$ by:

$$\|f,h\|^2=\langle f,f\mid h\rangle,\quad\forall f,h\in\mathcal{X}, \tag{2.7}$$

satisfies the following conditions:

(1) $\|f,g\|\geq 0$ and $\|f,g\|=0$ if and only if f and g are linearly dependent vectors.

(2) $\|f,g\|=\|g,f\|$ for all $f,g\in\mathcal{X}$.

(3) $\|\alpha f,g\|=|\alpha|\|f,g\|$ for all $f,g\in\mathcal{X}$ and $\alpha\in\mathbb{K}$.

(4)) $\|f+f',g\|\leq\|f,g\|+\|f',g\|$ for all $f,f',g\in\mathcal{X}$.

The function $\|\cdot,\cdot\|$ is called a 2-norm on $\mathcal{X}$.

$(\mathcal{X},\|\cdot,\cdot\|)$ is referred to as a linear 2-normed space.

Given a 2-inner product space $(\mathcal{X}, \langle\cdot,\cdot \mid \cdot\rangle)$, it is naturally considered as a linear 2-normed space $(\mathcal{X}, \|\cdot,\cdot\|)$ equipped with the 2-norm defined by (2.7). Using the properties stated above, we can prove the Cauchy-Schwarz inequality in a 2-inner product space.

Lemma 2.7. *[10] Let $(\mathcal{X}, \langle\cdot,\cdot \mid \cdot\rangle)$ be a 2-inner product space. Then, the following inequality holds:*

$$|\langle f, g \mid h\rangle|^2 \leq \langle f, f \mid h\rangle\langle g, g \mid h\rangle, \quad \forall f, g, h \in \mathcal{X}. \tag{2.8}$$

Lemma 2.8. *[1] Let $(\mathcal{X}, \langle\cdot,\cdot \mid \cdot\rangle)$ be a 2-inner product space and $f, h \in \mathcal{X}$. Then,*

$$\|f, h\| = sup\{|\langle f, g|h\rangle|; g \in \mathcal{X}, \|g, h\| = 1\}. \tag{2.9}$$

Example 2.9. [1, 2] Let $(\mathcal{X}, (\cdot,\cdot))$ be an inner product space. The $\mathbb{K}$-valued function defined on $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ by

$$\langle f, h|g\rangle = \begin{vmatrix} (f,h) & (f,g) \\ (g,h) & (g,g) \end{vmatrix} = (f,h)(g,g) - (g,h)(f,g), \tag{2.10}$$

is called the standard 2-inner product defined on $\mathcal{X}$.

$(\mathcal{X}, \langle\cdot,\cdot \mid \cdot\rangle)$ is known as the standard 2-inner product space.

Based on [1, 8], we introduce the definition of a 2-Hilbert space as follows:

Definition 2.10. [1, 8] Let $(\mathcal{X}, \langle\cdot,\cdot \mid \cdot\rangle)$ be a 2-inner product space. A sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ is said to be convergent to an element $f \in \mathcal{X}$ if

$$\lim_{n\to+\infty} \|f_n - f, \eta\| = 0, \quad \forall \eta \in \mathcal{X}.$$

A sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $N > 0$ such that

$$\|f_n - f_m, \eta\| < \varepsilon, \quad \forall m, n \geq N \quad and \quad \forall \eta \in \mathcal{X}.$$

A 2-inner product space $\mathcal{X}$ is called a 2-Hilbert space if it is complete, meaning that every Cauchy sequence in $\mathcal{X}$ converges to an element in $\mathcal{X}$.

In what follows, we will require the results from [1] as well as the subsequent notations:

- $(\mathcal{H}, \langle\cdot,\cdot \mid \cdot\rangle)$ is a 2-Hilbert space, and η is a fixed element in $\mathcal{H}$.
- $\mathcal{L}_\eta$ is the linear subspace spanned by η.
- $\mathcal{M}_\eta$ is the complement of $\mathcal{L}_\eta$ in $\mathcal{H}$, i.e., $\mathcal{L}_\eta \oplus \mathcal{M}_\eta = \mathcal{H}$.
- A semi-inner product is defined on $\mathcal{H}$ as follows:

$$(f, g)_\eta = \langle f, g \mid \eta\rangle, \quad \forall f, g \in \mathcal{H}. \tag{2.11}$$

- $\langle\cdot,\cdot \mid \eta\rangle$ induces an inner product on the quotient space $\mathcal{H}/\mathcal{L}_\eta$, defined by:

$$\langle f + \mathcal{L}_\eta, g + \mathcal{L}_\eta\rangle_\eta = (f, g)_\eta, \quad \forall f, g \in \mathcal{H}. \tag{2.12}$$

- Since $\mathcal{H}/\mathcal{L}_\eta \sim \mathcal{M}_\eta$, $\langle f, g\rangle_\eta$ defines an inner product on $\mathcal{M}_\eta$.
- $(\mathcal{M}_\eta, \|\cdot\|_\eta)$ is a normed space equipped with the following norm:

$$\|f\|_\eta^2 = (f, f)_\eta, \quad \forall f \in \mathcal{M}_\eta. \tag{2.13}$$

- $\mathcal{H}_\eta$ denotes the completion of $\mathcal{M}_\eta$ in $\mathcal{H}$.

Definition 2.11. [1] Let $(\mathcal{H}, \langle\cdot,\cdot \mid \cdot\rangle)$ be a 2-Hilbert space, and let η be a fixed element in $\mathcal{H}$. A sequence $F = \{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is called a 2-frame associated with η if there exist two constants A, B with $0 < A \leq B < \infty$ such that

$$A\|f, \eta\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n \mid \eta\rangle|^2 \leq B\|f, \eta\|^2, \quad \forall f \in \mathcal{H}. \tag{2.14}$$

The constants A, B are referred to as the lower and upper frame bounds, respectively for the 2-frame $\{f_n\}_{n\in\mathbb{N}}$.

$F=\{f_n\}_{n\in\mathbb{N}}$ is called a tight 2-frame if $A=B$.

$F=\{f_n\}_{n\in\mathbb{N}}$ is called a 2-Bessel sequence associated with η if only the right-hand side inequality is required in (2.14).

