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On Weakly S-J-Ideals of Commutative Rings

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Abstract In this paper, we introduce and systematically study the concept of weakly S-J-ideals in commutative rings, a generalization that merges the notions of weakly J-ideals and S-J-ideals. We investigate the fundamental properties of these ideals, particularly their behavior under ring homomorphism, quotient ring, finite intersection, direct product of ideals and localization. We further explore the subtle distinctions between weakly S-J-ideals and their related counterparts, such as S-J-ideals and weakly J-ideals, providing concrete examples to illustrate these differences. A crucial result demonstrates that a weakly S-J-ideal that fails to be an S-J-ideal must satisfy the strong structural condition $sI^2 = 0$.

1 Introduction

In this article, we focus on commutative rings with a nonzero identity, denoted by R. A subset S of R is called a multiplicatively closed subset (hereafter, m.c.s.), if $1 \in S$, $0 \notin S$ and it is closed under multiplication. The study of prime ideals and their generalizations has long been a cornerstone of commutative ring theory, providing essential tools for understanding the structure of rings. In recent decades, numerous variations of primality have been introduced by modifying the definition. These generalizations often involve multiplicative sets, Jacobson radicals, or nilpotent elements, leading to richer and more nuanced theories.

One fruitful direction of generalization has been the introduction of a multiplicatively closed subset S of a ring R, giving rise to concepts like S-prime ideals and S-Noetherian rings. This framework allows for a localized analysis of ideal properties. Another direction involves replacing the ideal I in the consequent of the implication with a larger, related ideal, such as the Jacobson radical $J(R)$, leading to the study of J-ideals and weakly J-ideals [5, 9, 11, 12, 15]. The concept of a J-ideal, introduced by Khashan and Bani-ata [11], generalizes the notion of an n-ideal [14] by utilizing the Jacobson radical instead. The paper investigates the fundamental properties of this new class of ideals and explores its relationship with classical ideal types such as prime ideals. Furthermore, this concept is extended to R-modules by defining and studying J-submodules thereby offering a new perspective in module theory, particularly in the context of multiplication modules. Yıldız et al. introduced the concept of a (2,J)-ideal as a generalization of J-ideals for products of three elements. Their central contribution is the key characterization theorem establishing that a proper ideal I is a (2,J)-ideal if and only if it is a (2,J)-primary ideal and additionally satisfies the condition $J(I) = J(R)$. This result precisely links the new structure to the related concept of (2,J)-primary ideals under a specific constraint on their Jacobson radicals [15]. In 2022, Khashan and Celikel introduced weakly J-ideals as a generalization of J-ideals by relaxing the defining condition to non-zero products. Their central contribution is the characterization that a weakly J-ideal fails to be a J-ideal if and only if its square is zero ($I^2 = 0$), a property that is thoroughly investigated across various ring constructions [12]. The authors in [1] defined S-J-ideals, which generalizes J-ideals relative to a multiplicatively closed

set S, by defining that for a fixed $s \in S$, the condition $ab \in I$ implies $sa \in J(R)$ or $sb \in I$. Their main contribution is a key characterization theorem stating that an ideal I is an S-J-ideal if and only if the colon ideal $(I : s)$ is a J-ideal for some $s \in S$, a result that is explored in both commutative and noncommutative settings.

This paper is situated at the confluence of these two research streams. We introduce the concept of a weakly S-J-ideal, a proper ideal I for which there exists a fixed $s \in S$ such that for any $a, b \in R$ with $0 \neq ab \in I$, we have either $sa \in J(R)$ or $sb \in I$. This definition naturally extends the notions of S-J ideals and weakly J-ideals (where $S = \{1\}$). Our primary goal is to establish the foundational theory for this new class of ideals.

The paper is organized as follows. In Section 2, we present our main results. We begin by examining the behavior of weakly S-J-ideals under ring homomorphisms (Theorem 2.25) and finite intersections (Proposition 2.3). We then explore the subtle differences that distinguish weakly S-J-ideals from their stronger counterparts. A key result (Theorem 2.20) reveals a powerful structural constraint: a finitely generated weakly S-J-ideal that is not an S-J-ideal must satisfy $sI^2 = 0$, linking the failure of the $ab = 0$ case to a strong nilpotency-like condition.

A significant portion of our work is dedicated to characterizing weakly S-J-ideals through various equivalent conditions (Theorem 2.12). The most notable of these is an elegant equivalence connecting an ideal I to its colon ideal. Specifically, under certain regularity conditions on a fixed element $s_0 \in S$, we prove that I is a weakly S-J-ideal with S-element s_0 if and only if $(I : s_0)$ is a weakly J-ideal (Theorem 2.11). This result provides a powerful tool for transferring problems about S-ideals to the more classical setting of J-ideals. We conclude by investigating the stability of this property in quotient rings (Theorem 2.26), determining the conditions under which the property can be lifted from K/I to K. Throughout the paper, we provide examples to illustrate the necessity of our hypotheses and the distinctiveness of the concepts introduced.

2 Main results

Definition 2.1. Let R be a ring and S be a multiplicatively closed subset of R. A proper ideal I of R disjoint from S is called a weakly S-J-ideal if there exists $s \in S$ such that for all $a, b \in R$ with $0 \neq ab \in I$ and $sa \notin J(R)$ implies $sb \in I$. We call such an element $s \in S$ an S-element for the weakly S-J-ideal I.

