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THE SPECTRUM AND WIENER INDEX OF THE NIL CLEAN DIVISOR GRAPH OF A FINITE COMMUTATIVE RING

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Abstract Let R be a finite commutative ring. An element of R is a nil clean element of R if it can be written as a sum of a nilpotent and an idempotent element. The nil clean divisor graph of R denoted by $G_{N_D}(R)$ is a graph with vertex set $\{x \neq 0 \in R : \exists\, y \neq 0, y \neq x \textit{ such that } xy \in NC(R)\}$, where $NC(R)$ is the set of all nil clean elements of R and two vertices x and y are adjacent in $G_{N_D}(R)$ if $xy \in NC(R)$. In this paper, we extend some of the results of [2] . We compute the spectrum of the nil clean divisor graph of some rings and propose the Wiener index of the nil clean divisor graph of a finite commutative ring R.

1 Introduction

In recent years, connecting a graph with algebraic structure is frequently used one. In this association, a lot of graph parameters are related to the algebraic properties and it becomes a great development of its theory. For example, any commutative ring R with unity can associate to a graph, whose vertex set is $\{x \neq 0 \in R : \exists\, y \neq 0, y \neq x \textit{ such that } xy \in NC(R)\}$ and two vertices x and y are adjacent if xy is a nil clean element of R. This graph is called the nil clean divisor graph of R, denoted by $G_{N_D}(R)$. In [2] they have studied the basic graph theoretic properties like girth, completeness, diameter, connectedness of the nil clean divisor graph. They also studied the nature of the nil clean divisor graph of Z_{2p} and Z_{3p}, where p is prime. Similar to nil clean divisor graph, nil clean graph [9] , clean graph [12], square element graph of a Near ring [13] are also studied. In research area, finding various spectra and topological indices of graphs form rings play a vital role. In [16] and [1], the authors deal with spectra and topological indices of some graphs form rings.

In this paper, we focus on spectrum and the parameter Wiener index which are most frequently used in research area. First we provide the necessary results and notations used throughout this paper. In next section, we express a nil clean divisor graph of a ring as a generalized composition of totally disconnected or complete graphs with the usage of an equivalence relation. Further we discuss about the spectrum of a nil clean divisor graph of some class of rings. In next section, we compute the Wiener index of the nil clean divisor graph of a ring.

Here, some of the necessary definitions, results and notations of graph theory which are used in this paper are discussed from [19] . A graph G is a pair $G = (V, E)$, where $V = V(G)$ and $E = E(G)$ are the set of vertices and edges respectively. Let G be a graph. Two distinct vertices $x, y \in V(G)$ are adjacent, if there is an edge between x and y. A subgraph $G^{'}$ of a graph G is a graph such that $V(G^{'}) \subset V(G)$ and $E(G^{'}) \subset E(G)$. If U is a subset of $V(G)$ then the subgraph of G induced by U, denoted by $G(U)$ is the graph with vertex set U and two vertices of $G(U)$ are adjacent iff they are adjacent in G. A graph in which any two distinct vertices are adjacent is said to be a complete graph. The distance between any two vertices x and y of G, denoted by $d(x, y)$ is the number of edges in the shortest $x - y$ path. A path in a graph is a sequence of

distinct vertices in which each vertex is adjacent to the next vertex of that sequence. A graph G is said to be connected if there is a path between any two vertices of G. Throughout this paper we focus on any finite commutative ring with unity whose nil clean divisor graph is connected.

2 Nil clean divisor graph as a generalized composition of graphs

In this section, we focus on expressing a nil clean divisor graph as a generalized composition of graphs, which contributes more advantages in computing the spectrum and Wiener index of the nil clean divisor graph.

We begin with the definition of the nil clean divisor graph of a commutative ring with unity.

Definition 2.1. [2] Let R be a ring. The nil clean divisor graph of R, denoted by $G_{N_D}(R)$ is defined to be the simple graph with vertex set $\{x \neq 0 \in R : \exists y \neq 0, y \neq x \text{ such that } xy \in NC(R)\}$, where $NC(R)$ is the set of all nil clean elements of R. Two vertices x and y of $G_{N_D}(R)$ are adjacent if xy is a nil clean element of R.

An element is said to be a nil clean element of R if it can be written as a sum of a nilpotent and an idempotent element of R.

