Palestine Journal of Mathematics

Vol.15(3), (2026) , 1287–1299 © Palestine Polytechnic University-PPU 2026

Quaternionic Essential S-Spectra and Some Perturbation Results involving Demi-compactness Classes of Right linear operators

Dorsaf Kouas

Communicated by Ayman Badawi

MSC 2020 Classifications: Primary 47A10; Secondary 47A53; Tertiary 47A55.

Keywords and phrases: Quaternionic Hilbert space, demi-compact operator, matrix operator, Fredholm and semi-Fredholm operators, perturbation theory, left and right essential S-spectra.

The author would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract This paper is devoted to study and investigate the class of quaternionic demi-compact and Fredholm right linear operators in order to give and establish some new perturbation results of the left and right essential S-spectra. More precisely, we give a new characterization of the essential S-spectra of right linear operators in terms of demi-compact classes. In addition, we apply the obtained results to study the essential S-spectra of block matrix of right linear operators via Frobenius-Schur's decomposition in quaternionic Hilbert spaces.

1 Introduction

In [6], with the paper of G. Birkhoff and J. von Neumann on the logic of quantum mechanics, the authors show that quantum mechanics can be formulated also on quaternionic case. So, several authors headed to study and give rise to the development of quaternionic Quantum Mechanics. However, the development of a theory of quaternionic linear operators has been handicapped by a lack of a precise definition of the spectrum of a quaternionic linear operator since then. The principal reason for this was that from the physical point of view, there were no precise clues for the notion of spectrum for a quaternionic linear operator. Only recently, this problem was solved with the discovery of the S-spectrum for quaternionic linear operators (see [9]). The S-spectrum has been introduced for the definition of the S-functional calculus that includes both the quaternionic functional calculus and a calculus for n-tuples of non-necessarily commuting operators. The notion of right spectrum for right linear quaternionic operators has been widely used in the literature, especially in the context of quaternionic quantum mechanics. Due to the noncommutativity of the quaternions things are different, if T is a quaternionic bounded linear operator acting on a quaternionic two-sided Banach space $U_{\mathbb{H}}$ and $q \in \mathbb{H}$, where $\mathbb{H}$ is the space of all quaternions. To define the spectrum of T, we have to consider the second order operator $T^2 - 2\,\mathrm{Re}(q)T + |q|^2 I$. Then, the S-spectrum of T is defined as those quaternions $q \in \mathbb{H}$ such that the operator $T^2 - 2\,\mathrm{Re}(q)T + |q|^2 I$ is not invertible in the space of all quaternionic bounded linear operators $\mathcal{B}(U_{\mathbb{H}})$ which is denoted by $\sigma^S(T)$.
In this paper, we interest in the quaternionic operator theory and the essential S-spectra via the perturbation theory. Also, we apply these results to block matrix of right linear operators by means of demi-compactness classes and Frobenius Schur's decomposition. Note that in [4] Ammar et all extended this concept in a right quaternionic Hilbert space. In particular, they established some spectral properties of demi-compact operators. Further, it should be recalled that the Fredholm theory and the essential S-spectrum are introduced and studied by B. Muraleetharan and K. Thirulogasanthar in [19, 20]. Several authors, have given important contributions to the development of the spectral theory in the quaternionic version, see [1, 2, 3, 5, 7, 8, 9, 10, 18].

Accordingly, in this work based on this concept, we characterize and study some results on the essential S-spectrum to bounded right linear operator and we apply the obtained results to block operator matrix. More specifically, we will work on a right quaternionic Hilbert spaces. Let us consider the 2×2-block right linear operator matrix

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are bounded right linear operators on right quaternionic Hilbert spaces. In this regard, one of the objective of this work is to study the essential S-spectra of $\mathcal{L}$ via the Frobenius-Schur factorization of the pseudoresolvent operator of $\mathcal{L}$ with bounded entries. The intimate relation between the spectral properties of the block operator matrix $R_q(\mathcal{L})$ and those of its Schur complement

$$\mathcal{S}(q) = R_q(D) - T_1 R_q(A)^{-1} T_2, \quad \text{for } q \notin \sigma^S(A)$$

is obvious from the so-called Frobenius-Schur decomposition which is the following:

$$R_q(\mathcal{L}) = \begin{pmatrix} I & 0 \\ T_1 R_q(A)^{-1} & I \end{pmatrix} \begin{pmatrix} R_q(A) & 0 \\ 0 & \mathcal{S}(q) \end{pmatrix} \begin{pmatrix} I & R_q(A)^{-1} T_2 \\ 0 & I \end{pmatrix},$$

where $T_1 = CA + (D - 2q_0)\, C$ and $T_2 = BD + (A - 2q_0)\, B$. This later was introduced by A. Ammar, D. Kouas and N. Lazrag (see [5]).
The remaining part of this paper is organized as follows. Section 2 is covered some basic concepts, definitions, notations and terminology on the S-spectral theory used through out this dissertation. Section 3 is entirely devoted to study some properties and perturbation results of Fredholm operators and demi-compact operators in a right quaternionic Hilbert space. Section 4 is dedicated to develop the essential S-spectrum of bounded right linear operator and some perturbation results in association with the demi-compactness classes. Section 5 is devoted to apply the obtained results for block right quaternionic linear operator matrix via Frobenius Shcur's factorization, then we shall give the study of some perturbations results of 2×2-block right linear operator matrices.

2 Preliminary and auxiliary results

Let us start this section by recalling the basic definitions of quaternionic S-spectral theory. The space of quaternions $\mathbb{H}$ is the four dimensional real algebra with 0 its null element and 1 its multiplicative identity. This space includes three imaginary units, which we indicate by i, j, k. They satisfy

$$i^2 = j^2 = k^2 = -1, ij = k = -ji, jk = i = -kj, ki = j = -ik.$$

The elements $1, i, j$, and k are assumed to form a real vector basis of $\mathbb{H}$, so that any element $q \in \mathbb{H}$ takes the form:

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$. Given $q \in \mathbb{H}$, the quaternionic conjugate of q is $\bar{q} = q_0 - iq_1 - jq_2 - kq_3$, the usual norm of q is $|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \in \mathbb{R}$, the real part of q is $\mathrm{Re}(q) = \frac{1}{2}(q + \bar{q}) = q_0 \in \mathbb{R}$ and $\mathrm{Im}(q) = \frac{1}{2}(q - \bar{q}) = iq_1 + jq_2 + kq_3$ is the imaginary part of q. Note that

$$\mathrm{Re}(qr) = \mathrm{Re}(q)r, \quad r \in \mathbb{R} \text{ and } q \in \mathbb{H}. \tag{2.1}$$

Let $U_{\mathbb{H}}^R$ be a right vector space, that is, an abelian group with a right scalar multiplication. The inner product is a function $\langle .,. \rangle from U_{\mathbb{H}}^R \times U_{\mathbb{H}}^R$ into $\mathbb{H}$ such that for every $u, v, w \in U_{\mathbb{H}}^R$ and $p, q \in \mathbb{H}$, the following properties hold:

- $\overline{\langle u, v \rangle} = \langle v, u \rangle$.
- $\langle u, u \rangle > 0$ unless $u = 0$.

- $\langle u, vp + wq\rangle = \langle u, v\rangle p + \langle u, w\rangle q$.
- $\langle uq, v\rangle = \bar{q}\langle u, v\rangle$.
- $\|u\| = \sqrt{\langle u, u\rangle}$.