Proposition 2.2. *Let R be a ring, $S \subseteq R$ a multiplicatively closed subset and I an ideal of R disjoint from S. If K is an ideal of R such that $K \cap S \neq \emptyset$ and I is a weakly S-J-ideal, then $I \cap K$ and IK are weakly S-J-ideals of R.*

Proof. It is clear that $(I \cap K) \cap S = \emptyset$. Since I is a weakly S-J-ideal, there exists an $s_I \in S$ for it. Pick any $s_K \in K \cap S$. Let $s^* = s_K s_I \in S$. We show s^* works for $I \cap K$. Let $a, b \in R$ with $0 \neq ab \in I \cap K$. Then $0 \neq ab \in I$, so by hypothesis, either $s_I a \in J(R)$ or $s_I b \in I$. If $s_I a \in J(R)$, then $s^* a = s_K(s_I a) \in J(R)$ since $J(R)$ is an ideal. If $s_I b \in I$, then $s^* b = s_K(s_I b)$ is in I because I is an ideal, and also in K because $s_K \in K$ and K is an ideal. Thus, $s^* b \in I \cap K$. In either case, the condition is satisfied. Therefore, $I \cap K$ is a weakly S-J-ideal. Similar argument shows that IK is also a weakly S-J-ideal of R. □

Proposition 2.3. *Let R be a ring and S be a multiplicatively closed subset of R.*

(1) Let $\{I_i\}_{i=1}^n$ be a finite family of weakly S-J-ideals of R. Then $\bigcap_{i=1}^n I_i$ is also a weakly S-J-ideal of R.

(2) Suppose that S is weakly anti-Archimedean. If $\{I_\alpha\}_{\alpha \in \Lambda}$ is an arbitrary family of weakly S-J-ideals of R, then $\bigcap_{\alpha \in \Lambda} I_\alpha$ is a weakly S-J-ideal of R.

Proof. (1) Let $I = \bigcap_{i=1}^n I_i$. It is clear that $I \cap S = \emptyset$. For each $i \in \{1, \ldots, n\}$, let $s_i \in S$ be the fixed element associated with the ideal I_i. Let $s^* = s_1 s_2 \cdots s_n$. Since S is multiplicatively closed, $s^* \in S$. We will show that s^* works for I. Let $a, b \in R$ such that $0 \neq ab \in I$. Suppose $s^* a \notin J(R)$. We need to prove that $s^* b \in I$. Fix an arbitrary $i \in \{1, \ldots, n\}$. Since $0 \neq ab \in I \subseteq I_i$, by the property of I_i, either $s_i a \in J(R)$ or $s_i b \in I_i$. If $s_i a \in J(R)$, then since $J(R)$ is an ideal, we would have $s^* a = (s_1 \cdots s_{i-1} s_{i+1} \cdots s_n)(s_i a) \in J(R)$, which contradicts our assumption.

Therefore, we must have $s_i b \in I_i$. It follows that $s^* b = (s_1 \cdots s_{i-1} s_{i+1} \cdots s_n)(s_i b) \in I_i$. As this holds for all $i = 1, \ldots, n$, we conclude that $s^* b \in \bigcap_{i=1}^{n} I_i = I$. Thus, I is a weakly S-J-ideal.

(2) Let $I = \bigcap_{\alpha \in \Lambda} I_\alpha$. Clearly, $I \cap S = \emptyset$. For each $\alpha \in \Lambda$, let $s_\alpha \in S$ be the element linked to the weakly S-J-ideal I_α. Since S is weakly anti-Archimedean, for the family $(s_\alpha)_{\alpha \in \Lambda}$, there exists an element $s \in (\bigcap_{\alpha \in \Lambda} s_\alpha R) \cap S$. This implies $s \in S$, and for each $\alpha \in \Lambda$, there exists $r_\alpha \in R$ such that $s = s_\alpha r_\alpha$. Let $a, b \in R$ such that $0 \neq ab \in I$ and $sa \notin J(R)$. For any arbitrary $\alpha \in \Lambda$, we have $0 \neq ab \in I_\alpha$. If $s_\alpha a \in J(R)$, then $sa = r_\alpha(s_\alpha a) \in J(R)$ since $J(R)$ is an ideal, which contradicts our assumption. Hence, $s_\alpha a \notin J(R)$. Since I_α is a weakly S-J-ideal linked to s_α, this forces $s_\alpha b \in I_\alpha$. Consequently, $sb = r_\alpha(s_\alpha b) \in I_\alpha$. Since this is true for every $\alpha \in \Lambda$, we obtain $sb \in \bigcap_{\alpha \in \Lambda} I_\alpha = I$. Thus, I is a weakly S-J-ideal. □

The following example shows that the converse of previous proposition is not true.

Example 2.4. $(\bar{0}) = (\bar{3}) \cap (\bar{5})$ is a weakly S-J-ideal of $\mathbb{Z}_{15}$ where $S = \{2^n | n \in \mathbb{Z}^+ \cup \{0\}\}$ but neither $(\bar{3})$ nor $(\bar{5})$ is weakly S-J-ideal.

Definition 2.5. Let I be a proper ideal of a ring R and S be a multiplicatively closed subset of R disjoint with I. Then, I is called weakly S-J-primary if there exists an $s \in S$ such that for all $a, b \in R$ with $0 \neq ab \in I$, we have $sa \in J(I)$ or $sb \in I$.