Let $F=\{f_n\}_{n\in\mathbb{N}}$ be a 2-frame for $\mathcal{H}$ associated with η, and let $\mathcal{H}_\eta$ be the completion of the inner product space $\mathcal{M}_\eta$. According to Lemma 5.1.2 in [3], the sequence $F=\{f_n\}_{n\in\mathbb{N}}$ is also a frame for $\mathcal{H}_\eta$ with the same bounds.
To summarize, we state the following theorem from [1]:

Theorem 2.12. *[1] Let $(\mathcal{H},\langle\cdot,\cdot\mid\cdot\rangle)$ be a 2-Hilbert space, and let η be a fixed element in $\mathcal{H}$. A sequence $F=\{f_n\}_{n\in\mathbb{N}}\subset\mathcal{H}$ is a 2-frame associated with η having bounds A and B if and only if it is a frame for the Hilbert space $\mathcal{H}_\eta$ with the same bounds A and B.*

Definition 2.13. [1] The 2-synthesis operator T_η and the 2-frame operator S_η of 2-Bessel sequence $F=\{f_n\}_{n\in\mathbb{N}}$ associated to η are defined as follows:

$$T_\eta:\ell^2(\mathbb{N})\to\mathcal{H}_\eta,\quad T_\eta(\{a_n\}_{n\in\mathbb{N}})=\sum_{n\in\mathbb{N}}a_nf_n,\quad\forall\{a_n\}_{n\in\mathbb{N}}\in\ell^2(\mathbb{N}),\tag{2.15}$$

$$S_\eta:\mathcal{H}_\eta\to\mathcal{H}_\eta,\quad S_\eta f=T_\eta T_\eta^*f=\sum_{n\in\mathbb{N}}\langle f,f_n\mid\eta\rangle f_n,\quad\forall f\in\mathcal{H}_\eta.\tag{2.16}$$

Moreover, if $F=\{f_n\}_{n\in\mathbb{N}}$ be a 2-frame associated with η, these operators are well-defined and bounded. Under this condition, S_η is also a self-adjoint and invertible operator.

3 Main results

In this section, we introduce the concept of g-2-frames in 2-Hilbert spaces and establish several key results; furthermore, we characterize these g-2-frames in terms of their associated operators.

3.1 G-2-frames in 2-Hilbert spaces

Firstly, we need to prove the Riesz representation theorem in 2-Hilbert space as if below in order to be able to define the adjoint of an operator of $B_\eta(\mathcal{H},H)$ thus, generalize 2-frames in 2-Hilbert spaces. For this reason, we remind some definitions from [8].

Definition 3.1. [8] Let $\mathcal{X}$ be a vector space over $\mathbb{K}$ with $\dim\mathcal{X}>1$, and let $\eta\in\mathcal{X}$. Two elements $x,y\in\mathcal{X}$ are said to be η-congruent if and only if $x-y\in\mathcal{L}_\eta$, where $\mathcal{L}_\eta$ is the linear subspace spanned by η.

Definition 3.2. [8] Let $\mathcal{H}$ be a 2-Hilbert space, let η be a fixed non-zero element in $\mathcal{H}$, and $(H,(\cdot,\cdot))$ be a Hilbert space. A linear operator $\tau:\mathcal{H}\to H$ is said to be η-bounded if there exists a constant $M>0$ such that:

$$\|\tau f\|\le M\|f,\eta\|,\quad\forall f\in\mathcal{H}.\tag{3.1}$$

This is equivalent to saying that its η-operator norm is finite, which is defined by:

$$\|\tau\|=\sup_{\|f,\eta\|=1}\|\tau f\|<+\infty.\tag{3.2}$$

We denote by $B_\eta(\mathcal{H},H)$ the space of all η-bounded linear operators from $\mathcal{H}$ into H.

Lemma 3.3. *Let $(\mathcal{H},\langle\cdot,\cdot\mid\cdot\rangle)$ be a 2-Hilbert space, η a fixed non-zero element of $\mathcal{H}$. We define a norm $\|\cdot\|_\eta$ on the quotient space $\tilde{\mathcal{H}}=\mathcal{H}/\mathcal{L}_\eta$ by:*

$$\|\tilde f\|_\eta=\|f+\mathcal{L}_\eta\|_\eta=d(f,\mathcal{L}_\eta)=\inf_{g\in\mathcal{L}_\eta}\|f-g,\eta\|=\|f,\eta\|,\quad where\tilde f\in\tilde{\mathcal{H}}.\tag{3.3}$$

Moreover, $(\tilde{\mathcal{H}},\|\cdot\|_\eta)$ is a Hilbert space.

Proof. First of all, $\|\cdot\|_\eta$ is well-defined; indeed, let $\tilde{f}, \tilde{f}' \in \tilde{\mathcal{H}}$ such that

$$\tilde{f} = \tilde{f}',$$

Applying (2.12), (3.3), and the fact that $f' - f \in \mathcal{L}_\eta$, we obtain:

$$\|\tilde{f}\|_\eta = \|f, \eta\| = \|f + f' - f, \eta\| = \|f', \eta\|) = \|\tilde{f}'\|_\eta.$$

Let us show that $\|\cdot\|_\eta$ defines a norm on the quotient space $\tilde{\mathcal{H}}$.
Let $\tilde{f} = f + \mathcal{L}_\eta$ be an element of $\tilde{\mathcal{H}}$ such that $\|\tilde{f}\|_\eta = 0$. By the definition of this norm, we have

$$\langle f + \mathcal{L}_\eta, f + \mathcal{L}_\eta\rangle_\eta = 0.$$

According to relation (2.12), the above equality is equivalent to:

$$\langle f, f\rangle_\eta = 0, \quad \text{i.e.,} \quad \tilde{f} \in \mathcal{L}_\eta.$$

Let $\lambda \in \mathbb{K}$ and let $\tilde{f} \in \tilde{\mathcal{H}}$, we have

$$\|\lambda\tilde{f}\|_\eta = \|\tilde{\lambda f}\|_\eta = \|\lambda f, \eta\| = |\lambda|\|f, \eta\| = |\lambda|\|\tilde{f}\|_\eta.$$

Soient $\tilde{f}, \tilde{f}' \in \tilde{\mathcal{H}}$, Since $\|f + f', \eta\| \leqslant \|f, \eta\| + \|f', \eta\|$, it follows that

$$\begin{aligned}\|\tilde{f} + \tilde{f}'\|_\eta &= \|f + f' + \mathcal{L}_\eta\|_\eta \\ &= \|f + f', \eta\| \\ &\leqslant \|f, \eta\| + \|f', \eta\| \\ &\leqslant \|\tilde{f}\|_\eta + \|\tilde{f}'\|_\eta.\end{aligned}$$