If $U_{\mathbb{H}}^R$ with the metric $\text{dist}(u, v) = \|u-v\|$ is a complete metric space, then $U_{\mathbb{H}}^R$ is said to be a right quaternionic Hilbert space. Obviously, if $U_{\mathbb{H}}^R$ and $V_{\mathbb{H}}^R$ be two right quaternionic Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$, then the product space $U_{\mathbb{H}}^R \times V_{\mathbb{H}}^R$ is still a right quaternionic Hilbert space with the following induced inner product, still denoted by $\langle\cdot,\cdot\rangle$ without any confusion:

$$\langle (u, u_1), (v, v_1)\rangle = \langle u, v\rangle + \langle u_1, v_1\rangle, \quad \text{for all } (u, u_1), (v, v_1) \in U_{\mathbb{H}}^R \times V_{\mathbb{H}}^R.$$

A right linear operator is a map $T : \mathcal{D}(T) \subseteq U_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ such that

$$T(up + vq) = (Tu)p + (Tv)q, \quad \text{if } u, v \in \mathcal{D}(T) \text{ and } p, q \in \mathbb{H},$$

where the domain $\mathcal{D}(T)$ of T is a right vector subspace of $U_{\mathbb{H}}^R$. The range and the kernel of T are defined respectively as

$$\begin{aligned}\text{ran}(T) &= \left\{y \in V_{\mathbb{H}}^R : Tu = v \text{ for } u \in \mathcal{D}(T)\right\},\\ \ker(T) &= \{u \in \mathcal{D}(T) : Tu = 0\}.\end{aligned}$$

Let T be a right linear operator, T is said to be invertible operator if, and only if, it is injective with a closed range. As in the complex case, if $T : \mathcal{D}(T) \subseteq U_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ is any right linear operator, one defines the norm of $\|T\|$ by

$$\|T\| = \sup_{u\in\mathcal{D}(T)\setminus\{0\}} \frac{\|Tu\|}{\|u\|} = \inf\{c \in \mathbb{R} : \|Tu\| \le c\|u\|, \text{ for all } u \in \mathcal{D}(T)\}.$$

T is said to be bounded right linear operator, if $\mathcal{D}(T) = U_{\mathbb{H}}^R$ and $\|T\| < \infty$. The collection of all bounded right linear operators as defined above is denoted by $\mathcal{B}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)$. If $U_{\mathbb{H}}^R = V_{\mathbb{H}}^R$, we write $\mathcal{B}\left(U_{\mathbb{H}}^R\right) = \mathcal{B}\left(U_{\mathbb{H}}^R, U_{\mathbb{H}}^R\right)$.
Note that $\mathcal{B}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)$ has a natural structure of real algebra, in which the sum is the usual pointwise sum, the product is the composition and the real scalar multiplication

$$\begin{array}{ccc}\mathcal{B}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right) \times \mathbb{R} & \longrightarrow & \mathcal{B}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)\\ (T, \lambda) & \longmapsto & T\lambda\end{array}$$

is defined by setting

$$(T\lambda)(u) = T(u)\lambda \tag{2.2}$$

In general, for $q \in \mathbb{H}$, we have qT is different from Tq. However, if $\lambda \in \mathbb{R}$, the operator λT and $T\lambda$ are equal and coincide with the operator $T\lambda$ defined in (2.2) (see [17, Proposition 3.1]). T is said compact if $\overline{T(U)}$ is compact in $U_{\mathbb{H}}^R$, where $U = \left\{u \in U_{\mathbb{H}}^R : \|u\| \le 1\right\}$. We denote the set of all compact operators from $U_{\mathbb{H}}^R$ to $V_{\mathbb{H}}^R$ by $\mathcal{B}_0\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)$. If $U_{\mathbb{H}}^R = V_{\mathbb{H}}^R$, the set $\mathcal{B}_0\left(U_{\mathbb{H}}^R, U_{\mathbb{H}}^R\right)$ is replaced by $\mathcal{B}_0\left(U_{\mathbb{H}}^R\right)$. Let $U_{\mathbb{H}}^R$ and $V_{\mathbb{H}}^R$ two quaternionic Hilbert space. Let $T \in \mathcal{B}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)$. The cokernel of T defined by

$$\text{coker}(T) = V_{\mathbb{H}}^R / \text{ran}(T).$$

The dimension of the cokernel is called the codimension, and it is denoted by $\text{codim}(T)$ and defined by $\text{codim}(T) = \dim\left(V_{\mathbb{H}}^R\right) - \dim(\text{ran}(T))$. The sets of left semi-Fredholm operators, right semi-Fredholm operators, semi-Fredholm operators and Fredholm operators from $U_{\mathbb{H}}^R$ to $V_{\mathbb{H}}^R$ are defined respectively by

$$\begin{aligned}\mathcal{F}_l\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right) &= \left\{T \in \mathcal{K}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right) : \dim(\ker(T)) < \infty \text{ and } \text{ran}(T) \text{ is closed}\right\},\\ \mathcal{F}_r\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right) &= \left\{T \in \mathcal{K}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right) : \text{codim}(T) < \infty \text{ and } \text{ran}(T) \text{ is closed}\right\},\\ \mathcal{F}_{lr}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right) &= \mathcal{F}_l\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right) \cup \mathcal{F}_r\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)\\ \mathcal{F}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right) &= \mathcal{F}_l\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right) \cap \mathcal{F}_r\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right).\end{aligned}$$

Also, the index of T is the integer

$$\text{ind}(T) = \dim(\ker(T)) - \text{codim}(T).$$

Theorem 2.1. [5] *Let $U_{\mathbb{H}}^R, V_{\mathbb{H}}^R$, and $W_{\mathbb{H}}^R$ be right quaternionic Hilbert spaces.*
(i) *If T and S are Fredholm operators, upper semi-Fredholm operators, lower semi-Fredholm operators, then TS is Fredholm operator, upper semi-Fredholm operator, lower semi-Fredholm operator, respectively, and*

$$\operatorname{ind}(ST) = \operatorname{ind}(T) + \operatorname{ind}(S).$$

(ii) *If T is Fredholm operator and K is compact operator, then $T+K$ is Fredholm operator and* $\operatorname{ind}(T+K) = \operatorname{ind}(T)$. ♢

Theorem 2.2. [5] *Let $U_{\mathbb{H}}^R, V_{\mathbb{H}}^R$ be right quaternionic Hilbert spaces and $T \in \mathcal{K}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)$ with dense domain. T is a Fredholm operator if, and only if, there exist $S_1, S_2 \in \mathcal{B}\left(V_{\mathbb{H}}^R, \mathcal{D}(T)\right)$, and compact operators K_1, K_2 on $U_{\mathbb{H}}^R$ and $V_{\mathbb{H}}^R$ respectively, such that*

$$S_1 T = I - K_1 \quad \text{on } \mathcal{D}(T),$$

$$T S_2 = I - K_2 \quad \text{on } V_{\mathbb{H}}^R.$$ ♢

Let $U_{\mathbb{H}}^R$ be a separable right Hilbert space. For a given right linear operator $T : \mathcal{D}(T) \subseteq U_{\mathbb{H}}^R \longrightarrow U_{\mathbb{H}}^R$ and $q \in \mathbb{H}$, we define the operator $R_q(T) : \mathcal{D}\left(T^2\right) \longrightarrow U_{\mathbb{H}}^R$ by

$$R_q(T) = T^2 - 2\operatorname{Re}(q)T + |q|^2 I.$$

In the literature, $R_q(T)$ is called pseudo-resolvent since it is not the resolvent operator of T but it is the one related to the notion of spectrum as we shall see in the next definition.