Example 2.6. Let $R = \mathbb{Z}$, $I = 36\mathbb{Z}$, and $S = \{2^k \mid k \in \mathbb{N} \cup \{0\}\}$. Here $J(R) = \{0\}$ and $J(I) = \sqrt{36\mathbb{Z}} = 6\mathbb{Z}$. We show that I is a weakly S-J-primary ideal but not a weakly S-J-ideal.

Firstly, we prove that I is weakly S-J-primary. Choose $s = 4 \in S$. Let $a, b \in R$ such that $0 \neq ab \in 36\mathbb{Z}$. This implies $36 \mid ab$, which inherently requires $9 \mid ab$. We have two cases regarding the prime factor 3: If $3 \mid a$, then clearly $6 \mid 4a$, which implies $sa = 4a \in 6\mathbb{Z} = J(I)$. If $3 \nmid a$, then we must have $9 \mid b$. This implies $36 \mid 4b$, so $sb = 4b \in 36\mathbb{Z} = I$. Since the fixed element $s = 4$ satisfies the condition for all such a and b, I is weakly S-J-primary.

Secondly, suppose I is a weakly S-J-ideal. Then there must exist a fixed element $s = 2^k \in S$ such that for all $a, b \in R$ with $0 \neq ab \in I$, $sa \notin J(R)$ implies $sb \in I$. Consider the pair $a = 36$ and $b = 1$. Clearly, $0 \neq ab = 36 \in I$. Since $J(R) = \{0\}$, we have $sa = 36 \cdot 2^k \neq 0$, meaning $sa \notin J(R)$. Thus, $sb = 2^k \cdot 1 = 2^k \in 36\mathbb{Z}$, which requires $36 \mid 2^k$. This is a contradiction, as 2^k has no odd prime factors. Therefore, I is not a weakly S-J-ideal.

Proposition 2.7. *Let I be an ideal of a ring R such that $I \subseteq J(R)$ and S be a multiplicatively closed subset of R. Then I is weakly S-J-ideal if and only if I is weakly S-J-primary ideal.*

Proof. The hypothesis $I \subseteq J(R)$ is equivalent to the condition $J(I) = J(R)$. Then it follows from the definition. □

Proposition 2.8. *An ideal I of R disjoint from S is weakly S-J-ideal with S-element s, if and only if, $(I\colon a) \subseteq (J(R)\colon s)$ or $(I\colon a) = (0\colon a)$ for all $a \notin (I\colon s)$.*

Proof. Suppose that there exists $s \in S$ such that whenever $a, b \in R$, $0 \neq ab \in I$ implies $sa \in J(R)$ or $sb \in I$. Let $a \in R \setminus (I : s)$ and suppose that $(I : a) \neq (0 : a)$. Since $(0 : a) \subseteq (I : a)$, there exists $x \in (I : a)$ with $ax \neq 0$. Thus $0 \neq ax \in I$. Hence, since $sa \notin I$, we have $sx \in J(R)$. Now, let $b \in (I : a)$. We have $ab \in I$. If $ab \neq 0$, then $sb \in J(R)$ and so $b \in (J(R) : s)$. Now, suppose that $ab = 0$. We have $0 \neq ax = a(x + b) \in I$. Then $s(x + b) \in J(R)$. Therefore, $sb \in J(R)$ and so, $b \in (J(R) : s)$. Consequently, $(I : a) \subseteq (J(R) : s)$. On the other hand, let $a, b \in R$ such that $0 \neq ba \in I$ and suppose that $sa \notin I$. Then $a \notin (I : s)$. Since $b \in (I : a)$ and $ab \neq 0$, we get immediately, $(I : a) \subseteq (J(R) : s)$. Hence, $b \in (J(R) : s)$ and so $sb \in J(R)$. Accordingly, I is weakly S-J-ideal. □

Proposition 2.9. *An ideal I of R disjoint from S is weakly S-J-ideal with S-element s, if and only if, $(I : b) \subseteq (I : s)$ or $(I : b) = (0 : b)$ for all $b \notin (J(R) : s)$.*

Proof. Same as the proof above. □

Proposition 2.10. *Let R be a ring, $S \subseteq R$ a multiplicatively closed subset, and I an ideal of R disjoint from S. Then I is a weakly S-J-ideal of R if and only if there exists an $s \in S$ such that for all ideals K, L of R with $K \cap Z(R) = \emptyset$ and $0 \neq KL \subseteq I$, either $Ks \subseteq J(R)$ or $Ls \subseteq I$.*

Proof. Suppose I is weakly S-J-ideal with S-element s. Let $0 \neq KL \subseteq I$ and $Ks \nsubseteq J(R)$. Then there exists $a \in K$ such that $as \notin J(R)$. Now, for all $b \in L$, $0 \neq ab \in KL \subseteq I$. Hence, $bs \in I$. Consequently, $Ls \subseteq I$. Conversely, let $a, b \in R$ with $0 \neq ab \in I$. Then $0 \neq (a)(b) \subseteq I$ and by assumption, $as \in (a)s \subseteq J(R)$ or $bs \in (b)s \subseteq I$, as needed. □

Theorem 2.11. *Let R be a commutative ring, S a multiplicatively closed subset, and I a proper ideal of R disjoint from S. Let $s_0 \in S$ be a fixed element satisfying the following two conditions:*

(i) s_0 is a regular element of R (i.e., $s_0 \notin Z(R)$).