Let us prove that $\tilde{\mathcal{H}}$ is complete.
Let $\{\tilde{g}_n\}_{n\geqslant 0}$ be a Cauchy sequence in $\tilde{\mathcal{H}}$; therefore, for every $n \in \mathbb{N}$, there exists an index $\varphi(n)$ such that

$$\text{for } p, q \geqslant \varphi(n),\ \|\tilde{g}_p - \tilde{g}_q\|_\eta \leqslant \tfrac{1}{2^n}.$$

Noting that such a function φ can always be chosen, it suffices to replace φ with a function φ' defined by

$$\varphi'(n+1) = max\{\varphi(n+1), \varphi'(n) + 1\}$$

It follows that $\forall n \in \mathbb{N}, \|\tilde{g}_{\varphi(n+1)} - \tilde{g}_{\varphi(n)}\|_\eta \leqslant \frac{1}{2^n}$, that is,

$$\inf_{f\in\mathcal{L}_\eta} \|(g_{\varphi(n+1)} - g_{\varphi(n)}) - f, \eta\| \leqslant \frac{1}{2^n}.$$

Soit $f_n \in \mathcal{L}_\eta$ such that

$$\|(g_{\varphi(n+1)} - g_{\varphi(n)}) - f_n, \eta\| \leqslant \|\tilde{g}_{\varphi(n+1)} - \tilde{g}_{\varphi(n)}\|_\eta + \frac{1}{2^n},$$

and letting $h_n = g_{\varphi(n)} + \sum_{i=1}^{n-1} f_i$, we obtain

$$\begin{aligned}\sum_{n\geq 0} \|h_{n+1} - h_n, \eta\| &= \sum_{n\geqslant 0} \| \left(g_{\varphi(n+1)} + \sum_{i=1} f_i\right) - \left(g_{\varphi(n)} + \sum_{i=1}^{n-1} f_i\right), \eta\| \\ &= \sum_{n\geqslant 0} \|(g_{\varphi(n+1)} - g_{\varphi(n)}) - f_n, \eta\| \leqslant \sum_{n\geqslant 0} \frac{1}{2}^{n-1} < \infty\end{aligned}$$

hence, the series $\sum_{n\geqslant 0}(h_{n+1} - h_n)$ converges in $(\mathcal{H}, \|\cdot, \eta\|)$, which is a 2-Hilbert space; that is,

$$\lim_{n\to+\infty} \|h_n - a, \eta\| = 0, a \in \mathcal{H},$$

We conclude that

$$\|\tilde{g}_{\varphi(n+1)} - \tilde{a}, \eta\| = \inf_{f\in\mathcal{L}_\eta} \|(g_{\varphi(n+1)} - a) - f,\ \eta\| \leqslant \|(g_{\varphi(n+1)} - a) - \sum_{i=1}^{n-1} f_i, \eta\| = \|h_n - a, \eta\|$$

that is,

$$\|\tilde{g}_{\varphi(n+1)} - \tilde{a}\|_\eta \to 0,$$

et comme $\tilde{g}_n$ is a Cauchy sequence in $(\tilde{\mathcal{H}}, \|\cdot\|_\eta)$, alors

$$\lim_{n\to+\infty} \|\tilde{g}_n - \tilde{a}\|_\eta = 0,$$

which yields the result. □

Lemma 3.4. *Let $(\mathcal{H}, \langle\cdot,\cdot \mid \cdot\rangle)$ be a 2-Hilbert space, let η be a fixed element in $\mathcal{H}$, and let $(H, (\cdot,\cdot))$ be a Hilbert space. If $\tau \in B_\eta(\mathcal{H}, H)$, then the mapping $\tilde{\tau} : \tilde{\mathcal{H}} \to H$ defined by*

$$\tilde{\tau}(\tilde{f}) = \tilde{\tau}(f + \mathcal{L}_\eta) = \tau(f), \quad \forall f \in \mathcal{H}, \tag{3.4}$$

is well-defined and belongs to $B(\tilde{\mathcal{H}}, H)$.

Proof. Since, $\mathcal{L}_\eta \oplus \mathcal{M}_\eta = \mathcal{H}$, we have for all $f, g \in \mathcal{H}$, there exist two unique elements $f_\eta, g_\eta \in \mathcal{M}_\eta$, and $\alpha, \beta \in \mathbb{K}$ such that

$$f = f_\eta + \alpha\eta, \quad g = g_\eta + \beta\eta.$$

$\overline{\tau}$ is well-defined; Let $f, g \in \mathcal{H}$ such that

$$f + \mathcal{L}_\eta = g + \mathcal{L}_\eta.$$

This implies that $f - g \in \mathcal{L}_\eta$. Assuming f_η, g_η are the representative elements such that $f_\eta - g_\eta \in \mathcal{M}_\eta \cap \mathcal{L}_\eta$, and since $\mathcal{M}_\eta \cap \mathcal{L}_\eta = \{0\}$, we obtain

$$f_\eta = g_\eta.$$

Consequently, Applying τ yields

$$\tau f_\eta = \tau g_\eta,$$

which means that

$$\tilde{\tau}(f_\eta + \mathcal{L}_\eta) = \tilde{\tau}(g_\eta + \mathcal{L}_\eta).$$

Noting that $f_\eta + \mathcal{L}_\eta = f + \mathcal{L}_\eta$ and $g_\eta + \mathcal{L}_\eta = g + \mathcal{L}_\eta$, we conclude that

$$\tilde{\tau}(f + \mathcal{L}_\eta) = \tilde{\tau}(g + \mathcal{L}_\eta).$$

This ensures that $\tilde{\tau}$ is well-defined. Moreover, the linearity of $\tilde{\tau}$ follows directly from the linearity of τ. Furthermore, we can show that $\tilde{\tau}$ is a bounded operator. Indeed, for all $f \in \mathcal{H}$, and by using (2.12), we have:

$$\begin{aligned}\|\tilde{\tau}(f + \mathcal{L}_\eta)\|^2 &= \|\tau f_\eta\|^2 \\ &\leq \|\tau\|^2 \|f_\eta\|_\eta^2 \\ &\leq \|\tau\|^2 \|f + \mathcal{L}_\eta\|_\eta^2\end{aligned}$$