Schwenk [14] introduced the generalized composition of graphs, denoted as $H[G_1, G_2, \ldots G_k]$.

Definition 2.2. [14] Let $G_1, G_2, \ldots, G_k$ be k pairwise disjoint graphs and let H be a graph with k number of vertices, say $\{v_1, v_2, \ldots, v_k\}$. Then the *generalized composition graph* $G = H[G_1, G_2, \ldots, G_k]$ is a graph with vertex set $V(G_1) \cup V(G_2) \cup \ldots V(G_k)$ and the edge set consists of all the edges of $G_1, G_2, \ldots G_k$ together with all possible edges from $V(G_i)$ to $V(G_j)$ only when v_i and v_j are adjacent in H. Here, the graph H is called the base graph of G and the graphs $G_i^{'s}$ are called the factors of G.

Remark 2.3. The nil clean divisor graph of $\mathbb{Z}_7$ is the generalized composition $\overline{K}_2[K_2, K_2]$ where K_n is a complete graph with n vertices and $\overline{K}_n$ is an empty graph on n vertices.

Let R be any finite commutative ring with unity, whose $G_{N_D}(R)$ is connected. We can write $G_{N_D}(R)$ as a generalized composition of graphs. Let $x \in R$ then define $NC_D(x) = \{y \neq 0 : xy \in NC(R)\}$. If x^2 is a nil clean element then $x \in NC_D(x)$. Define a relation $\sim$ on R such that $x \sim y$ if and only if $NC_D(x) = NC_D(y)$. This relation is an equivalence relation. Let $C_1, C_2, \cdots, C_k$ be the equivalence classes of this relation with representatives $c_1, c_2, \cdots c_k$ respectively. We have the following properties of that equivalence classes.

Lemma 2.4. *Among those representatives, two of them say c_i and c_j are adjacent in $G_{N_D}(R)$ if and only if every element of C_i is adjacent to every element of C_j.*

Proof. Let c_i and c_j be adjacent in $G_{N_D}(R)$, then $c_ic_j \in NC(R)$. Thus $c_i \in NC_D(c_j)$. Let $x \in C_i$ and $y \in C_j$, then $NC_D(y) = NC_D(c_j)$. Therefore $c_i \in NC_D(y)$, that is $c_iy \in NC(R)$. Thus $y \in NC_D(c_j)$. Since $x \in C_i$, $NC_D(x) = NC_D(c_i)$. Therefore $y \in NC_D(x)$. That is $xy \in NC(R)$. Hence every element of C_i is adjacent to every element of C_j. Conversely, if every element of C_i is adjacent to every element of C_j then $c_ic_j \in NC(R)$. Hence c_i and c_j are adjacent in $G_{N_D}(R)$. □

The subgraph of $G_{N_D}(R)$ induced by C_i is denoted as $G(C_i)$. We can easily observe the nature of $G(C_i)$ for each i.

Corollary 2.5. *The subgraph $G(C_i)$ is complete if and only if $c_i^2 \in NC(R)$. Otherwise $G(C_i)$ becomes totally disconnected. For $i \neq j$, a vertex of $G(C_i)$ is adjacent to either all or none of the vertices of $G(C_j)$ in $G_{N_D}(R)$.*

Proof. If $c_i \in NC_D(c_i)$, then $c_i \in NC_D(x)$ for every x in $G(C_i)$. Thus $G(C_i)$ becomes complete. If $G(C_i)$ is complete then $c_i \in NC_D(x)$ for every x in $G(C_i)$. Thus $c_i \in NC_D(c_i)$. From above lemma, it is possible that either every element of C_i is adjacent to every element of C_j or there is no join form any element of C_i to any element of C_j in $G_{N_D}(R)$. □

In a finite commutative ring with unity, every element is either a unit or a zero divisor. Therefore it is not possible that nil clean divisor of an element is empty. It may happen that $NC_D(x) = \{x\}$. In this case, $\{x\}$ is one of the equivalence nil clean divisor classes. Discard the singleton equiv nil clean divisor classes whose nil clean divisor is itself only. Let $C_1, C_2, \ldots C_k$ be such equiv nil clean divisor classes. Let $G_1, G_2, \ldots G_k$ be the subgraphs induced by $C_1, C_2, \ldots C_k$ respectively.