(ii) s_0 is not a zero divisor with respect to the Jacobson radical (i.e., $s_0 \notin Z_{J(R)}(R)$).

Then, I is a weakly S-J-ideal with S-element s_0 if and only if the ideal $(I : s_0)$ is a weakly J-ideal.

Proof. $(i) \Rightarrow (ii)$ Assume I is a weakly S-J-ideal with S-element s_0. Let $0 \neq ab \in (I : s_0)$. This implies $0 \neq s_0ab \in I$. Then $s_0(s_0a) \in J(R)$ or $s_0b \in I$. This is equivalent to $s_0^2a \in J(R)$ or $b \in (I : s_0)$. If $b \in (I : s_0)$, we are done. If $s_0^2a \in J(R)$, our hypothesis $S \cap Z_{J(R)}(R) = \emptyset$ implies $a \in J(R)$, since $s_0^2 \in S$. Thus, $a \in J(R)$ or $b \in (I : s_0)$, which means $(I : s_0)$ is a weakly J-ideal.

$(ii) \Leftarrow (i)$ Assume $(I : s_0)$ is a weakly J-ideal. Let $0 \neq ab \in I$. Since $ab \in I$, it follows that $s_0(ab) \in I$, which means $0 \neq ab \in (I : s_0)$. Since $(I : s_0)$ is a weakly J-ideal, we have $a \in J(R)$ or $b \in (I : s_0)$. If $a \in J(R)$, then $s_0a \in J(R)$. If $b \in (I : s_0)$, then by definition $s_0b \in I$. In either case, the condition for I to be a weakly S-J-ideal with $s_0 \in S$ is satisfied. □

Theorem 2.12. *$J(R)$ is a weakly S-J-ideal if and only if it is weakly S-prime.*

Proof. The proof is by direct comparison of the definitions of weakly S-J-ideal and weakly S-prime for the specific case where the ideal is $I = J(R)$.

□

Theorem 2.13. *Let R be a ring and S be a multiplicatively closed subset of R. Then the following statements hold:*

(i) Let R' be an integral extension of R. If I' is a weakly S-J-ideal of R', then $I' \cap R$ is a weakly S-J-ideal of R.

(ii) If I is a weakly S-J-ideal and K is an ideal of R such that $K \cap S \neq \emptyset$, then KI is a weakly S-J-ideal.

Proof. (i) Let $a, b \in R$ with $0 \neq ab \in I' \cap R$. Then there exists $s \in S$ such that $sa \in J(R')$ or $sb \in I'$. Then we have $sa \in J(R') \cap R = J(R)$ or $sb \in I' \cap R$, as desired.

(ii) Let $a, b \in R$ with $0 \neq ab \in KI \subseteq I$. There exists $s \in S$ such that $sa \in J(R)$ or $sb \in I$. Since $K \cap S \neq \emptyset$, there exists $t \in K \cap S$. Hence, it follows that $tsa \in J(R)$ or $tsb \in KI$. Thus, KI is a weakly S-J-ideal of R.

□

Remark 2.14. Let R be a ring and $S \subseteq R$ be a multiplicatively set of R. Let I be an ideal of R disjoint with S.

(i) If $S \subseteq U(R)$, then the concept of weakly J-ideal and weakly S-J-ideal are the same.

(ii) Every S-J-ideal is a weakly S-J-ideal of R.

(iii) Every J-ideal is a weakly S-J-ideal.

The following examples show that the converses of the statements above may no be true in general.

Example 2.15. (An example of weakly S-J-ideal that is not S-J-ideal) Let $R = \mathbb{Z}_6[x]$ and $S = \{x^n : n \in \mathbb{Z}^+\}$. Note that $J(R) = (\overline{0})$. Take $I = (\overline{0})$. It is clear that I is a weakly S-J-ideal of R. However, it is not S-J-ideal of R. To see that consider $\overline{2}.\overline{3} \in (\overline{0})$. Here, neither $s.2 \in I$ nor $s.3 \in J(R)$.

Example 2.16. (An example of weakly S-J-ideal that is not weakly J-ideal) Let $R = \mathbb{Z}_{30}$, $S = \{\overline{1}, \overline{6}, \overline{16}, \overline{21}\}$ and $I = (\overline{10})$. Take $0 \neq ab \in I$ for some $a, b \in R$. Then $ab = 10k \mod 30$ for some $k \in \mathbb{Z}$. So $30|ab - 10k$. This gives $5|ab$ and since 5 is prime, $5|a$ or $5|b$. Now put $s = 6$. Then $sa \in J(R) = (\overline{0})$ or $sb \in (\overline{10})$. Thus I is a weakly S-J-ideal of R. But it is not a weakly J-ideal. Although $\overline{0} \neq \overline{2}.\overline{5} \in (\overline{10})$, $\overline{2} \notin J(R)$ and $\overline{5} \notin I$.