Taking the square root on both sides yields:

$$\|\tilde{\tau}(\tilde{f})\| \leq \|\tau\| \|\tilde{f}\|_\eta.$$

Hence, $\tilde{\tau}$ is bounded, meaning that $\tilde{\tau} \in B(\tilde{\mathcal{H}}, H)$. □

Theorem 3.5. *Let $(\mathcal{H}, \langle\cdot,\cdot \mid \cdot\rangle)$ be a 2-Hilbert space, let η be a fixed element in $\mathcal{H}$, and $\tau \in B_\eta(\mathcal{H}, \mathbb{K})$ a η-bounded linear functional. Then, there exists a unique element $g_\eta \in \mathcal{H}$ such that:*

$$\tau(f) = \langle f, g_\eta \mid \eta\rangle, \quad \forall f \in \mathcal{H}. \tag{3.5}$$

Proof. By Lemma 3.3 the quotient space $\tilde{\mathcal{H}} = \mathcal{H}/\mathcal{L}_\eta$ is a Hilbert space. Applying Lemma 3.4 yields

$$\overline{\tau} \in B(\tilde{\mathcal{H}}, \mathbb{K}).$$

Then, by applying the Riesz Representation Theorem to $\overline{\tau} \in B(\tilde{\mathcal{H}}, \mathbb{K})$, there exists an element $g \in \mathcal{H}$ unique up to η-congruence such that:

$$\overline{\tau}(f + \mathcal{L}_\eta) = \langle f + \mathcal{L}_\eta, g + \mathcal{L}_\eta \mid \eta \rangle, \quad \forall f \in \mathcal{H}.$$

Since $g \in \mathcal{H}$, there exists an unique element $g_\eta \in \mathcal{M}_\eta$ and $\alpha \in \mathbb{K}$ such that

$$g = g_\eta + \alpha\eta, \quad g + \mathcal{L}_\eta = g_\eta + \mathcal{L}_\eta.$$

Using the relation above and (2.12), we obtain

$$\begin{aligned} \forall f \in \mathcal{H}, \quad \tau(f) &= \overline{\tau}(f + \mathcal{L}_\eta) \\ &= \langle f + \mathcal{L}_\eta, g + \mathcal{L}_\eta | \eta \rangle \\ &= \langle f + \mathcal{L}_\eta, g_\eta + \mathcal{L}_\eta | \eta \rangle \\ &= \langle f, g_\eta | \eta \rangle, \end{aligned}$$

which ensures that there exists a unique element $g_\eta \in \mathcal{M}_\eta$ such that

$$\tau(f) = \langle f, g_\eta \mid \eta \rangle, \quad \forall f \in \mathcal{H}.$$

This completes the proof of the theorem. □

Secondly, we introduce the concept of g-2-frames in a 2-Hilbert space $\mathcal{H}$ and establish some of their key properties.

Definition 3.6. Let $(\mathcal{H}, \langle \cdot, \cdot \mid \cdot \rangle)$ be a 2-Hilbert space, let η be a fixed element in $\mathcal{H}$, and let $\{H_i\}_{i\in\mathbb{N}}$ be a family of separable Hilbert spaces. A sequence $\Lambda = \{\Lambda_i \in B_\eta(\mathcal{H}, H_i) : i \in \mathbb{N}\}$ is called a generalized 2-frame, or simply a g-2-frame for $\mathcal{H}$, with respect to $\{H_i\}_{i\in\mathbb{N}}$ (or briefly w.r.t $\{H_i\}_{i\in\mathbb{N}}$) associated with η if there exist two constants A, B with $0 < A \leq B < \infty$ such that

$$A\|f, \eta\|^2 \leq \sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2 \leq B\|f, \eta\|^2, \quad \forall f \in \mathcal{H}. \tag{3.6}$$

The constants A, B are referred to as the lower and upper frame bounds, respectively for the g-2-frame $\{\Lambda_i\}_{i\in\mathbb{N}}$.

$\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ is called a tight g-2-frame if $A = B$.

$\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ is called a g-2-Bessel sequence associated with η if only the right-hand side inequality is required in (3.6).

Theorem 3.7. *A 2-frame is equivalent to a g-2-frame whenever $H_i = \mathbb{C}$, for each $i \in \mathbb{N}$.*

Proof. Let $F = \{f_i\}_{i\in\mathbb{N}}$ a 2-frame associated with $\eta \in \mathcal{H}$ with bounds A and B, then from (2.14), we have

$$A\|f, \eta\|^2 \leqslant \sum_{i\in\mathbb{N}} |\langle f, f_i | \eta \rangle|^2 \leqslant B\|f, \eta\|^2 \quad \forall f \in \mathcal{H}.$$

For each $i \in \mathbb{N}$, we consider the linear functional

$$\Lambda_i : \mathcal{H} \to \mathbb{C}, \quad \Lambda_i(f) = \langle f, f_i \mid \eta \rangle.$$

This functional is bounded and satisfies $\|\Lambda_i\| \leq \|f_i, \eta\|$, where the norm is defined by

$$\|\Lambda_i\| = \sup \{|\Lambda_i(f)| \mid f \in \mathcal{H}, \ \|f, \eta\| = 1\}.$$

Consequently, there exist two constants A, B with $0 < A \leq B < \infty$ such that

$$A\|f, \eta\|^2 \leq \sum_{i\in\mathbb{N}} |\Lambda_i(f)|^2 \leq B\|f, \eta\|^2 \quad \forall f \in \mathcal{H}.$$

Conversely, suppose that $\Lambda = \{\Lambda_i \in B_\eta(\mathcal{H}, \mathbb{C}) : i \in \mathbb{N}\}$ is a g-2-frame for $\mathcal{H}$ w.r.t $\mathbb{C}$ associated with η. It follows from equations (3.6) that there exist two constants A, B with $0 < A \leq B < \infty$ such that

$$A\|f, \eta\|^2 \leq \sum_{i\in\mathbb{N}} |\Lambda_i(f)|^2 \leq B\|f, \eta\|^2 \quad \forall f \in \mathcal{H}.$$