Lemma 2.6. *The subgraph H of $G_{N_D}(R)$ induced by the representatives $c_1, c_2, \ldots c_k$ is connected.*

Proof. Let c_i and c_j be two distinct vertices of H. If $c_ic_j \in NC(R)$ then c_i and c_j are adjacent. Otherwise there is a $c_i - c_j$ path in $G_{N_D}(R)$, which gives a $c_i - c_j$ walk in H. Thus H contains a $c_i - c_j$ path. Hence H is connected. □

Proposition 2.7. $G_{N_D}(R) = H[G_1, G_2, \ldots G_k]$

Proof. Proof follows from corollary 2.5 . □

3 Spectrum of a nil clean divisor graph

In this section, we compute the spectrum of a nil clean divisor graph of any finite commutative ring, which can be expressed as the generalized composition of either complete or totally disconnected graphs.

Lemma 3.1. *The spectrum of the nil clean divisor graph of $\mathbb{Z}_p$, where p is a prime number are* $\{\underbrace{1, 1, \ldots 1}_{(\frac{p-3}{2})times}, \underbrace{-1, -1, \ldots -1}_{(\frac{p-3}{2})times}\}$

Proof. The nil clean elements of $\mathbb{Z}_p$ are $\{0, 1\}$ and the self inverse elements of $\mathbb{Z}_p$ are $\{1, p-1\}$. Therefore $V(G_{N_D}(\mathbb{Z}_p)) = \{2, 3, \ldots p-3, p-2\}$. Therefore the nil clean divisor graph of $\mathbb{Z}_p$ is the union of K_2 in $\frac{p-3}{2}$ times. The adjacency matrix of $\mathbb{Z}_p$ becomes

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

Hence the spectrum of $G_{N_D}(\mathbb{Z}_p)$ are $\{\underbrace{1, 1, \ldots 1}_{(\frac{p-3}{2})\text{times}}, \underbrace{-1, -1, \ldots -1}_{(\frac{p-3}{2})\text{times}}\}$. □

Remark 3.2. The nil clean divisor graph of $\mathbb{Z}_{2^n}$ is a complete graph with $2^n - 1$ number of vertices. Thus the spectrum of $G_{N_D}(\mathbb{Z}_{2^n})$ are $\{2^n - 2, \underbrace{-1, -1, \ldots, -1}_{(2^n-2)\text{times}}\}$. In next, we investigate about the spectrum of the nil clean divisor graph of $\mathbb{Z}_{2p}$, where $p > 2$ is a prime number. Here the nil clean elements of $\mathbb{Z}_{2p}$ are $\{0, 1, p, p+1\}$ and the vertex set of $G_{N_D}(\mathbb{Z}_{2p})$ is $\{1, 2, \ldots, 2p-1\}$. $G_{N_D}(\mathbb{Z}_{2p})$ can be written as a generalized composition of some graphs, which are either complete or totally disconnected. After some computations, we get $NC_D(1) = NC_D(p+1), NC_D(2) = NC_D(p+2), \ldots NC_D(p-1) = NC_D(2p-1)$. Thus the equiv classes become $\{1, p+1\}, \{2, p+2\}, \ldots, \{p-1, 2p-1\}, \{p\}$. Thus there are $p-1$ number of equiv classes in which each equiv class has exactly 2 elements and $\{p\}$ is an equiv class whose nil clean divisor is not itself. Therefore we need not discard the equiv class $\{p\}$. Let the representatives be $1, 2, \ldots, p-1, p$ respectively. In the following lemma, we give the adjacency matrix of the base graph H of $G_{N_D}(\mathbb{Z}_{2p})$.

Lemma 3.3. *In the generalized composition of $G_{N_D}(\mathbb{Z}_{2p})$, the adjacency matrix of the base graph H is*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & & 1\dots & 1 & 1 & 1 \end{bmatrix}$$

Proof. By above discussion $V(H) = \{1, 2, \dots p-1, p\}$. From [2] , p is adjacent to every other element of $G_{N_D}(\mathbb{Z}_{2p})$, 1 is adjacent to p, $p+1$, and $p-1$ is adjacent to p , $2p-1$ in $G_{N_D}(\mathbb{Z}_{2p})$. Therefore in H, p is adjacent to every other element, 1 and $p-1$ are adjacent to p only and every other element is adjacent to exactly one element of H other than p. Hence the adjacency matrix of the base graph H becomes

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & & 1\dots & 1 & 1 & 1 \end{bmatrix}$$

□

Further, we discuss about the subgraphs of $G_{N_D}(\mathbb{Z}_{2p})$ induced by that p number of equivalence classes and we derive the spectrum of $G_{N_D}(\mathbb{Z}_{2p})$.