Definition 2.3. Let $T \in \mathcal{B}\left(U_{\mathbb{H}}^R\right)$.
(i) The S-resolvent set is given by

$$\begin{aligned}\rho^S(T) &= \left\{q \in \mathbb{H} : R_q(T) \quad \text{has an inverse in} \quad \mathcal{B}\left(U_{\mathbb{H}}^R\right)\right\} \\ &= \left\{q \in \mathbb{H} : \ker\left(R_q(T)\right) = \{0\} \quad \text{and} \quad \operatorname{ran}\left(R_q(T)\right) = U_{\mathbb{H}}^R\right\},\end{aligned}$$

and hence the S-spectrum (also called spherical spectrum) is given by

$$\begin{aligned}\sigma^S(T) &= \mathbb{H} \backslash \rho^S(T) \\ &= \left\{q \in \mathbb{H} : R_q(T) \text{ has no inverse in } \mathcal{B}\left(U_{\mathbb{H}}^R\right)\right\} \\ &= \left\{q \in \mathbb{H} : \ker\left(R_q(T)\right) \neq \{0\} \text{ or } \operatorname{ran}\left(R_q(T)\right) \neq U_{\mathbb{H}}^R\right\}.\end{aligned}$$

(ii) The approximate S-point spectrum of T :

$$\sigma_{ap}^S(T) = \left\{q \in \mathbb{H} : \text{ there is } \{\varphi_n\}_{n=1}^{\infty} \text{ such that } \|\varphi_n\| = 1 \text{ and } \|R_q(T)\varphi_n\| \longrightarrow 0\right\}.$$

(iii) The defect S-spectrum of T :

$$\sigma_{\delta}^S(T) = \left\{q \in \mathbb{H} : R_q(T) \text{ is not surjective }\right\}.$$ ♢

Let $T \in \mathcal{B}\left(U_{\mathbb{H}}^R\right)$, various notions of the essential S-spectra have been defined in the literature. In this work, we are concerned with the following S-essential spectra:

$$\begin{aligned}\sigma_{el}^S(T) &= \left\{q \in \mathbb{H} : R_q(T) \notin \mathcal{F}_l\left(U_{\mathbb{H}}^R\right)\right\} \\ \sigma_{er}^S(T) &= \left\{q \in \mathbb{H} : R_q(T) \notin \mathcal{F}_r\left(U_{\mathbb{H}}^R\right)\right\} \\ \sigma_{elr}^S(T) &= \left\{q \in \mathbb{H} : R_q(T) \notin \mathcal{F}_{lr}\left(U_{\mathbb{H}}^R\right)\right\} \\ \sigma_{e}^S(T) &= \left\{q \in \mathbb{H} : R_q(T) \notin \mathcal{F}\left(U_{\mathbb{H}}^R\right)\right\} \\ \sigma_{ess}^S(T) &= \bigcap_{K \in \mathcal{B}_0\left(U_{\mathbb{H}}^R\right)} \sigma^S(T+K) \\ \sigma_{eap}^S(T) &= \bigcap_{K \in \mathcal{B}_0\left(U_{\mathbb{H}}^R\right)} \sigma_{ap}^S(T+K) \\ \sigma_{e\delta}^S(T) &= \bigcap_{K \in \mathcal{B}_0\left(U_{\mathbb{H}}^R\right)} \sigma_{\delta}^S(T+K),\end{aligned}$$

where $\sigma_{ap}^{S}(T)$ and $\sigma_{\delta}^{S}(T)$ are given in Definition 2.1.
In what follows, we recall a new characterization of essential S-spectrum in a separable right Hilbert space $U_{\mathbb{H}}^{R}$ (see [5]).

Theorem 2.4. [5] *Let $T \in \mathcal{B}\left(U_{\mathbb{H}}^{R}\right)$. Then,*

$$\sigma_{ess}^{S}(T) = \mathbb{H}\backslash\left\{q \in \mathbb{H} : R_q(T) \in \mathcal{F}\left(U_{\mathbb{H}}^{R}\right) \text{ and } \operatorname{ind}\left(R_q(T)\right) = 0\right\}. \quad \diamondsuit$$

In this section, our purpose is to establish and recall some results about demi-compact operators, which will be needed in the sequel.

Definition 2.5. [4] Let $U_{\mathbb{H}}^{R}$ be a right quaternionic Hilbert space and $T : \mathcal{D}(T) \subseteq U_{\mathbb{H}}^{R} \longrightarrow U_{\mathbb{H}}^{R}$ be a right linear operator on $U_{\mathbb{H}}^{R} \cdot T$ is said to be demi-compact if for every bounded sequence (u_n) in $\mathcal{D}(T) \subseteq U_{\mathbb{H}}^{R}$ such that $u_n - Tu_n \longrightarrow u \in U_{\mathbb{H}}^{R}$, there is a convergent subsequence of (u_n). $\diamondsuit$

The set of demi-compact operators on $U_{\mathbb{H}}^{R}$ is denoted by $\mathcal{DC}\left(U_{\mathbb{H}}^{R}\right)$. Let us give an example of demi-compact operator in queternionic setting.

Example 2.6. Let $\ell^2(\mathbb{H}) = \left\{(u_n)_{n\geq 1} \text{ such that } u_n \in \mathbb{H} \text{ and } \sum_{n=1}^{+\infty} |u_n|^2 < \infty\right\}$ be the quaternionic Hilbert space of all square-summable sequences in $\mathbb{H}$ and let $s \in \mathbb{S}$ (see [9]). Let us consider the right linear operator

$$\begin{aligned} T: \quad & \ell^2(\mathbb{H}) \longrightarrow \\ & (u_n) \longmapsto \left(\frac{n-1}{n} s u_n\right). \end{aligned}$$

Then, T is demi-compact. $\diamondsuit$

We recall the following results established in [5].

Proposition 2.7. *Let $T \in \mathcal{K}\left(U_{\mathbb{H}}^{R}\right)$. If $T \in \mathcal{B}_0\left(U_{\mathbb{H}}^{R}\right)$, then $T \in \mathcal{DC}\left(U_{\mathbb{H}}^{R}\right)$.* $\diamondsuit$

Remark 2.8. In general, we have $\mathcal{B}_0\left(U_{\mathbb{H}}^{R}\right) \subsetneq \mathcal{DC}\left(U_{\mathbb{H}}^{R}\right)$. $\diamondsuit$

The following result is a connection between demi-compact operators and left semi-Fredholm operator in the quaternionic setting.

Theorem 2.9. *Let $T \in \mathcal{K}\left(U_{\mathbb{H}}^{R}\right)$ with dense domain. Then, the following statement are equivalent*
(i) T is demi-compact.
(ii) $I - T$ is a left semi-Fredholm operator. $\diamondsuit$

Theorem 2.10. *Let $T \in \mathcal{B}\left(U_{\mathbb{H}}^{R}\right)$. If $\alpha T \in \mathcal{DC}\left(U_{\mathbb{H}}^{R}\right)$, for each $\alpha \in [0,1]$, then $I - T \in \mathcal{F}\left(U_{\mathbb{H}}^{R}\right)$ with ind $(I - T) = 0$.* $\diamondsuit$

Let $T \in \mathcal{B}\left(U_{\mathbb{H}}^{R}\right)$. We define the set $\Lambda_{U_{\mathbb{H}}^{R}}$ by:

$$\Lambda_{U_{\mathbb{H}}^{R}} = \left\{S \in \mathcal{B}\left(U_{\mathbb{H}}^{R}\right) : S\mu \text{ is demi-compact for every } \mu \in [0,1]\right\}.$$

Now, we extend the following definition in the quaternionic setting.

Definition 2.11. Let $T \in \mathcal{B}\left(U_{\mathbb{H}}^{R}, V_{\mathbb{H}}^{R}\right)$. We call that T has a left (resp. right) inverse modulo compact operator if there exist $T_l \in \mathcal{B}\left(V_{\mathbb{H}}^{R}, U_{\mathbb{H}}^{R}\right)$ (resp. $T_r \in \mathcal{B}\left(V_{\mathbb{H}}^{R}, U_{\mathbb{H}}^{R}\right)$) and $K \in \mathcal{B}_0\left(U_{\mathbb{H}}^{R}\right)$) (resp. $K_1 \in \mathcal{B}_0\left(U_{\mathbb{H}}^{R}\right)$) such that $T_l T = I - K$ (resp. $T T_r = I - K_1$). $\diamondsuit$

3 Some properties of Demi-compact and Frdholm right linear operators

We begin this section by giving some perturbation results of demi-compact and Fredholm operators in a right quaternionic Hilbert space.