Applying Theorem 3.5 to each $\Lambda_i \in B_\eta(\mathcal{H}, \mathbb{C})$ shows that there exists a unique $f_i \in \mathcal{H}$ such that

$$\Lambda_i(f) = \langle f, f_i \mid \eta\rangle \quad \forall f \in \mathcal{H}.$$

then, there exist two constants A, B with $0 < A \leq B < \infty$ such that

$$A\|f, \eta\|^2 \leq \sum_{i\in\mathbb{N}} |\langle f, f_i/\eta\rangle|^2 \leq B\|f, \eta\|^2 \quad \forall f \in \mathcal{H},$$

which completes the proof. □

Proposition 3.8. *Let $(H, (\cdot,\cdot))$ be a Hilbert space, let η be a fixed element in H with $\|\eta\| = 1$, and let $\{H_i\}_{i\in\mathbb{N}}$ be a family of separable Hilbert spaces. Every g-2-frame for the standard 2-inner product space $(H, \langle\cdot,\cdot \mid \cdot\rangle)$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$ associated with η is a g-frame for $L_\eta^\perp$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$, where L_η is the subspace generated by η.*

Proof. If $\Lambda = \{\Lambda_i \in B_\eta(\mathcal{H}, H_i) : i \in \mathbb{N}\}$ is g-2-frame for the standard 2-inner product space $(H, \langle\cdot,\cdot \mid \cdot\rangle)$, w.r.t $\{H_i\}_{i\in\mathbb{N}}$ associated with η, then (3.6) implies, there exist two constants A and B with $0 < A \leq B < \infty$ such that

$$A\left(\|f\|^2 - |(f, \eta)|^2\right) \leq \sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2 \leq B\left(\|f\|^2 - |(f, \eta)|^2\right), \quad \forall f \in H.$$

Consequently, if $f \in L_\eta^\perp$, then $(f, \eta) = 0$, which reduces the inequality to standard g-frame bounds. Therefore, $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-frame for the Hilbert space $L_\eta^\perp$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$, where

$$L_\eta^\perp = \{f \in H \mid (f, g) = 0, \forall g \in L_\eta\}.$$

□

Theorem 3.9. *Let $(\mathcal{H}, \langle\cdot,\cdot \mid \cdot\rangle)$ be a 2-Hilbert space, let η be a fixed element in $\mathcal{H}$, and let $\{H_i\}_{i\in\mathbb{N}}$ be a family of separable Hilbert spaces. A sequence $\Lambda = \{\Lambda_i \in B_\eta(\mathcal{H}, H_i) : i \in \mathbb{N}\}$ is a g-2-frame for $\mathcal{H}$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$ associated with η if and only if Λ is a g-frame for a Hilbert space $\mathcal{H}_\eta$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$.*

Proof. By considering the operator $\tilde{\Lambda}_i \in B(\tilde{\mathcal{H}}, H_i)$ defined in (3.4), we have

$$\Lambda_i(f) = \tilde{\Lambda}_i(f + \mathcal{L}_\eta) = \tilde{\Lambda}_i(f_\eta + \mathcal{L}_\eta), \quad \forall f = f_\eta + \alpha\eta \in \mathcal{H}.$$

Applying (3.6) to $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$, there exist two constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f, \eta\|^2 \leq \sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2 \leq B\|f, \eta\|^2, \quad \forall f \in \mathcal{H},$$

which is equivalent to

$$A\|f_\eta + \alpha\eta, \eta\|^2 \leq \sum_{i\in\mathbb{N}} \|\tilde{\Lambda}_i(f_\eta + \mathcal{L}_\eta)\|^2 \leq B\|f_\eta + \alpha\eta, \eta\|^2, \quad \forall f_\eta \in \mathcal{M}_\eta.$$

By virtue of (2.12), this simplifies to

$$A\|f_\eta\|_\eta^2 \leq \sum_{i\in\mathbb{N}} \|\Lambda_i f_\eta\|^2 \leq B\|f_\eta\|_\eta^2, \quad \forall f_\eta \in \mathcal{M}_\eta.$$

Since $\mathcal{M}_\eta$ is dense in $\mathcal{H}_\eta$, for any $f \in \mathcal{H}_\eta$ there exists a sequence $(f_n) \subset \mathcal{M}_\eta$ such that $\lim_{n\to+\infty} \|f_n - f\|_\eta = 0$. Using the standard triangle inequality and the frame bounds on $\mathcal{M}_\eta$, we obtain:

$$\begin{aligned} A\|f\|_\eta^2 &\leq 2A\|f_n\|_\eta^2 + 2A\|f - f_n\|_\eta^2 \\ &\leq 2\sum_{i\in\mathbb{N}} \|\Lambda_i f_n\|^2 + 2A\|f - f_n\|_\eta^2 \\ &= 2\sum_{i\in\mathbb{N}} \|\Lambda_i(f_n - f) + \Lambda_i f\|^2 + 2A\|f - f_n\|_\eta^2 \\ &\leq 4\sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2 + 4\sum_{i\in\mathbb{N}} \|\Lambda_i(f_n - f)\|^2 + 2A\|f - f_n\|_\eta^2 \\ &\leq 4\sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2 + (4B + 2A)\|f - f_n\|_\eta^2. \end{aligned}$$

Taking the limit as $n \to +\infty$, the term $\|f - f_n\|_\eta^2$ vanishes, yielding:

$$\frac{A}{4}\|f\|_\eta^2 \leq \sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2, \quad \forall f \in \mathcal{H}_\eta.$$

The upper bound $\sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2 \leq 2B\|f\|_\eta^2$ follows similarly by density. Consequently, $\Lambda = \{\Lambda_i \in B_\eta(\mathcal{H}, H_i)\}_{i\in\mathbb{N}}$ is a g-frame for $\mathcal{H}_\eta$ with respect to $\{H_i\}_{i\in\mathbb{N}}$. □

Remark 3.10. By Theorem 3.9 every question about g-2-frame for 2-Hilbert space $\mathcal{H}$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$ associated with η, can be solved as a question about g-frame for the Hilbert space $\mathcal{H}_\eta$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$.

3.2 Operators for g-2-frames

This subsection is dedicated to introducing the synthesis and frame operators for g-2-frames in 2-Hilbert spaces. First, we will prove the existence of the adjoint operator for a η-bounded linear operator from $\mathcal{H}$ to H, where η is a fixed element in $\mathcal{H}$.

Proposition 3.11. *Let $(\mathcal{H}, \langle\cdot,\cdot \mid \cdot\rangle)$ be a 2-Hilbert space, let η be a fixed element of $\mathcal{H}$, and let $(H, (\cdot,\cdot))$ be a Hilbert space. For every $\tau \in B_\eta(\mathcal{H}, H)$, there exists a unique operator $\tau^* \in B(H, \mathcal{H})$ such that*

$$(\tau(f), g) = \langle f, \tau^*(g) \mid \eta\rangle, \quad \forall f \in \mathcal{H}, \quad \forall g \in H. \tag{3.7}$$

Moreover, $\|\tau^*\| = \|\tau\|$.