Lemma 3.4. *In that p number of equiv classes $C(1), C(2), \dots C(p-1), C(p)$ of $G_{N_D}(\mathbb{Z}_{2p})$, the subgraphs induced by $C(1)$, $C(p-1)$ and $C(p)$ are complete and all other induced subgraphs are totally disconnected.*

Proof. We have $C(1) = \{1, p+1\}$, $C(p-1) = \{p-1, 2p-1\}$, $C(p) = \{p\}$. Since $1.(p+1) = p+1 \in NC(\mathbb{Z}_{2p})$, we get the subgraph induced by $C(1)$ is a complete graph K_2. Since $(p-1)(2p-1) = p+1 \in NC(\mathbb{Z}_{2p})$. Therefore the subgraph induced by $C(p-1)$ is a complete graph K_2 and the subgraph induced by $C(p)$ is K_1. Suppose there exists $m \in \mathbb{Z}_{2p}$ such that $1 < m < p-1$ and the induced subgraph of $C(m)$ is complete. It is clear that $C(m) = \{m, p+m\}$, thus $m(p+m) \in NC(\mathbb{Z}_{2p})$. If $m(p+m) = 0$ then both m and $p+m$ are zero divisors of $\mathbb{Z}_{2p}$ and hence $\gcd(m, 2p) \neq 1$ and $\gcd(p+m, 2p) \neq 1$. Thus m and $p+m$ are even. Therefore p must be even, which gives a contradiction. If $m(p+m) = 1$ then both m and $p+m$ are units of $\mathbb{Z}_{2p}$ and hence $\gcd(m, 2p) = 1$ and $\gcd(p+m, 2p) = 1$. Thus m and $p+m$ are odd. Therefore p must be even, which gives a contradiction. Suppose $m(p+m) = p$, in this case if m is odd then p must be even and if m is even then also p must be even, gives contradiction. Suppose $m(p+m) = p+1$ then $pm + m^2 = p+1$. That is $-p(1-m) = (1-m)(1+m)$, in this case $1-m = 0$ or $-p = 1+m$. That is $m = 1$ or $m = p-1$, which gives a contradiction . Hence all other induced subgraphs are totally disconnected. □

To calculate the spectrum of $G_{N_D}(\mathbb{Z}_{2p})$, we use the following result from [8] as a main tool.

Remark 3.5. [8] Let $G = H[G_1, G_2, \dots G_k]$ be the H-join of the graphs $G_1, G_2, \dots G_k$ and let H be a graph of order k. Assume that ρ_{ij} be the $(i,j)^{th}$ entry of the adjacency marix of H. Let n_i be the order of G_i, A_i be the adjacency matrix of G_i and each G_i be r_i-regular, for $1 \leq i \leq k$. Then r_i is an eigen value of G_i and Spec$(A(G))$ = $(\cup(\text{spec}(A_i) \setminus \{r_i\})) \cup \text{spec}(\tilde{A}(G))$, where

$$\tilde{A}(G) = \begin{bmatrix} r_1 & \sqrt{n_1 n_2}\rho_{1,2} & \cdots & \sqrt{n_1 n_k}\rho_{1,k} \\ \sqrt{n_2 n_1}\rho_{2,1} & r_2 & \cdots & \sqrt{n_2 n_k}\rho_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ \sqrt{n_k n_1}\rho_{k,1} & \sqrt{n_k n_2}\rho_{k,2} & \cdots & r_k \end{bmatrix}$$

Theorem 3.6. *The spectrum of* $G_{N_D}(\mathbb{Z}_{2p})$ *are* $\{-1, -1, \underbrace{0, 0, \dots 0}_{(p-2)times}, 1, 1, \underbrace{2, 2, \dots 2}_{(\frac{p-2}{2})times}, \underbrace{-2, -2, \dots -2}_{(\frac{p-2}{2})times}\}$.