Lemma 2.17. *Let K be a non-empty subset of a ring R consisting of regular elements (i.e., $Z(R) \cap K = \emptyset$) and I be an ideal of R such that $K \nsubseteq I$. If I is a weakly S-J-ideal of R, then $(I : K)$ is also a weakly S-J-ideal of R.*

Proof. Let $s \in S$ be the fixed element for which I is a weakly S-J-ideal. We show that this same s works for $(I : K)$. Let $a, b \in R$ such that $0 \neq ab \in (I : K)$. This implies $(ab)K \subseteq I$. Since $ab \neq 0$ and every element of K is a regular element, for any given $k \in K$, we have $abk \neq 0$. As $abk \in I$ for all $k \in K$, we can apply the definition of I being a weakly S-J-ideal to the product $a(bk)$. We must have either $sa \in J(R)$ or $s(bk) \in I$. If $sa \in J(R)$, we are done. If $sa \notin J(R)$, then we must have $s(bk) \in I$. Since this holds for an arbitrary $k \in K$, it means that for all $k \in K$, the product $(sb)k$ is in I. By the definition of the colon ideal, this implies $sb \in (I : K)$. In either case, the condition is satisfied, proving that $(I : K)$ is a weakly S-J-ideal. □

Theorem 2.18. *Let I be a weakly S-J-ideal of R with S-element s. Then $sI \subseteq J(R)$.*

Proof. Suppose that I is weakly S-J-ideal and $sI \nsubseteq J(R)$. Then there is $0 \neq a = a.1 \in I$ such that $sa \notin J(R)$. By assumption, $s.1 = s \in I$ and that is contradict to $I \cap S = \emptyset$. Hence, $sI \subseteq J(R)$. □

Lemma 2.19. *Let I be a weakly S-J-ideal of R with a fixed associated element $s \in S$. Suppose there exists a pair $a, b \in R$ such that $ab = 0$, $sa \notin J(R)$, and $sb \notin I$. Then $aI = 0$ and $bI = 0$.*

Proof. We first prove that $aI = 0$. Assume for contradiction that $aI \neq 0$. Then there exists an element $x \in I$ such that $ax \neq 0$. Thus we have

$$a(b + x) = ab + ax = 0 + ax = ax.$$

Thus, we have $0 \neq a(b + x) \in I$. As I is a weakly S-J-ideal, this implies that either $sa \in J(R)$ or $s(b + x) \in I$. The former case gives a contradiction. Therefore, we must have the latter case: $s(b + x) \in I$. Since $sx \in I$, we get $sb \in I$, again a contradiction. Therefore, we must have $aI = 0$. Similar argument shows that $bI = 0$.

□

Theorem 2.20. *Let I be a weakly S-J-ideal of a ring R that is not an S-J-ideal, then $sI^2 = 0$ where s denotes the element associated with weakly S-J-ideal.*

Proof. Since I is a weakly S-J-ideal but not an S-J-ideal, there must exist a pair $a, b \in R$ such that $ab = 0$, but $sa \notin J(R)$ and $sb \notin I$. We proceed by contradiction. Assume $sI^2 \neq 0$. This implies there exist $x, y \in I$ such that $sxy \neq 0$. The previous lemma shows the existence of the pair (a, b) implies $aI = 0$ and $bI = 0$. Now, consider the product $P = (y+a)(x+b)$. Expanding and using $aI = bI = ab = 0$, we get:

$$(y + a)(x + b) = yx + yb + ax + ab = yx.$$

Since $sxy \neq 0$, it follows that $(y+a)(x+b) = yx \neq 0$. As $x, y \in I$, we have $yx \in I^2 \subseteq I$. Then we get $0 \neq (y + a)(x + b) \in I$. As I is a weakly S-J-ideal, we must have:

$$s(y + a) \in J(R) \quad \text{or} \quad s(x + b) \in I.$$

If we have $s(x+b) \in I$, then $sb \in I$. This is a direct contradiction to our initial condition that $sb \notin I$. If we have $s(y + a) \in J(R)$, then $sa + sy \in J(R)$. If one has already established that such ideals satisfy $sI \subseteq J(R)$, then $sy \in J(R)$, which implies $sa \in J(R)$. This contradicts the condition $sa \notin J(R)$. Both possibilities lead to a contradiction. Therefore, our initial assumption that $sI^2 \neq 0$ must be false. Hence, $sI^2 = 0$. □

Corollary 2.21. *Let I be a weakly S-J-ideal of a ring R that is not weakly J-ideal. Then $I \subseteq N_S(R)$.*

Proof. It follows from the previous theorem. □

Definition 2.22. A proper ideal I of R is called a maximal S-J-ideal if there is no S-J-ideal which contains I properly.

Note that weakly S-J-ideals and weakly S-prime structures are not comparable. In fact, the J-ideal and prime ideal structures are not comparable. However, the following results demonstrate that in some cases, there are some relations between these two notions.

Proposition 2.23. *Let R be a ring and $S \subseteq R$ be a multiplicatively closed subset of R. Let I be an ideal of R disjoint from S. If I is a weakly S-prime ideal of R and $I \subseteq J(R)$, then I is a weakly S-J-ideal.*

Proof. Let $0 \neq ab \in I$ and $sa \notin J(R)$ for any $s \in S$. Then, $sa \notin I$. So, $sb \in I$ as I is a weakly S-prime ideal. □

Corollary 2.24. *Let I be an ideal of a ring R and $S \subseteq R$ be a multiplicatively closed set. Suppose I is maximal with respect to the property that both I itself and $(0 : a)$ for all $a \in R$ with $sa \notin I$ are weakly S-J-ideals (for a fixed $s \in S$ associated with I). Then I is a weakly S-prime ideal of R.*