Proof. For each fixed element $g \in H$, we consider the bounded linear functional ψ_g defined on the quotient space $\mathcal{H}/\mathcal{L}_\eta$ as follows:

$$\psi_g : \mathcal{H}/\mathcal{L}\eta \to \mathbb{C}, \quad \psi_g(f + \mathcal{L}\eta) = (\tilde{\tau}(f + \mathcal{L}\eta), g). \tag{3.8}$$

By applying the Riesz Representation Theorem to $\psi_g \in B(\tilde{\mathcal{H}}, \mathbb{C})$, there exists a unique $h_g \in \mathcal{H}$ up to η-congruence such that

$$\psi_g(f + \mathcal{L}_\eta) = \langle f + \mathcal{L}_\eta, h_g + \mathcal{L}_\eta|\eta\rangle \quad \forall f \in \mathcal{H}. \tag{3.9}$$

Using (3.4), we have

$$(\tau(f), g) = \langle f + \mathcal{L}_\eta, h_g + \mathcal{L}_\eta|\eta\rangle \quad \forall f \in \mathcal{H}.$$

Since $h_g \in \mathcal{H}$, there exists a unique element $(h_g)_\eta \in \mathcal{M}_\eta$ such that $h_g = (h_g)_\eta + \alpha\eta$ for some $\alpha \in \mathbb{K}$. We then define the mapping τ^* as follows:

$$\tau^* : H \to \mathcal{H}, \quad \tau^*(g) = (h_g)\eta. \tag{3.10}$$

To show that τ^* is well-defined, let $g, g' \in H$ such that $g = g'$. This implies that

$$hg - h_{g'} \in \mathcal{L}\eta,$$

which is equivalent to

$$(hg)\eta - (hg')\eta \in \mathcal{L}\eta.$$

Since $\mathcal{H} = \mathcal{L}_\eta \oplus \mathcal{M}_\eta$, it follows that $(h_g)_\eta - (h_{g'})_\eta \in \mathcal{L}_\eta \cap \mathcal{M}_\eta = \{0\}$, which yields

$$(h_g)\eta = (hg')_\eta.$$

This proves that $\tau^*(g) = \tau^*(g')$, confirming that τ^* is well-defined.
To show that τ^* is a linear operator, let $g, g' \in H$ and $\alpha, \beta \in \mathbb{K}$. Then, for any $f \in \mathcal{H}$, we have

$$\begin{aligned}
\langle f + \mathcal{L}_\eta, (h_{\alpha g+\beta g'})_\eta + \mathcal{L}_\eta|\eta\rangle &= (\tau f, \alpha g + \beta g') \\
&= \overline{\alpha}\langle f + \mathcal{L}_\eta, (h_g)_\eta + \mathcal{L}_\eta|\eta\rangle + \overline{\beta}\langle f + \mathcal{L}_\eta, (h_{g'})_\eta + \mathcal{L}_\eta|\eta\rangle \\
&= \overline{\alpha}\langle f, \tau^*(g) + \mathcal{L}_\eta|\eta\rangle + \overline{\beta}\langle f, \tau^*(g') + \mathcal{L}_\eta|\eta\rangle \\
&= \langle f, \alpha\tau^*(g) + \mathcal{L}_\eta|\eta\rangle + \langle f, \beta\tau^*(g') + \mathcal{L}_\eta|\eta\rangle \\
&= \langle f + \mathcal{L}_\eta, \alpha\tau^*(g) + \beta\tau^*(g') + \mathcal{L}_\eta|\eta\rangle,
\end{aligned}$$

since $\alpha\tau^*(g) + \beta\tau^*(g') \in \mathcal{M}_\eta$ and due to the uniqueness of $(h_{\alpha g+\beta g'})_\eta \in \mathcal{M}_\eta$, we get

$$\tilde{\tau}^*(\alpha g + \beta g') = \alpha\tilde{\tau}^*(g) + \beta\tilde{\tau}^*(g'), \quad \forall g, g' \in \mathcal{H} \quad \forall \alpha, \beta \in \mathbb{K}.$$

Using (2.12), we have

$$(\tau(f), g) = \langle f + \mathcal{L}_\eta, \tau^*(g) + \mathcal{L}_\eta|\eta\rangle = \langle f, \tau^*(g)|\eta\rangle, \quad \forall f \in \mathcal{H}, \quad \forall g \in H, \tag{3.11}$$

show that τ^* is bounded and that $\|\tau^*\| = \|\tau\|$. Recall that

$$\|\tau\| = \sup\{\|\tau(f)\| \mid f \in \mathcal{H}, |f, \eta\| = 1\}, \quad \|\tau^*\| = \sup\{\|\tau^*(g), \eta\| \mid g \in H, \|g\| = 1\}.$$

Substituting $f = \tau^*(g)$ into (3.11) and applying (2.3), and then substituting $g = \tau(f)$ into (3.11) and applying (2.8), we obtain:

$$\begin{cases} \|\tau^*(g), \eta\|^2 &= \langle \tau^*(g), \tau^*(g)|\eta\rangle \\ &= (\tau(\tau^* g), g) \\ &\leqslant \|\tau(\tau^* g)\|\|g\| \\ &\leqslant \|\tau\|\|\tau^*(g), \eta\|\|g\|. \end{cases} \qquad \begin{cases} \|\tau f\|^2 &= (\tau f, \tau f) \\ &= \langle f, \tau^*(\tau f)|\eta\rangle \\ &\leqslant \|f, \eta\|\|\tau^*(\tau f), \eta\| \\ &\leqslant \|f, \eta\|\|\tau^*\|\|\tau f)\|. \end{cases}$$

This proves that $\|\tau^*\| \leqslant \|\tau\|$ and $\|\tau\| \leqslant \|\tau^*\|$, which implies that $\|\tau^*\| = \|\tau\|$. Consequently, τ^* is a bounded operator, known as the η-adjoint of τ, and satisfies:

$$(\tau(f), g) = \langle f, \tau^*(g)|\eta\rangle, \forall f \in \mathcal{H}, \forall g \in H,$$

□

Remark 3.12. Let $\Lambda = \{\Lambda_i \in B_\eta(\mathcal{H}, H_i) : i \in \mathbb{N}\}$ be a g-2-Bessel sequence for $\mathcal{H}$ with respect to $\{H_i\}_{i\in\mathbb{N}}$ associated with η. For each $i \in \mathbb{N}$, there exists a unique η-adjoint operator $\Lambda_i^* \in B_\eta(H_i, \mathcal{H}))$ satisfying

$$(\Lambda_i(f), g) = \langle f, \Lambda_i^*(g) \mid \eta\rangle, \quad \forall f \in \mathcal{H}, \quad \forall g \in H_i. \tag{3.12}$$

Moreover, $\|\Lambda_i^*\| = \|\Lambda_i\|$.