Proof. By above argument, $G_{N_D}(\mathbb{Z}_{2p})$ can be expressed as the generalized composition of graphs as follows :
$G_{N_D}(\mathbb{Z}_{2p}) = H[K_2, K_2, \underbrace{\overline{K}_2, \overline{K}_2, \dots \overline{K}_2}_{(p-3)\text{times}}, K_1]$, where H is the base graph of $G_{N_D}(\mathbb{Z}_{2p})$. The spectrum of K_2 are $\{1, -1\}$ and the spectrum of $\overline{K}_2$ are $\{0, 0\}$. By above result, spec$G_{N_D}(\mathbb{Z}_{2p}) = \{-1, -1, \underbrace{0, 0, \dots 0}_{(p-2)\text{times}}\} \cup \text{spec}\tilde{A}(G)$, where

$$\tilde{A}(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 2 & \dots & 0 & 0 & \sqrt{2} \\ 0 & 0 & 2 & 0 & \dots & 0 & 0 & \sqrt{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 2 & \sqrt{2} \\ 0 & 0 & 0 & 0 & \dots & 2 & 0 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \dots & \sqrt{2} & \sqrt{2} & 0 \end{bmatrix}$$

The spectrum of $\tilde{A}(G)$ are $\{1, 1, \underbrace{2, 2, \dots 2}_{(\frac{p-2}{2})\text{times}}, \underbrace{-2, -2, \dots -2}_{(\frac{p-2}{2})\text{times}}\}$. Thus the spectrum of $G_{N_D}(\mathbb{Z}_{2p})$ are $\{-1, -1, \underbrace{0, 0, \dots 0}_{(p-2)\text{times}}, 1, 1, \underbrace{2, 2, \dots 2}_{(\frac{p-2}{2})\text{times}}, \underbrace{-2, -2, \dots -2}_{(\frac{p-2}{2})\text{times}}\}$. □

Further, we discuss about the spectrum of the nil clean divisor graph of $(\mathbb{Z}_{p^2}, \oplus, \odot)$, where $p > 2$ is a prime number. Here the vertex set of $G_{N_D}(\mathbb{Z}_{p^2})$ are $\{1, 2, \dots p^2 - 1\}$ and the nil clean elements of $\mathbb{Z}_{p^2}$ are $\{0, 1, p, p+1, 2p, 2p+1, \dots (p-1)p, (p-1)p+1\}$. $G_{N_D}(\mathbb{Z}_{p^2})$ can be expressed as a generalized composition of some graphs which are either complete or totally disconnected. By the definition of $NC_D(x)$, we get $NC_D(1) = NC_D(p+1) = NC_D(2p+1) = \dots = NC_D((p-1)p+1), NC_D(2) = NC_D(p+2) = NC_D(2p+2) = \dots = NC_D((p-1)p+2) \dots NC_D(p) = NC_D(2p) = NC_D(3p) = \dots = NC_D((p-1)p)$. Therefore the equiv classes become $\{1, p+1, 2p+1, \dots, (p-1)p+1\}, \{2, p+2, 2p+2, \dots (p-1)p+2\}, \dots \{p, 2p, 3p, \dots, p^2 - p\}$. Thus there are p number of equiv classes, in which each has exactly p number of elements except the last equiv class which has $p-1$ number of elements. Let $\{1, 2, \dots p-1, p\}$ be the respective representatives. We start with the following lemma, which gives the adjacency matrix of the base graph H of $G_{N_D}(\mathbb{Z}_{p^2})$.

Lemma 3.7. *The adjacency matrix of the base graph H in the generalized composition of the nil clean divisor graph of $\mathbb{Z}_{p^2}$ is*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{bmatrix}$$