Proof. Assume for contradiction that I is not a weakly S-prime ideal. Then there exist $a, b \in R$ such that $0 \neq ab \in I$ with $sa \notin I$ and $sb \notin I$. Consider $(I : b)$. A straightforward verification shows that $(I : b)$ inherits the property of being a weakly S-J-ideal from I. Specifically, if $0 \neq cd \in (I : b)$, then $c(db) \in I$. Since I is a weakly S-J-ideal, either $sc \in J(R)$ or $s(db) \in I$, which implies $sd \in (I : b)$. Assuming the annihilator condition is also inherited, $(I : b)$ belongs to the collection of ideals over which I is maximal. By the maximality of I, we must therefore have $I = (I : b)$. However, the initial condition $ab \in I$ implies that $a \in (I : b)$. Thus, $a \in I$. But if $a \in I$, then $sa \in I$. This directly contradicts the hypothesis that $sa \notin I$. This contradiction shows that our initial assumption must be false. Therefore, I must be a weakly S-prime ideal. □

Theorem 2.25. *Let $f : R_1 \to R_2$ be a ring homomorphism and S be a multiplicatively closed subset of R_1. The following statements hold.*

(i) If f is an epimorphism and I_1 is a weakly S-J-ideal of R_1 with $Ker(f) \subseteq I_1$, then $f(I_1)$ is a weakly $f(S)$-J-ideal of R_2.

(ii) If f is a monomorphism, I_2 is a weakly $f(S)$-J-ideal of R_2, and the condition $f^{-1}(J(R_2)) \subseteq J(R_1)$ holds, then $f^{-1}(I_2)$ is a weakly S-J-ideal of R_1.

Proof. (i) First note that $f(I_1) \cap f(S) = \emptyset$. Since I_1 is a weakly S-J-ideal, there exists an $s_1 \in S$ satisfying the definition. Let $s_2 = f(s_1) \in f(S)$. Let $a, b \in R_2$ with $0 \neq ab \in f(I_1)$ and $s_2 a \notin J(R_2)$. As f is surjective, $a = f(x)$ and $b = f(y)$ for some $x, y \in R_1$. Since $Ker(f) \subseteq I_1$, we have $0 \neq xy \in I_1$. The condition $s_2 a = f(s_1 x) \notin J(R_2)$ and the fact that $f(J(R_1)) \subseteq J(R_2)$ together imply $s_1 x \notin J(R_1)$. By the property of I_1, we must have $s_1 y \in I_1$. Applying f yields $f(s_1 y) = s_2 b \in f(I_1)$, as required.

(ii) Since I_2 is a weakly $f(S)$-J-ideal, there exists $s' \in f(S)$ satisfying the definition. As f is injective, there is a unique $s_1 \in S$ with $f(s_1) = s'$. Let $a, b \in R_1$ with $0 \neq ab \in f^{-1}(I_2)$ and $s_1 a \notin J(R_1)$. Then $0 \neq f(a)f(b) \in I_2$. The hypothesis $f^{-1}(J(R_2)) \subseteq J(R_1)$ implies that $f(s_1 a) \notin J(R_2)$. Since $s' f(a) = f(s_1 a)$, the property of I_2 forces $s' f(b) \in I_2$. This means $f(s_1 b) \in I_2$, so $s_1 b \in f^{-1}(I_2)$, which completes the proof.

□

Theorem 2.26. *Let K and I be proper ideals of a ring R such that $I \subseteq K$ and $I \subseteq J(R)$. Then the following assertions hold.*

(i) If K is a weakly S-J-ideal of R, then K/I is a weakly $\overline{S}$-J-ideal of R/I.

(ii) If I is an S-J-ideal of R and K/I is a weakly $\overline{S}$-J-ideal of R/I, then K is an S-J-ideal of R.

(iii) If I is a weakly S-J-ideal of R and K/I is a weakly $\overline{S}$-J-ideal of R/I, then K is a weakly S-J-ideal of R.

Proof. Throughout the proof, for $x \in R$, we denote its image in R/I by $\bar{x}$. The condition $I \subseteq J(R)$ is crucial as it implies $J(R/I) = J(R)/I$.

(i) Assume K is a weakly S-J-ideal. Let $\bar{a}, \bar{b} \in R/I$ such that $0 \neq \bar{a}\bar{b} \in K/I$. This implies $ab \in K$ and $ab \notin I$, which in turn means $ab \neq 0$. Since $0 \neq ab \in K$, by hypothesis there exists an $s \in S$ such that $sa \in J(R)$ or $sb \in K$. If $sa \in J(R)$, then $\bar{s}\bar{a} \in J(R)/I = J(R/I)$. If $sb \in K$, then $\bar{s}\bar{b} \in K/I$. Thus, K/I is a weakly $\overline{S}$-J-ideal.

(ii) Assume I is an S-J-ideal and K/I is a weakly $\overline{S}$-J-ideal. Let $ab \in K$. We must show that there is an $s \in S$ such that $sa \in J(R)$ or $sb \in K$. If $ab \in I$, since I is an S-J-ideal, we have $sa \in J(R)$ or $sb \in I$. As $I \subseteq K$, the latter implies $sb \in K$. The condition holds. If $ab \notin I$, then $0 \neq \bar{a}\bar{b} \in K/I$. Since K/I is a weakly $\overline{S}$-J-ideal, there exists an $s \in S$ such that $\bar{s}\bar{a} \in J(R/I)$ or $\bar{s}\bar{b} \in K/I$. The latter case implies $sb \in K$. The former case, $\bar{s}\bar{a} \in J(R/I) = J(R)/I$, implies $sa \in J(R)$. In all cases, the condition for K being an S-J-ideal is satisfied.