Lemma 3.13. *Let $\Lambda = \{\Lambda_i \in B_\eta(\mathcal{H}, H_i) : i \in \mathbb{N}\}$ be a g-2-Bessel sequence for $\mathcal{H}$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$ associated with a fixed element $\eta \in \mathcal{H}$ with bounds A and B. Then, the synthesis operator, T_Λ of Λ, defined by*

$$T_\Lambda : \left(\bigoplus_{i\in\mathbb{N}} H_i\right)_{\ell^2} \to \mathcal{H}_\eta, \quad T_\Lambda(h_i)_{i\in\mathbb{N}} = \sum_{i\in\mathbb{N}} \Lambda_i^* h_i, \tag{3.13}$$

is well-defined and bounded.

Proof. *For* $n < m$, we have

$$\|\sum_{i=1}^{m}\Lambda_i^* h_i - \sum_{i=1}^{n}\Lambda_i^* h_i\|_\eta^2 = \|\sum_{i=n+1}^{m}\Lambda_i^* h_i\|_\eta^2.$$

Using (**??**), (2.8), (2.9), and (3.6), we obtain:

$$\begin{aligned}
\|\sum_{i=n+1}^{m}\Lambda_i^* h_i\|_\eta^2 &\leqslant \sup\{|\sum_{i=n+1}^{m}\langle\Lambda_i^* h_i, g|\eta\rangle|^2 \mid g\in\mathcal{H}_\eta, \|g\|_\eta = 1\}\\
&\leqslant \sup\{\sum_{i=n+1}^{m}|\langle\Lambda_i^* h_i, g|\eta\rangle|^2 \mid g\in\mathcal{H}_\eta, \|g\|_\eta = 1\}\\
&\leqslant \sup\{\sum_{i=n+1}^{m}|\langle g, \Lambda_i^* h_i|\eta\rangle|^2 \mid g\in\mathcal{H}_\eta, \|g\|_\eta = 1\}\\
&\leqslant \sup\{\sum_{i=n+1}^{m}|\,(\Lambda_i g, h_i)|^2 \mid g\in\mathcal{H}_\eta, \|g\|_\eta = 1\}\\
&\leqslant \sum_{i=n+1}^{m}\|h_i\|^2 \sup\{\sum_{i=n}^{m}\|\Lambda_i g\|^2 \mid g\in\mathcal{H}_\eta, \|g\|_\eta = 1\}\\
&\leqslant B\sum_{i=n+1}^{m}\|h_i\|^2.
\end{aligned}$$

Consequently, the series $\sum_{i\in\mathbb{N}}\Lambda_i^* h_i$ is well-defined as an element of $\mathcal{H}_\eta$. Since $(h_i)_{i\in\mathbb{N}}$ is a sequence in $\left(\bigoplus_{i\in\mathbb{N}}H_i\right)_{\ell^2}$, an argument similar to the one used above shows that

$$\|T_\Lambda\| \leq \sqrt{B}. \tag{3.14}$$

□

Next, we can determine the adjoint operator T_Λ^* of T_Λ as follows:

$$T_\Lambda^* \quad H_\eta \to \left(\bigoplus_{i\in\mathbb{N}}H_i\right)_{\ell^2}, \quad T_\Lambda^*(f) = (\Lambda_i f)_{i\in\mathbb{N}}. \tag{3.15}$$

From (3.6), we have

$$\sum_{i\in\mathbb{N}}\|\Lambda_i f\|^2 \leq B\|f,\eta\|^2, \quad \forall f\in\mathcal{H}. \tag{3.16}$$

This allows us to conclude that T_Λ^* is well-defined, from which we deduce that $\|T_\Lambda^*\| \leq \sqrt{B}$.

Definition 3.14. Let $\Lambda = \{\Lambda_i \in B_\eta(\mathcal{H}, H_i) : i\in\mathbb{N}\}$ be a g-2-Bessel sequence for $\mathcal{H}$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$ associated with a fixed element $\eta\in\mathcal{H}$ with bounds A and B. The g-2-frame operator S_Λ associated with Λ is defined by

$$S_\Lambda : \mathcal{H}_\eta \to \mathcal{H}_\eta, \quad S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i\in\mathbb{N}}\Lambda_i^*\Lambda_i f. \tag{3.17}$$

The operator $S_\Lambda = T_\Lambda T_\Lambda^*$ is linear by construction. Furthermore, S_Λ is bounded. Indeed, using (2.9) and (3.6), we obtain:

$$\|S_\Lambda\| = \sup_{\|f,\eta\|=1}|\langle S_\Lambda f, f|\eta\rangle| = \sup_{\|f,\eta\|=1}|\sum_{i\in\mathbb{N}}(\Lambda_i f, \Lambda_i f)| \leqslant \sup_{\|f,\eta\|=1}\sum_{i\in\mathbb{N}}\|\Lambda_i f\|^2 \leqslant B$$

Hence, S_Λ is a bounded operator with $\|S_\Lambda\| \leqslant B$. Using (3.6), we obtain:

$$A\|f,\eta\| \leqslant \|S_\Lambda f,\eta\| \leqslant B\|f,\eta\|, \quad \forall f\in\mathcal{H}_\eta. \tag{3.18}$$

Theorem 3.15. *Let $\Lambda = \{\Lambda_i \in B_\eta(\mathcal{H}, H_i) : i \in \mathbb{N}\}$ be a g-2-frame sequence for $\mathcal{H}$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$ associated with a fixed element $\eta \in \mathcal{H}$ with bounds A and B, its g-2-frame operator S_Λ is self-adjoint and invertible operator.*

Proof. It is easy to check that S_Λ is self-adjoint, for any $f, g \in \mathcal{H}_\eta$, we get

$$(S_\Lambda f, g)_\eta = \sum_{i\in\mathbb{N}} (\Lambda_i^*\Lambda_i f, g)_\eta = \sum_{i\in\mathbb{N}} (f, \Lambda_i^*\Lambda_i g)_\eta = (f, S_\Lambda g)_\eta.$$