Proof. From above argument, $V(H) = \{1, 2, \dots, p-1, p\}$. Let $x \in V(H)$, then $px \in NC(\mathbb{Z}_{p^2})$. Thus p is adjacent to all other vertices of H. In $G_{N_D}(\mathbb{Z}_{p^2})$, 1 is adjacent to all nil clean elements only. Therefore in H, 1 is adjacent to p only. Let $a \in V(H) - \{1, p\}$. Since $sp \in NC(\mathbb{Z}_{p^2})$ for $0 \leq s \leq p-1$, we get a is adjacent to sp in $G_{N_D}(\mathbb{Z}_{p^2})$, where $0 \leq s \leq p-1$. If a is adjacent to y in $G_{N_D}(\mathbb{Z}_{p^2})$ then $ay \cong sp \ (modp^2)$ and $ay \cong sp+1 \ (modp^2)$, where $0 \leq s \leq p-1$. If $ay \cong sp \ (modp^2)$ then $ay - sp = \lambda p^2$. Now $a(y+p) = (a+s)p + \lambda p^2$. Thus a is adjacent to $y+p$. Like this if $ay \cong sp+1 \ (modp^2)$ then $a(y+p) = (a+s)p + 1 + \lambda p^2$. That is a is adjacent to $y+p$. Proceeding like this, we get if a is adjacent to y in $G_{N_D}(\mathbb{Z}_{p^2})$, then a is adjacent to $\{p+y, 2p+y, \dots (p-1)p+y\}$. Since $a \in V(G_{N_D}(\mathbb{Z}_{p^2}))$, there exists $b \in \mathbb{Z}_{p^2}$ such that $ab \in NC(\mathbb{Z}_{p^2})$. Thus if $a \in V(H) - \{1, p\}$ with $a^2 \notin NC(\mathbb{Z}_{p^2})$ then a is adjacent to $\{p, 2p, 3p, \dots, (p-1)p, y, p+y, 2p+y, \dots, (p-1)p+y\}$. Here, $(p-1)^2 = (p^2-2)p + 1 \in NC(\mathbb{Z}_{p^2})$. Hence every element of H other than $1, p, p-1$ is adjacent to exactly one of the elements of the same set. Therefore the adjacency matrix of H becomes

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{bmatrix}$$

□

In next, we deal with the subgraphs of $G_{N_D}(\mathbb{Z}_{p^2})$ induced by that p number of equivalence classes and extend the spectrum of $G_{N_D}(\mathbb{Z}_{p^2})$.

Lemma 3.8. *Among that p number of equiv classes $C(1), C(2), \dots C(p-1), C(p)$ of $G_{N_D}(\mathbb{Z}_{p^2})$, the subgraphs induced by $C(1), C(p-1)$ and $C(p)$ are complete and all other induced subgraphs are totally disconnected.*

Proof. We already have $C(1) = \{1, p+1, 2p+1, \dots, (p-1)p+1\}$, $C(p-1) = \{p-1, p+p-1, 2p+p-1, \dots (p-1)p+(p-1)\}$ and $C(p) = \{p, 2p, 3p, \dots (p-1)p\}$. Since $1(p+1) = p+1 \in NC(\mathbb{Z}_{p^2})$, the subgraph induced by $C(1)$ is the complete graph K_p. Since $(p-1)(p+p-1) = p^2 - 3p + 1 \in NC(\mathbb{Z}_{p^2})$, the subgraph induced by $C(p-1)$ is the complete graph K_p. Since $p(2p) = 0 \in NC(\mathbb{Z}_{p^2})$, the subgraph induced by $C(p)$ is the complete graph K_{p-1}.

Suppose there exists m with $1 < m < p-1$, such that the subgraph induced by $C(m)$ is complete, then $m(p+m) \in NC(\mathbb{Z}_{p^2})$. That is $mp + m^2 \in NC(\mathbb{Z}_{p^2})$. It is possible only when $m^2 = 0$ or 1. If $m^2 = 0$ then $m = 0$ or p, which gives a contradiction. If $m^2 = 1$ then $m = 1$ or $m = p^2 - 1$, which also gives a contradiction. Thus $C(m)$ is totally disconnected, where $1 < m < p-1$. □

Theorem 3.9. *The spectrum of $G_{N_D}(\mathbb{Z}_{p^2})$ are*

$$\{\underbrace{-1, -1, \dots, -1}_{(3p-4)\text{times}}, \underbrace{0, 0, \dots, 0}_{((p-3)(p-1))\text{times}}, p-1, p-1, \underbrace{p, p, \dots p}_{(\frac{p-2}{2})\text{times}}, \underbrace{-p, -p, \dots, -p}_{(\frac{p-2}{2})\text{times}}\}.$$