Theorem 3.1. *Let $T \in \mathcal{F}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)$. Assume that $F \in \mathcal{B}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)$ such that $-FG$ and $-GF$ are demi-compact 1 -set-contractive for some operator G which is an inverse of T modulo compact operator. Then, $T+F$ is a Fredholm operator with* $\operatorname{ind}(T+F) = \operatorname{ind}(T)$. ♢

Proof. Based on the fact that $-FT_0$ and $-T_0F$ are demi-compact 1-set-contractive for some operator T_0 which is an inverse of T modulo compact operator and since T is a Fredholm operator, then by Theorem 2.2 there is a bounded right linear operator $T_0 : V_{\mathbb{H}}^R \longrightarrow U_{\mathbb{H}}^R$ such that

$$\begin{aligned} T_0T &= I - F_1 \\ TT_0 &= I - F_2, \end{aligned}$$

where F_1 and F_2 are compact operators. This implies that T_0 is an inverse of T modulo compact operator. Now, we have

$$\begin{aligned} T_0(T+F) &= I - F_1 + T_0F \\ &= I - L_1 \end{aligned}$$

and

$$\begin{aligned} (T+F)T_0 &= I - F_2 + FT_0 \\ &= I - L_2 \end{aligned}$$

where $L_1 = F_1 + T_0F$ and $L_2 = F_2 + FT_0$. Based on $-FT_0$ and $-T_0F$ are demi-compact 1-set-contractive and F_1 and F_2 are compact, we obtain that L_1 and L_2 are demi-compact 1 -set-contractive operators. Hence, by using of Theorem 2.1 (ii), we infer that $T+F$ is a Fredholm operator. Now, it remains to show that $\operatorname{ind}(T+F) = \operatorname{ind}(T)$. Since T is Fredholm, then by Theorem 2.2, there is a bounded right linear operator $A : V_{\mathbb{H}}^R \longrightarrow U_{\mathbb{H}}^R$ such that

$$\begin{aligned} AT &= I - K_1 \\ TA &= I - K_2 \end{aligned}$$

where K_1 and K_2 are compact operators with $\operatorname{ran}(K_1) = \ker(T) \subset U_{\mathbb{H}}^R$. Furthermore, we have that A is also a Fredholm operator. Therefore, it follows from Theorem 2.1(i) that

$$\operatorname{ind}(A(T+F)) = \operatorname{ind}(A) + \operatorname{ind}(T+F).$$

Setting $T_1 = K_1 - AF$. If we consider T_1 as a demi-compact 1-set-contractive operator, then by applying Theorem 2.10, we deduce that $I - T_1$ is a Fredholm operator and $\operatorname{ind}(I - T_1) = 0$. Let us assume that

$$\operatorname{ind}(A(T+F)) = \operatorname{ind}(I - T_1) \tag{3.1}$$

Also, we have

$$\operatorname{ind}(T+F) = -\operatorname{ind}(A).$$

Again, by Theorem 2.1 (i), we infer that

$$\operatorname{ind}(AT) = \operatorname{ind}(A) + \operatorname{ind}(T).$$

The fact that $\operatorname{ind}(AT) = \operatorname{ind}(I - T_1) = 0$ implies that $\operatorname{ind}(T) = -\operatorname{ind}(A)$. Thus,

$$\operatorname{ind}(T+F) = \operatorname{ind}(T).$$

Consequently, our purpose is to prove (3.1). Based on $\operatorname{ran}(K_1) = \ker(T) \subset U_{\mathbb{H}}^R$ and AF is a right linear operator from $U_{\mathbb{H}}^R$ to $U_{\mathbb{H}}^R$, we obtain that

$$\operatorname{ran}(T_1) \subset U_{\mathbb{H}}^R. \tag{3.2}$$

So, we can see that $\ker(A(T+F)) \subset \ker(I - T_1)$. For the reverse inclusion, if $x \in \ker(I - T_1)$, then it follows from (3.2) that $x = T_1x \in U_{\mathbb{H}}^R$. This leads to

$$\ker(A(T+F)) = \ker(I - T_1). \tag{3.3}$$

Now, since $I - T_1$ is Fredholm, there exist a finite dimensional subspace U of $U_{\mathbb{H}}^R$ such that

$$U_{\mathbb{H}}^R = \text{ran}\,(I - T_1) \oplus U. \tag{3.4}$$

Or, we have $\text{ran}(A(T+F)) \subset \text{ran}\,(I - T_1)$. Conversely, if $y \in \text{ran}\,(I - T_1)$, then $y = x - T_1 x$, for some $x \in U_{\mathbb{H}}^R$. Based on $\text{ran}\,(T_1) \subset U_{\mathbb{H}}^R$, then we get $y = A(T+F)x$, which implies that $\text{ran}\,(I - T_1) = \text{ran}(A(T+F))$. This leads to

$$U_{\mathbb{H}}^R = \text{ran}(A(T+F)) \oplus U. \tag{3.5}$$

Combining (3.4) and (3.5), we conclude that

$$\text{codim}(A(T+F)) = \dim U = \text{codim}\,(I - T_1)\,.$$

This allows us to deduce that (3.1), which completes the proof. □

With the same proof, we prove the following result.

Proposition 3.2. *Let $T \in \mathcal{F}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)$. Assume that $F \in \mathcal{B}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)$ such that $-\alpha FG$ is demi-compact, for every $\alpha \in [0,1]$ and for some operator G which is an inverse of T modulo compact operator. Then, $T+F$ is a Fredholm operator with* $\text{ind}(T+F) = \text{ind}(T)$. ♢

In the following, we give a quaternionic version of Theorem 5.31 and 5.32 [21], which are needed to prove our first main result.

Proposition 3.3. *Let $U_{\mathbb{H}}^R, V_{\mathbb{H}}^R$, and $W_{\mathbb{H}}^R$ be right quaternionic Hilbert spaces.*
(i) If $T \in \mathcal{B}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right), S \in \mathcal{B}\left(V_{\mathbb{H}}^R, W_{\mathbb{H}}^R\right)$ and $ST \in \mathcal{F}_r\left(U_{\mathbb{H}}^R, W_{\mathbb{H}}^R\right)$. Then, $S \in \mathcal{F}_r\left(V_{\mathbb{H}}^R, W_{\mathbb{H}}^R\right)$.
(ii) If $T \in \mathcal{B}\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right), S \in \mathcal{B}\left(V_{\mathbb{H}}^R, W_{\mathbb{H}}^R\right)$ and $ST \in \mathcal{F}_l\left(U_{\mathbb{H}}^R, W_{\mathbb{H}}^R\right)$. Then, $T \in \mathcal{F}_l\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)$. ♢

Proof. (i) Using the fact that $ST \in \mathcal{F}_r\left(U_{\mathbb{H}}^R, W_{\mathbb{H}}^R\right)$, then $\dim\left(\text{ran}(ST)^\perp\right) < \infty$. This implies that there is a subspace W such that $\dim W < \infty$ and $W_{\mathbb{H}}^R = \text{ran}(ST)^\perp \oplus W$. Also, since $\text{ran}(ST) \subset \text{ran}(S)$, $\text{ran}(S)$ is closed and $\text{ran}(S)^\perp \subset \text{ran}(ST)^\perp$, then we have

$$\dim \text{ran}(S)^\perp < \infty.$$

Consequently, $S \in \mathcal{F}_r\left(V_{\mathbb{H}}^R, W_{\mathbb{H}}^R\right)$.
(ii) Since, $ST \in \mathcal{F}_l\left(U_{\mathbb{H}}^R, W_{\mathbb{H}}^R\right)$, then by using [5, Remark 3.1], we have $T^*S^* \in \mathcal{F}_r\left(W_{\mathbb{H}}^R, U_{\mathbb{H}}^R\right)$. In view of (i), we get $T^* \in \mathcal{F}_r\left(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R\right)$ which means that $T \in \mathcal{F}_l\left(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R\right)$. □

4 Stability and perturbation results of essential S-spectra

In this section, we give some new perturbation results and the connection between the essential S-spectrum of the sum of two quaternionic bounded right linear operators and the essential S-spectrum of each of these right linear operators.