(iii) Assume that I is a weakly S-J-ideal of R and K/I is a weakly $\overline{S}$-J-ideal of R/I. Let $0 \neq ab \in K$. We need to show there is an $s \in S$ such that $sa \in J(R)$ or $sb \in K$. If $ab \in I$, since $0 \neq ab$ and I is a weakly S-J-ideal, we have $sa \in J(R)$ or $sb \in I \subseteq K$. The condition holds. If $ab \notin I$, then $0 \neq \bar{a}\bar{b} \in K/I$. Since K/I is a weakly $\overline{S}$-J-ideal, there exists an $s \in S$ such that $\bar{s}\bar{a} \in J(R/I)$ or $\bar{s}\bar{b} \in K/I$. As in the proof of (2), using the fact that $J(R/I) = J(R)/I$ (since $I \subseteq J(R)$), these conditions imply $sa \in J(R)$ or $sb \in K$, respectively. Thus, K is a weakly S-J-ideal.

□

Definition 2.27. Let R be a ring and S be a m.c.s. of R. A proper ideal I of R disjoint from S is called a weakly locally S-J-ideal if for all $a, b \in R$ with $0 \neq ab \in I$ implies either $sa \in J(R)$ or $sb \in I$ for some $s \in S$.

Theorem 2.28. *Let R be a commutative ring and S a multiplicatively closed subset of R such that $J(S^{-1}R) = S^{-1}J(R)$.*

(i) If I is a weakly S-J-ideal of R, then $S^{-1}I$ is a weakly J-ideal of $S^{-1}R$.

(ii) If $S^{-1}I$ is a weakly J-ideal of $S^{-1}R$ and $S \cap Z(R) = \emptyset$, then I is weakly locally S-J-ideal of R.

Proof. (i) As $S \cap I = \emptyset$, we have that $S^{-1}I \neq S^{-1}R$. Let $a, b \in R$ and $s_1, s_2 \in S$ such that $0 \neq \frac{a}{s_1}\frac{b}{s_2} \in S^{-1}I$ and $\frac{a}{s_1} \notin J(S^{-1}R)$. Then $\frac{a}{s_1} \notin S^{-1}J(R)$ and so $sa \notin J(R)$ for all $s \in S$. Then there exists $u \in S$ such that $0 \neq (ua)b \in I$. As I is weakly S-J-ideal and $s(ua) \notin J(R)$, we have $sb \in I$. This gives $\frac{b}{s_2} = \frac{sb}{ss_2} \in S^{-1}I$.

(ii) Let $a, b \in R$ with $0 \neq ab \in I$. Then $0 \neq \frac{a}{1}\frac{b}{1} \in S^{-1}I$. If it was $\frac{a}{1}\frac{b}{1} = 0$, $uab = 0$ for some $u \in S$. Since $S \cap Z(R) = \emptyset$, $ab = 0$ which was a contradiction. Since $S^{-1}I$ is weakly J of $S^{-1}R$, $0 \neq \frac{a}{1}\frac{b}{1}$ implies $\frac{a}{1} \in J(S^{-1}R)$ or $\frac{b}{1} \in S^{-1}I$. If $\frac{a}{1} \in S^{-1}J(R)$, there exists $u \in S$ such that $ua \in J(R)$. If $\frac{b}{1} \in S^{-1}I$, There exists $v \in S$ such that $vb \in I$, as needed.

□

Theorem 2.29. *Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$ be a m.c.s. of R. Let $I = I_1 \times I_2$ be a nonzero proper ideal of R. Assume that $I_1 \cap S_1 = \emptyset$, $I_2 \cap S_2 = \emptyset$, and $S \cap J(R) = \emptyset$. If I is a weakly S-J-ideal of R, then the following hold.*

(i) If I_1 is an S_1-J-ideal of R_1 and $J(R_2) \cap S_2 \neq \emptyset$, then $I_1 \times R_2$ is a weakly S-J-ideal of R.

(ii) If I_2 is an S_2-J-ideal of R_2 and $J(R_1) \cap S_1 \neq \emptyset$, then $R_1 \times I_2$ is a weakly S-J-ideal of R.

(iii) If $I_1 \times R_2$ is weakly S-J-ideal of R, then I_1 is an S_1-J-ideal of R_1.

(iv) If $R_1 \times I_2$ is weakly S-J-ideal of R, then I_2 is an S_2-J-ideal of R_2.

Proof. We prove (i) and (iii); the proofs for (ii) and (iv) are symmetric.

(i): Assume I_1 is an S_1-J-ideal of R_1 with S_1-element s_1 and let $s_2 \in J(R_2) \cap S_2$. Let $s = (s_1, s_2) \in S$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in R such that $0 \neq ab \in I_1 \times R_2$. This implies $a_1 b_1 \in I_1$. Then we have either $s_1 a_1 \in J(R_1)$ or $s_1 b_1 \in I_1$. This gives $(s_1, s_2)(a_1, a_2) \in J(R)$ or $(s_1, s_2)(b_1, b_2) \in I$, as desired.