Then S_Λ is a self-adjoint operator and we have, for any $f \in \mathcal{H}_\eta$

$$(S_\Lambda f, f)_\eta = (\sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i f, f)_\eta = \sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2. \tag{3.19}$$

Using (3.19) and (3.6), we obtain

$$A|f,\eta|^2 \leqslant \langle S_\Lambda f, f \mid \eta\rangle \leqslant B|f,\eta|^2, \quad \forall f \in \mathcal{H}_\eta. \tag{3.20}$$

Furthermore, we find that:

$$\begin{aligned}((I - \frac{1}{B}S_\Lambda)f, f)_\eta &= (f,f)_\eta - \frac{1}{B}(S_\Lambda f, f)_\eta \\ &= \|f\|_\eta^2 - \frac{1}{B}\sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2 \\ &\leq (1 - \frac{A}{B})\|f\|_\eta^2 \\ &\leq \frac{B-A}{B}\|f\|_\eta^2 \\ &\leq \|f\|_\eta^2,\end{aligned}$$

Then, we obtain:

$$\|I - \frac{1}{B}S_\Lambda\| = \sup_{\|f,\eta\|=1} |\langle (I - \frac{1}{B}S_\Lambda)f, f|\eta\rangle| \leqslant \frac{B-A}{B} < 1. \tag{3.21}$$

This implies, by the Neumann series theorem, that the operator S_Λ is invertible Substituting $S_\Lambda^{-1}f$ for f in (3.18), we obtain:

$$\frac{1}{B}\|f\|_\eta \leqslant \|S_\Lambda^{-1}f\| \leqslant \frac{1}{A}\|f\|_\eta, \quad \forall f \in \mathcal{H}_\eta. \tag{3.22}$$

and

$$\frac{1}{B}(f,f)_\eta \leqslant (S_\Lambda^{-1}f, f)_\eta \leqslant \frac{1}{A}(f,f)_\eta \quad \forall f \in \mathcal{H}_\eta. \tag{3.23}$$

□

Corollary 3.16. *Let $\Lambda = \{\Lambda_i \in B_\eta(\mathcal{H}, H_i) : i \in \mathbb{N}\}$ be a g-2-frame sequence for $\mathcal{H}$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$ associated with a fixed element $\eta \in \mathcal{H}$, and let S_Λ be its associated g-2-frame operator. Then, every $f \in \mathcal{H}_\eta$ can be represented as:*

$$f = S_\Lambda S_\Lambda^{-1} f = \sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i S_\Lambda^{-1} f. \tag{3.24}$$

$$f = S_\Lambda^{-1} S_\Lambda f = \sum_{i\in\mathbb{N}} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f. \tag{3.25}$$

Proposition 3.17. *Let $\Lambda = \{\Lambda_i \in B_\eta(\mathcal{H}, H_i) : i \in \mathbb{N}\}$ be a g-2-frame sequence for $\mathcal{H}$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$ associated with a fixed element $\eta \in \mathcal{H}$, and let S_Λ be its associated g-2-frame operator. Then, the sequence $\tilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i\in\mathbb{N}}$ is also a g-2-frame for $\mathcal{H}$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$ associated with η, where*

$$\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}, \quad \forall i \in \mathbb{N}. \tag{3.26}$$

This sequence $\tilde{\Lambda}$ is called the canonical dual g-2-frame of Λ.

Proof. Using (2.3), (3.23) , and (3.24), we have For any $f \in \mathcal{H}_\eta$,

$$\begin{cases} \|f\|_\eta^2 & = (f, \sum_{i\in\mathbb{N}} \Lambda_i^* \bar{\Lambda}_i f)_\eta \\ & = \sum_{i\in\mathbb{N}} (\Lambda_i f, \bar{\Lambda}_i f)_\eta \\ & \leqslant (\sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2)^{\frac{1}{2}} (\sum_{i\in\mathbb{N}} \|\bar{\Lambda}_i f\|^2)^{\frac{1}{2}} \\ & \leqslant \sqrt{B}\|f\|_\eta (\sum_{i\in\mathbb{N}} \|\bar{\Lambda}_i f\|^2)^{\frac{1}{2}}. \end{cases} \quad \begin{cases} \sum_{i\in\mathbb{N}} \|\bar{\Lambda}_i f\|^2 & = \sum_{i\in\mathbb{N}} (\Lambda_i S_\Lambda^{-1} f, \Lambda_i S_\Lambda^{-1} f)_\eta \\ & = (\sum_{i\in\mathbb{N}} \Lambda_i^* \bar{\Lambda}_i f, S_\Lambda^{-1} f)_\eta \\ & = (f, S_\Lambda^{-1} f)_\eta \\ & \leqslant \frac{1}{A}\|f\|_\eta^2. \end{cases}$$

which implies that:

$$\frac{1}{B}\|f\|_\eta^2 \leq \sum_{i\in\mathbb{N}} \|\bar{\Lambda}_i f\|^2 \leq \frac{1}{A}\|f\|_\eta^2, \quad \forall f \in \mathcal{H}_\eta.$$

Hence $\bar{\Lambda}$ is a g-frame for H_η w.r.t $\{H_i\}_{i\in\mathbb{N}}$ with frame bounds $\frac{1}{B}$ and $\frac{1}{A}$. Then $\bar{\Lambda}$ is a g-2-frame for $\mathcal{H}$ w.r.t $\{H_i\}_{i\in\mathbb{N}}$ associated to η with frame bounds $\frac{1}{B}$ and $\frac{1}{A}$. □

4 Conclusion remarks

In this paper, we have introduced and studied g-2-frames in 2-Hilbert spaces. We have demonstrated that these new structures preserve the essential properties of classical g-frames, while successfully generalizing the existing concept of 2-frames. By bringing these ideas into spaces with a 2-inner product, this work provides a broader and more flexible theoretical framework for studying the structure of these spaces.

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Author information

S. Tatouti, Geometry and Applications (LAGA), Department of Mathematics, Faculty of Sciences, Ibn Tofail University, Kenitra, Morocco.
E-mail: samira.tatouti@uit.ac.ma

N. Bounader, Geometry and Applications (LAGA), Department of Mathematics, Faculty of Sciences, Ibn Tofail University, Kenitra, Morocco.
E-mail: n.bounader@live.fr

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