Theorem 4.1. *Let $T, S \in \mathcal{B}\left(U_{\mathbb{H}}^R\right), U_{\mathbb{H}}^R$ is a right quaternionic Hilbert space. If for every $q \in \mathcal{F}_l\left(U_{\mathbb{H}}^R\right)$, there exists T_{ql} (resp. T_{qr}) a left (resp. right) inverse modulo compact operator of $R_q(T)$ such that ST_{ql} and (resp. $T_{qr}S$) is demi-compact and $R_q(S)K + ST + TS - |q|^2 I$ (resp. $K'R_q(S) + ST + TS - |q|^2 I$) is compact, then*

$$\sigma_{el}^S(T+S) = \sigma_{el}^S(T). \qquad ♢$$

Proof. Our purpose is to prove $\sigma_{el}^S(T+S) \subseteq \sigma_{el}^S(T)$. Let $q \in \mathbb{H}$. If T_{ql} is a left inverse modulo compact operators of $R_q(T)$, then there exist a compact operator K such that

$$T_{ql} R_q(T) = I - K. \tag{4.1}$$

Also, we can write $R_q(T+S)$ in the following form:

$$\begin{aligned} R_q(T+S) &= (T+S)^2 - 2\,\text{Re}(q)(T+S) + |q|^2 I \\ &= R_q(T) + R_q(S) + TS + ST - |q|^2 I. \end{aligned}$$

Therefore, according to (4.1) we obtain that

$$R_q(T+S) = [I + R_q(S)T_{ql}]\, R_q(T) + R_q(S)K + TS + ST - |q|^2 I. \tag{4.2}$$

In the same way, there exists T_{qr} a right inverse modulo compact operator of $R_q(T)$. So, we can write

$$R_q(T+S) = R_q(T)\,[I + T_{qr}R_q(S)] + K_1 R_q(S) + TS + ST - |q|^2 I, \tag{4.3}$$

where K_1 is a compact operator. Now, let $q \notin \sigma_{el}^S(T)$, then $R_q(T) \in \mathcal{F}_l\left(U_{\mathbb{H}}^R\right)$. Set T_{ql} (resp. T_{qr}) be a left (resp. right) inverse modulo compact operator of $R_q(T)$. Hence, (4.2) and (4.3) hold. Using the fact that ST_{ql} (resp. $T_{qr}S$) is demi-compact implies from Theorem 2.9 that

$$(I + ST_{ql})\,(\text{resp}\,.\,(I + T_{qr}S)) \in \mathcal{F}_l\left(U_{\mathbb{H}}^R\right).$$

By referring to Theorem 2.1 and (4.2) (resp. (4.3)), we infer that

$$(I + ST_{ql})\, R_q(T)\,(\text{resp}\,.R_q(T)\,(I + T_{qr})) \in \mathcal{F}_l\left(U_{\mathbb{H}}^R\right).$$

Finally, based on the hypotheses $R_q(S)K + ST + TS - |q|^2 I$ (resp. $K'R_q(S) + ST + TS - |q|^2 I$) is compact, it follows from Theorem 2.1 that $R_q(T+S) \in \mathcal{F}_l\left(U_{\mathbb{H}}^R\right)$ and so $q \notin \sigma_{el}^S(T+S)$. For the reverse inclusion, let $q \in \mathbb{H}$ such that $q \notin \sigma_{el}^S(T+S)$. If T_{ql} is a left inverse modulo compact operator of $R_q(T)$, then there exist a compact operator K such that (4.1) holds. Hence, we can write (4.2). Also, by the hypothesis $R_q(T+S) \in \mathcal{F}_l\left(U_{\mathbb{H}}^R\right)$, we can deduce that

$$R_q(T+S) - R_q(S)K - TS - ST + |q|^2 I = [I + R_q(S)T_{ql}]\, R_q(T) \in \mathcal{F}_l\left(U_{\mathbb{H}}^R\right).$$

In view of Proposition 3.3 (ii), we obtain that $R_q(T) \in \mathcal{F}_l\left(U_{\mathbb{H}}^R\right)$. As a result, $q \notin \sigma_{el}^S(T)$, as desired. □

Theorem 4.2. *Let $T, S \in \mathcal{B}\left(U_{\mathbb{H}}^R\right)$ and $U_{\mathbb{H}}^R$ is a right quaternionic Hilbert space. If for every $q \in \sigma_\star^S(T)$, where $\star \in \{$ er, elr, e, ess, eap, e $\delta\}$ there exists T_{ql} (resp. T_{qr}) a left (resp. right) inverse modulo compact operator of $R_q(T)$ such that $ST_{ql} \in \Lambda$ (resp. $T_{qr}S \in \Lambda$) and $R_q(S)K + ST + TS - |q|^2 I$ (resp. $K'R_q(S) + ST + TS - |q|^2 I$) is compact, then*

$$\sigma_\star^S(T+S) = \sigma_\star^S(T). \qquad \diamondsuit$$

Proof. By using of (4.2) and (4.3), we just give the proof of the case $\star = e\delta$ because it is the same in the other cases. Let us suppose that $q \notin \sigma_{e\delta}^S(T)$, then by using of [**?**, Theorem 2.12]), we obtain that

$$R_q(T) \in \mathcal{F}_r\left(U_{\mathbb{H}}^R\right) \text{ and } \operatorname{ind}\left(R_q(T)\right) \geq 0.$$

Setting now, T_{ql} a left inverse modulo compact operator of $R_q(T)$, then (4.2) holds. Using the fact that $ST_{ql} \in \Lambda$, we get

$$(I - ST_{ql}) \in \mathcal{F}\left(U_{\mathbb{H}}^R\right) \text{ and } \operatorname{ind}\left(R_q(T)\right) = 0.$$

This implies from Theorem 2.1 and (4.2) that

$$R_q(T+S) \in \mathcal{F}_r\left(U_{\mathbb{H}}^R\right) \text{ and } \operatorname{ind}\left(R_q(T+S)\right) \geq 0.$$

Therefore, $q \notin \sigma_{e\delta}^S(T+S)$ which is equivalent to saying that $\sigma_{e\delta}^S(T+S) \subseteq \sigma_{e\delta}^S(T)$. In the same manner, we can show the reverse inclusion. □

5 Application to block matrix of right linear operators

In this section, the main tool is that to study the essential S-spectra of a 2×2 block right linear operator matrix defined by

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.1}$$

where A, B, C and D are bounded right linear operators involving demi-compactness perturbation and by means of the Frobenius-Schur's decomposition in right quaternionic Hilbert spaces. In this regard, let us recall the Frobenius-Schur's decomposition of $\mathcal{L}$ (see [5]).