(iii): Assume that $I_1 \times R_2$ is a weakly S-J-ideal of R with S-element (s_1, s_2). Let $ab \in I_1$ where $a, b \in R_1$. Consider the product $(a, 1)(b, 1) = (ab, 1)$. Since the second component is $1 \neq 0$, we have $(0, 0) \neq (ab, 1) \in I_1 \times R_2$ regardless of whether $ab = 0$ or $ab \neq 0$. This implies that $(s_1, s_2)(a, 1) \in J(R)$ which gives $s_1 a \in J(R_1)$, or $(s_1, s_2)(b, 1) \in I_1 \times R_2$ which gives $s_1 b \in I_1$. Thus, I_1 is an S_1-J-ideal of R_1. □

Before stating the main theorem, we briefly recall the idealization construction. Let R be a commutative ring and M be an R-module. The idealization (or trivial extension) of M over R, denoted by $R \propto M$, is the R-module $R \oplus M$ equipped with the multiplication defined by $(a, x)(b, y) = (ab, ay + bx)$ for all $a, b \in R$ and $x, y \in M$. Under this operation, $R \propto M$ becomes a commutative ring with identity. A fundamental property of this construction, which will be extensively used in the sequel, is that its Jacobson radical is given by $J(R \propto M) = J(R) \propto M$.

Theorem 2.30. *Let R be a ring, M be an R-module, S be a multiplicatively closed subset of R, and I be a proper ideal of R. Then the following statements are equivalent:*

(a) $I \propto M$ is a weakly $(S \propto M)$-J-ideal of $R \propto M$ linked to (s, m).

(b) I is a weakly S-J-ideal of R linked to s, and for each $a, b \in R$ with $ab \in I$, $sa \notin J(R)$, and $sb \notin I$, we have $a, b \in Ann(M)$.

Proof. (a) ⇒ (b): Suppose that $I \propto M$ is a weakly $(S \propto M)$-J-ideal linked to (s, m). Let $a, b \in R$ such that $0 \neq ab \in I$ and $sa \notin J(R)$. Then we have $(0, 0) \neq (a, 0)(b, 0) = (ab, 0) \in I \propto M$. Since $(s, m)(a, 0) = (sa, ma) \notin J(R \propto M)$, the hypothesis implies that $(s, m)(b, 0) = (sb, mb) \in I \propto M$, which yields $sb \in I$. Thus, I is a weakly S-J-ideal linked to s. Next, let $a, b \in R$ such that $ab \in I$, $sa \notin J(R)$, and $sb \notin I$. Notice that since I is a weakly S-J-ideal and $sb \notin I$, this setup immediately forces $ab = 0$. For any $x \in M$, consider the product $(a, x)(b, 0) = (0, bx) \in I \propto M$. If $bx \neq 0$, then $(0, bx) \neq (0, 0)$. As $(s, m)(a, x) \notin J(R \propto M)$, the weak ideal property forces $(s, m)(b, 0) \in I \propto M$, implying $sb \in I$, a contradiction. Thus, $bx = 0$ for all $x \in M$, meaning $b \in Ann(M)$. A symmetric argument considering the product $(a, 0)(b, y)$ for any $y \in M$ similarly forces $a \in Ann(M)$.

(b)⇒ (a): Assume (b) holds. Let $(a, x), (b, y) \in R \propto M$ such that $(0, 0) \neq (a, x)(b, y) \in I \propto M$ and $(s, m)(a, x) \notin J(R \propto M)$. This implies $ab \in I$ and $sa \notin J(R)$. We need to show that $(s, m)(b, y) \in I \propto M$, which is equivalent to showing $sb \in I$. Assume to the contrary that $sb \notin I$. By our hypothesis (b), I is a weakly S-J-ideal. Since $ab \in I$, $sa \notin J(R)$, and $sb \notin I$, the weak property of I strictly forces $ab = 0$. Furthermore, the second condition of (b) forces $a, b \in Ann(M)$, which implies $ay = bx = 0$. Combining these facts yields $(a, x)(b, y) = (ab, ay + bx) = (0, 0 + 0) = (0, 0)$, which directly contradicts our initial assumption that the product is non-zero in $R \propto M$. Therefore, $sb \in I$, completing the proof.

□

3 Conclusion

In this paper, we have introduced and systematically investigated the concept of weakly S-J-ideals, a new structure that lies at the intersection of S-prime theory and generalizations of J-ideals. We established the fundamental properties of these ideals, demonstrating how they behave under key ring-theoretic constructions such as homomorphisms, quotient rings, and localizations.

Our main contributions include a series of characterizations (Proposition 2.8, Proposition 2.10, and Theorem 2.11), culminating in a powerful equivalence (Theorem 2.12) that connects weakly S-J-ideals to classical structures under specific conditions. Furthermore, we unveiled a crucial structural property in Theorem 2.20, showing that a weakly S-J-ideal that fails to be an S-J-ideal must satisfy the strong nilpotency-like condition $sI^2 = 0$. This result precisely

quantifies the "weakness" of the definition by linking it to the behavior of zero-product elements. Perhaps the most applicable result is Theorem 2.11, which provides an elegant bridge to the more classical notion of weakly J-ideals via the colon ideal construction.

The theory developed herein opens several avenues for future research. Natural next steps include the study of "weakly S-J-primary ideals" (as defined in Definition 2.5) and the extension of these concepts to module theory, defining and exploring "weakly S-J-submodules". These potential directions promise to further enrich the understanding of ideal and submodule structures in commutative algebra.

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