Theorem 5.1. *Suppose that* $A : \mathcal{D}(A) \subseteq U_{\mathbb{H}}^R \to U_{\mathbb{H}}^R, B : \mathcal{D}(B) \subseteq V_{\mathbb{H}}^R \to U_{\mathbb{H}}^R$, $C : \mathcal{D}(C) \subseteq U_{\mathbb{H}}^R \to V_{\mathbb{H}}^R$, *and* $D : \mathcal{D}(D) \subseteq V_{\mathbb{H}}^R \to V_{\mathbb{H}}^R$ *are right linear operators satisfies* $\text{ran}(C) \subset \ker(B), \text{ran}(B) \subset \ker(C), \mathcal{D}(A) \subset \mathcal{D}(C)$ *and* $\mathcal{D}(B) \subset \mathcal{D}\left(D^2\right)$. *If* $q \in \rho^S(A)$, *then we have*

$$R_q(\mathcal{L}) = \begin{pmatrix} I & 0 \\ \mathcal{G}(q) & I \end{pmatrix} \begin{pmatrix} R_q(A) & 0 \\ 0 & \mathcal{S}(q) \end{pmatrix} \begin{pmatrix} I & \mathcal{F}(q) \\ 0 & I \end{pmatrix} \tag{5.2}$$

where $\mathcal{G}(q) = T_1 R_q(A)^{-1}, \mathcal{F}(q) = R_q(A)^{-1} T_2, \mathcal{S}(q) = R_q(D) - T_1 R_q(A)^{-1} T_2$, $T_1 = CA + (D - 2q_0)\, C$ *and* $T_2 = BD + (A - 2q_0)\, B$. ♢

Remark 5.2. For $q \in \rho^S(A)$. The matrix operator $\mathcal{L}$ can be factorized in the Frobenius-Schur sense by

$$R_q(\mathcal{L}) = \mathcal{X}(q)\mathcal{Y}(q)\mathcal{Z}(q)$$

where $\mathcal{X}(q) = \begin{pmatrix} I & 0 \\ \mathcal{G}(q) & I \end{pmatrix}$ $\mathcal{Y}(q) = \begin{pmatrix} R_q(A) & 0 \\ 0 & \mathcal{S}(q) \end{pmatrix}$ and $\mathcal{Z}(q) = \begin{pmatrix} I & \mathcal{F}(q) \\ 0 & I \end{pmatrix}$. Furthermore, we have the following equalities:

$$\begin{aligned} \dim\left(\ker\left(R_q(A)\right)\right) &= \text{codim}\left(R_q(A)\right) = 0 \\ \dim(\ker(\mathcal{Y}(q))) &= \dim(\ker(\mathcal{S}(q))) \\ \text{codim}(\mathcal{Y}(q)) &= \text{codim}(\mathcal{S}(q)). \end{aligned}$$

♢

Now, we take the following conditions:
$(\mathcal{H}1)$: For every $q \notin \sigma_\star^S(\mathcal{S}(p))$, where $(\star \in \{$ el, er, elr, e, ess, eap, e $\delta\})$, there exists $\mathcal{S}_q l \in \mathcal{B}\left(V_{\mathbb{H}}^R\right)$ a left inverse modulo compact operator of $R_q(\mathcal{S}(p))$.
$(\mathcal{H}2)$: For every $q \notin \sigma_\star^S(\mathcal{S}(p))$, where $(\star \in \{$ el, er, elr, e, ess, eap, e $\delta\})$, there exists $\mathcal{S}_q r \in \mathcal{B}\left(V_{\mathbb{H}}^R\right)$ a right inverse modulo compact operator of $R_q(\mathcal{S}(p))$.

Proposition 5.3. *Let* $q \in \rho^S(A)$ *and* $p \in \mathbb{H}$ *such that* $\text{Re}(p) = \text{Re}(q), |q| \neq |p|$. *Assume that the hypotheses of Theorem 5.1 hold.*

(i) *Let the condition* $(\mathcal{H}1)$ *be satisfied. If* $\left(|q|^2 - |p|^2\right)(I + \mathcal{G}(q)\mathcal{F}(p))S_{ql} \in \mathcal{DC}\left(V_{\mathbb{H}}^R\right)$. *Then,*

$$\sigma_{el}^S(\mathcal{S}(q)) = \sigma_{el}^S(\mathcal{S}(p)).$$

(ii) *Let the condition* $(\mathcal{H}2)$ *be satisfied. If* $\left(|q|^2 - |p|^2\right)(I + \mathcal{G}(q)\mathcal{F}(p))S_{qr} \in \Lambda_{V_{\mathbb{H}}^R}$. *Then,*

$$\sigma_{er}^S(\mathcal{S}(q)) = \sigma_{er}^S(\mathcal{S}(p)).$$ ♢

Proof. (i) Let $q \in \rho^S(A)$ and $p \in \mathbb{H}$ such that $\text{Re}(p) = \text{Re}(q), |q| \neq |p|$. Since, we can write:

$$\begin{aligned} \mathcal{S}(q) - \mathcal{S}(p) &= R_q(D) - T_1 R_q(A)^{-1} T_2 - R_p(D) + T_1 R_p(A)^{-1} T_2 \\ &= R_q(D) - R_p(D) + T_1 \left[R_p(A)^{-1} - R_q(A)^{-1}\right] T_2 \\ &= |q|^2 - \left|p^2\right| + \left(|q|^2 - \left|p^2\right|\right) T_1 R_q(A)^{-1} R_p(A)^{-1} T_2 \\ &= \left(|q|^2 - \left|p^2\right|\right)\left[I + T_1 R_q(A)^{-1} R_p(A)^{-1} T_2\right] \\ &= \left(|q|^2 - \left|p^2\right|\right)\left[I + \mathcal{G}(q)\mathcal{F}(p)\right]. \end{aligned}$$

Then, we get

$$\mathcal{S}(q) = \mathcal{S}(p) + \left(|q|^2 - \left|p^2\right|\right)\left[I + \mathcal{G}(q)\mathcal{F}(p)\right].$$

Based on the hypothesis ($\mathcal{H}1$) and the fact that $\left(|q|^2 - \left|p^2\right|\right)\left[I + \mathcal{G}(q)\mathcal{F}(p)\right]\mathcal{S}_{ql} \in \mathcal{DC}\left(V_{\mathbb{H}}^R\right)$ it follows from Theorem 4.1 that $\sigma_{el}^S(\mathcal{S}(q)) = \sigma_{el}^S(\mathcal{S}(p))$.

(ii) Since the condition $(\mathcal{H}2)$ holds true and $\left(|q|^2 - |p|^2\right)(I + \mathcal{G}(q)\mathcal{F}(p))S_{qr} \in \Lambda_{V_{\mathbb{H}}^R}$, then by applying of Theorem 3.2, we get $\sigma_{\text{er}}^S\left(\mathcal{S}(q)\right) = \sigma_{\text{er}}^S\left(\mathcal{S}(p)\right)$. □

The following result is an application of Theorem 4.1 to block operator matrix $\mathcal{L}$, this is equivalent to prove the connection between S-essential spectra of $\mathcal{L}$ and the S-essential spectra of its entries.

Theorem 5.4. *Let $q \in \rho^S(A)$ and $p \in \mathbb{H}$ such that $\mathrm{Re}(p) = \mathrm{Re}(q), |q| \neq |p|$. Assume that the hypotheses of Theorem 5.1 hold.*
(i) Let the condition $(\mathcal{H}1)$ be satisfied. If $\left(|q|^2 - |p|^2\right)(I + \mathcal{G}(q)\mathcal{F}(p))S_{ql} \in \mathcal{DC}\left(V_{\mathbb{H}}^R\right)$. Then,

$$\sigma_{el}^S(\mathcal{L}) \cap \rho^S(A) = \sigma_{el}^S(\mathcal{S}(p)) \cap \rho^S(A).$$

(ii) Let the condition $(\mathcal{H}2)$ be satisfied. If $\left(|q|^2 - |p|^2\right)(I + \mathcal{G}(q)\mathcal{F}(p))S_{qr} \in \Lambda_{V_{\mathbb{H}}^R}$. Then,

$$\sigma_{er}^S(\mathcal{L}) \cap \rho^S(A) = \sigma_{er}^S(\mathcal{S}(p)) \cap \rho^S(A). \qquad \diamondsuit$$

Proof. (i) Let $q \in \rho^S(A)$ and $p \in \mathbb{H}$ such that $\mathrm{Re}(p) = \mathrm{Re}(q), |q| \neq |p|$. Since $\mathcal{X}(q)$ and $\mathcal{Z}(q)$ are bounded with bounded inverse. Hence,

$$\dim\ \ker\left(R_q(\mathcal{L})\right) = \ \dim\ \ker(\mathcal{Y}(q)).$$

However, according to Remark 5.2, we obtain that

$$\dim\ker\left(R_q(\mathcal{L})\right) = \dim\ker\left(R_q(\mathcal{S}(q))\right).$$

Therefore, $R_q(\mathcal{L})$ is an upper-semi Fredholm operator if, and only, if $R_q(\mathcal{S}(q))$ is so. Now, based on the condition $(\mathcal{H}1)$, then by referring to Proposition 5, we infer that $q \in \sigma_{el}^S(\mathcal{L})$. This implies that $q \in \sigma_{el}^S(\mathcal{S}(p))$. As a consequence,

$$\sigma_{el}^S(\mathcal{L}) \cap \rho^S(A) = \sigma_{el}^S(\mathcal{S}(p)) \cap \rho^S(A).$$

(ii) The same reasoning as (i), we prove that $\sigma_{er}^S(\mathcal{L}) \cap \rho^S(A) = \sigma_{er}^S(\mathcal{S}(p)) \cap \rho^S(A)$. In particular, we use the condition $(\mathcal{H}2)$ and Proposition 5.3 (ii). □

Proposition 5.5. *Let $q \in \rho^S(A)$ and $p \in \mathbb{H}$ such that $\mathrm{Re}(p) = \mathrm{Re}(q), |q| \neq |p|$. Assume that the hypotheses of Theorem 5.1 hold. Let the conditions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ hold true. If $\left(|q|^2 - |p|^2\right)(I + \mathcal{G}(q)\mathcal{F}(p))S_{qr} \in \Lambda_{V_{\mathbb{H}}^R}$. Then,*

$$\sigma_\star^S(\mathcal{S}(q)) = \sigma_\star^S(\mathcal{S}(p)), \ \textit{where} \star \in \{\ \textit{elr}\ ,\ \textit{e}\ ,\ \textit{ess, eap, e}\ \delta\}. \qquad \diamondsuit$$

Proof. The proof follows from Proposition 5.3 (i) and (ii). In particular, we use Theorem 4.2. □

Theorem 5.6. *Let $q \in \rho^S(A)$ and $p \in \mathbb{H}$ such that $\mathrm{Re}(p) = \mathrm{Re}(q), |q| \neq |p|$. Assume that the hypotheses of Theorem 5.1 hold. Let the conditions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ hold true. If $\left(|q|^2 - |p|^2\right)(I + \mathcal{G}(q)\mathcal{F}(p))S_{qr} \in \Lambda_{V_{\mathbb{H}}^R}$. Then,*

$$\sigma_\star^S(\mathcal{L}) \cap \rho^S(A) = \sigma_\star^S(\mathcal{S}(p)) \cap \rho^S(A), \quad \textit{where} \star \in \{\ \textit{elr}\ ,\ \textit{e}\ ,\ \textit{ess, eap, e}\ \delta\}. \qquad \diamondsuit$$

Proof. In view of Remark 5.2, we have

$$\dim\ker\left(R_q(\mathcal{L})\right) = \dim\ker\left(R_q(\mathcal{S}(q))\right)$$

and

$$\mathrm{codim}\left(R_q(\mathcal{L})\right) = \mathrm{codim}\left(R_q(\mathcal{S}(q))\right).$$

Then, we get

$$\mathrm{ind}\left(R_q(\mathcal{L})\right) = \mathrm{ind}\left(R_q(\mathcal{S}(q))\right).$$

Therefore,

$$\sigma_\star^S(\mathcal{L}) \cap \rho^S(A) = \sigma_\star^S(\mathcal{S}(p)) \cap \rho^S(A), \quad \text{where} \star \in \{\ \text{elr}\ ,\ \text{e}\ ,\ \text{ess}\ ,\ \text{eap}\ ,\ \text{e}\ \delta\}.$$

□

Lemma 5.7. *Let $A \in \mathcal{DC}\left(U_{\mathbb{H}}^{R}\right)$ and $D \in \mathcal{DC}\left(V_{\mathbb{H}}^{R}\right)$. We consider the matrix of right linear operators*

$$\mathcal{M} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

Then,
(i) $\mathcal{M} \in \mathcal{DC}\left(U_{\mathbb{H}}^{R} \times V_{\mathbb{H}}^{R}\right)$.
(ii) *If $D \in \mathcal{B}_0\left(V_{\mathbb{H}}^{R}\right)$. Then, $\mathcal{L} \in \mathcal{DC}\left(U_{\mathbb{H}}^{R} \times V_{\mathbb{H}}^{R}\right)$, for all $C \in \mathcal{B}\left(U_{\mathbb{H}}^{R}, V_{\mathbb{H}}^{R}\right)$.* ♢

Proof. Let $(u_n)_n = (x_n, y_n)_n$ be a bounded sequence in $U_{\mathbb{H}}^{R} \times V_{\mathbb{H}}^{R}$ such that

$$v_n = (\mathcal{I} - \mathcal{M})u_n \longrightarrow v \text{ with } (v_n)_n = ((I-A)x_n, (I-D)y_n)_n \text{ and } v = (x, y).$$

Hence, we get

$$\begin{pmatrix} I-A & 0 \\ 0 & I-D \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}.$$

This implies that, $(I-A)x_n \longrightarrow x$ and $(I-D)y_n \longrightarrow y$. In what follows, we give the S-essential spectrum of the sum of two block operators matrix defined on a right quaternionic Hilbert space $U_{\mathbb{H}}^{R} \times V_{\mathbb{H}}^{R}$ through the S-essential spectrum of each of these right linear operators. So, let us consider the two right bounded linear operators $A_1 : \mathcal{D}(A_1) \subset U_{\mathbb{H}}^{R} \longrightarrow U_{\mathbb{H}}^{R}$ and $D_1 : \mathcal{D}(D_1) \subset V_{\mathbb{H}}^{R} \longrightarrow V_{\mathbb{H}}^{R}$. Now, we define the block operator matrix $\mathcal{T}$ by

$$\mathcal{T} = \begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix}. \tag{5.3}$$

□

Theorem 5.8. *Let $\mathcal{L}$ and $\mathcal{T}$ are given respectively in* (5.1) *and* (5.3) *such that $\mathcal{T}$ and $\mathcal{L}$ are anti-commute. If for every $q \notin \sigma_{ess}^{S}(\mathcal{T})$, there exist I_1 and I_2 respectively an inverse of $R_q(A_1)$ and $R_q(D_1)$ satisfies the two following statements:*
(i) $\alpha\left(R_q(A) + |q|^2\right) I_1 \in \mathcal{DC}\left(U_{\mathbb{H}}^{R}\right), \alpha\left(R_q(D) + |q|^2\right) I_2 \in \mathcal{DC}\left(V_{\mathbb{H}}^{R}\right)$, *for every $\alpha \in [0,1]$.*
(ii) $\left((A - 2q_0)B + BD\right) I_2 \in \mathcal{B}_0\left(V_{\mathbb{H}}^{R}, U_{\mathbb{H}}^{R}\right)$. *Then,*

$$\sigma_{ess}^{S}(\mathcal{T} + \mathcal{L}) \subseteq \sigma_{ess}^{S}(\mathcal{T}).$$

♢

Proof. Let $q \notin \sigma_{\text{ess}}^{S}(\mathcal{T})$, then $R_q(\mathcal{T})$ is a Fredholm operator and $\text{ind}\left(R_q(\mathcal{T})\right) = 0$. First of all, we prove that $\mathcal{I} = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}$ is an inverse modulo compact operator. In fact, we have

$$\begin{aligned} \mathcal{I}R_q(\mathcal{T}) &= \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} R_q(A_1) & 0 \\ 0 & R_q(D_1) \end{pmatrix} \\ &= \begin{pmatrix} I_1 R_q(A_1) & 0 \\ 0 & I_2 R_q(D_1) \end{pmatrix}. \end{aligned}$$

Based on the hypthesis I_1 and I_2 respectively an inverse of $R_q(A_1)$ and $R_q(D_1)$, then there exist $K_1 \in \mathcal{B}_0\left(U_{\mathbb{H}}^{R}\right)$ and $K_2 \in \mathcal{B}_0\left(V_{\mathbb{H}}^{R}\right)$ such that

$$\begin{aligned} I_1 R_q(A_1) &= I - K_1 \\ I_2 R_q(D_1) &= I - K_2. \end{aligned}$$

As a consequence, we obtain that

$$\mathcal{I}R_q(\mathcal{T}) = \begin{pmatrix} I - K_1 & 0 \\ 0 & I - K_2 \end{pmatrix} = \mathbb{I} - \mathcal{K}.$$

where $\mathcal{K} = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \in \mathcal{B}_0\left(U_{\mathbb{H}}^R \times V_{\mathbb{H}}^R\right)$. In addition, since $\operatorname{ran}(C) \subset \ker(B)$ and $\operatorname{ran}(B) \subset \ker(C)$, we get

$$\begin{aligned}\left(R_q(\mathcal{L}) + |q|^2\mathbb{I}\right)\mathcal{I} &= \begin{pmatrix} R_q(A) + |q|^2 I & (A - 2q_0)\,B + BD \\ CA + (D - 2q_0)\,C & R_q(D)|q|^2 I \end{pmatrix} \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} \\ &= \begin{pmatrix} \left(R_q(A) + |q|^2\right) I_1 & \left((A - 2q_0)\,B + BD\right) I_2 \\ \left(CA + (D - 2q_0)\,C\right) I_1 & \left(R_q(D)|q|^2\right) I_2 \end{pmatrix} \\ &= \begin{pmatrix} \left(R_q(A) + |q|^2\right) I_1 & T_1 I_2 \\ T_2 I_1 & \left(R_q(D) + |q|^2\right) I_2 \end{pmatrix}\end{aligned}$$

where $T_1 = (A - 2q_0)\,B + BD$ and $T_2 = CA + (D - 2q_0)\,C$. In view of (i) and (ii), it follows from Lemma 5.7 that $\alpha \begin{pmatrix} \left(R_q(A) + |q|^2\right) I_1 & T_1 I_2 \\ T_2 I_1 & \left(R_q(D) + |q|^2\right) I_2 \end{pmatrix}$ is a demi-compact for every $\alpha \in [0, 1]$. In Proposition 3.2, if we take $T = R_q(\mathcal{T})$ and $F = R_q(\mathcal{L}) + |q|^2\mathbb{I}$, then we deduce that $R_q(\mathcal{T}) + R_q(\mathcal{L}) + |q|^2\mathbb{I}$ is Fredholm and $\operatorname{ind}\left(R_q(\mathcal{T}) + R_q(\mathcal{L}) + |q|^2\mathbb{I}\right) = \operatorname{ind}\left(R_q(\mathcal{T})\right) = 0$. Based on the fact that $\mathcal{T}$ and $\mathcal{L}$ are anti-commute, hence

$$R_q(\mathcal{T} + \mathcal{L}) = R_q(\mathcal{T}) + R_q(\mathcal{L}) + |q|^2\mathbb{I}.$$

This implies that $R_q(\mathcal{T} + \mathcal{L})$ is Fredholm and $\operatorname{ind}\left(R_q(\mathcal{T} + \mathcal{L})\right) = \operatorname{ind}\left(R_q(\mathcal{T})\right) = 0$. Thus, $q \notin \sigma_{\text{ess}}^S(\mathcal{T} + \mathcal{L})$. □

References

[1] D. Alpay, F. Colombo and I. Sabadini. Quaternionic de Branges spaces and characteristic operator function, SpringerBriefs in Mathematics, Springer, Cham, (2020).

[2] D. Alpay, F. Colombo, J. Gantner and I. Sabadini. A new resolvent equation for the S-functional calculus, J. Geom. Anal., 25, 1939-1968, (2015).

[3] D. Alpay, F. Colombo and D.P. Kimsey. The spectral theorem for quaternionic unbounded normal operators based on the S-spectrum, J. Math. Phys., 57, no. 2, 023503, 27 pp, (2016).

[4] A. Ammar, S. Fakhfakh and A. Jeribi. Demicompact operators and ellipsoid spectra on quaternionic Hilbert spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 114, no. 3, 149, (2020).

[5] A. Ammar, D. Kouas and N. Lazrag. Demicompactness results and Frobenuis Schur's decomposition for block operator matrices in right quaternionic Hilbert spaces. Complex Anal. Oper. Theory 17, no. 2, Paper No. 19, 46 pp, (2023).

[6] Birkhoff, G. and J. von Neumann. The logic of quantum mechanics. Ann. of Math. (2) 37, no. 4, 823–843, (1936).

[7] F. Colombo and I. Sabadini. On the formulations of the quaternionic functional calculus. J. Geom. Phys. 60, no. 10, 1490-1508, (2010).

[8] F. Colombo, I. Sabadini and D. C. Struppa. Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions. Progress in Mathematics, 289. Birkhauser Springer Basel AG, Basel, vi+221 pp. ISBN: 978-3-0348-0109-6, (2011).

[9] F. Colombo, J. Gantner and D. P. Kimsey. Spectral theory on the S-spectrum for quaternionic operators. Operator Theory: Advances and Applications, 270. Birkhauser Springer, Cham, ix+356 pp. ISBN: 978-3-030-03073-5; 978-3-030-03074-2, (2018).

[10] F. Colombo and J. Gantner. Quaternionic closed operators, fractional powers and fractional diffusion processes, Operator Theory: Advances and Applications, 274. Birkhäuser/Springer, Cham, viii+322 pp, (2019).

[11] F. Colombo and I. Sabadini. On some properties of the quaternionic functional calculus, J. Geom. Anal., 19(3), pp. 601-627, (2009).

[12] F. Colombo, I. Sabadini, D. C. Struppa. Entire slice regular functions. SpringerBriefs in Mathematics. Springer, Cham, v+118 pp. ISBN: 978-3-319-49264-3; 978-3-319-49265-0, (2016).

[13] F. Colombo, I. Sabadini and D.C. Struppa. Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions, Progress in Mathematics, 289. Birkhäuser/Springer Basel AG, Basel. vi+221 pp, (2011).

[14] F. Colombo, J. Gantner and S. Pinton. An Introduction to Hyperholomorphic Spectral Theories and Fractional Powers of Vector Operators, Adv. Appl. Clifford Algebr, 31, no. 3, Paper No. 45, (2021).

[15] J. Gantner. Operator Theory on One-Sided Quaternionic Linear Spaces: Intrinsic S-Functional Calculus and Spectral Operators, Mem. Amer. Math. Soc., 267, no. 1297, iii+101 pp, (2020).

[16] G. Gentili, C. Stoppato and D. C. Struppa. Regular functions of a quaternionic variable. Springer Nature, (2022).

[17] R. Ghiloni, V. Moretti and A. Perotti. Continuous slice functional calculus in quaternionic Hilbert spaces. Rev. Math. Phys. 25, no. 4, 1350006, 83 pp, (2013).

[18] D. Kouas. On the local S-spectra with Bishop's property of operators matrices in a right quaternionic Hilbert space. Rend. Circ. Mat. Palermo (2) 74, no. 1, Paper No. 69, 14 pp, (2025).

[19] B. Muraleetharan and K. Thirulogasanthar. Fredholm operators and essential S-spectrum in the quaternionic setting. J. Math. Phys. 59, no. 10, 103506, 27 pp, (2018).

[20] B. Muraleetharan and K. Thirulogasanthar. Weyl and Browder S-spectra in a right quaternionic Hilbert space. J. Geom. Phys. 135, 7–20, (2019).

[21] Schechter M. Principles of functionnal analysis. New York: Academic Press; (1971).

Author information

Dorsaf Kouas, Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax, Soukra Road Km 3.5, B. P. 1171, 3000, Sfax, Tunisia.
E-mail: dorsaf.kouas@fsegs.usf.tn

Received: 2025-10-25
Accepted: 2026-05-21