Proof. From above discussion,$G_{N_D}(\mathbb{Z}_{p^2})$ can be written as the generalized composition of graphs as follows : $G_{N_D}(\mathbb{Z}_{p^2}) = H[K_p, K_p, \underbrace{\overline{K}_p, \overline{K}_p, \dots \overline{K}_p}_{(p-3)\text{times}}, K_{p-1}]$, where H is the base graph of $G_{N_D}(\mathbb{Z}_{p^2})$. The spectrum of K_p are $\{p-1, \underbrace{-1, -1, \dots -1}_{(p-1)\text{times}}\}$ and the spectrum of $\overline{K_p}$ are $\{0, 0, \dots 0\}$. Therefore Spec $G_{N_D}(\mathbb{Z}_{p^2}) =$
$\{\{\underbrace{-1, -1, \dots, -1}_{(p-1)\text{times}}\}, \{\underbrace{-1, -1, \dots, -1}_{(p-1)\text{times}}\}, \underbrace{\{\underbrace{0, 0, \dots, 0}_{(p-1)\text{times}}\}, \cdots, \{\underbrace{0, 0, \dots, 0}_{(p-1)\text{times}}\}}_{(p-3)\text{times}}, \{\underbrace{-1, -1, \dots, -1}_{(p-2)\text{times}}\}\} \cup \text{spec}\tilde{A}(G)$
, where $\tilde{A}(G)$ is as follows :

$$\tilde{A}(G) = \begin{bmatrix} p-1 & 0 & 0 & 0 & \dots & 0 & 0 & \sqrt{p^2-p} \\ 0 & p-1 & 0 & 0 & \dots & 0 & 0 & \sqrt{p^2-p} \\ 0 & 0 & 0 & p & \dots & 0 & 0 & \sqrt{p^2-p} \\ 0 & 0 & p & 0 & \dots & 0 & 0 & \sqrt{p^2-p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & p & \sqrt{p^2-p} \\ 0 & 0 & 0 & 0 & \dots & p & 0 & \sqrt{p^2-p} \\ \sqrt{p^2-p} & \sqrt{p^2-p} & \sqrt{p^2-p} & \sqrt{p^2-p} & \dots & \sqrt{p^2-p} & \sqrt{p^2-p} & p-2 \end{bmatrix}$$

Here the spectrum of $\tilde{A}(G)$ are $\{p-1, p-1, \underbrace{p, p, \dots p}_{(\frac{p-2}{2})\text{times}}, \underbrace{-p, -p, \dots -p}_{(\frac{p-2}{2})\text{times}}\}$. Hence the spectrum of $G_{N_D}(\mathbb{Z}_{p^2})$ are $\{\underbrace{-1, -1, \dots, -1}_{(3p-4)\text{times}}, \underbrace{0, 0, \dots, 0}_{((p-3)(p-1))\text{times}}, p-1, p-1, \underbrace{p, p, \dots p}_{(\frac{p-2}{2})\text{times}}, \underbrace{-p, -p, \dots, -p}_{(\frac{p-2}{2})\text{times}}\}$.

□

4 Wiener index of a nil clean divisor graph

From this section, we consider only the equiv nil clean divisor classes wose nil clean divisor is non-singleton. We focus on the distance between any two vertices of a nil clean divisor graph and we deduce the Wiener index of a nil clean divisor graph, which is connected.

Lemma 4.1. *If $x \in C_i$ and $y \in C_j$, then*

$$d_{G_{N_D}(R)}(x, y) = \begin{cases} 1 \text{ if } i = j \text{ and } c_i^2 \in NC(R) \\ 2 \text{ if } i = j \text{ and } c_i^2 \notin NC(R) \\ d_H(c_i, c_j) \text{ if } i \neq j \end{cases}$$

Proof. If $i = j$ and $c_i^2 \in NC(R)$ then $c_i \in NC_D(c_i)$. Since $x, y \in C_i$, $NC_D(x) = NC_D(y) = NC_D(c_i)$. Thus x is adjacent to y in $G_{N_D}(R)$ and hence in this case, $d_{G_{N_D}(R)}(x, y) = 1$. If $i = j$ and $c_i^2 \notin NC(R)$ then $c_i \notin NC_D(c_i)$. Suppose x is adjacent to y then $x \in NC_D(y)$. Since $x, y \in C_i$, $NC_D(x) = NC_D(y) = NC_D(c_i)$. Thus $x \in NC_D(c_i)$ and hence $c_i \in NC_D(c_i)$ which is a contradiction. Therefore x is not adjacent to y. By 2.5, $d_{G_{N_D}(R)}(x, y) = 2$.
Let us assume that $i \neq j$.Suppose $c_i \sim e_1 \sim e_2 \sim \cdots \sim e_{n-1} \sim c_j$ is the shortest $c_i - c_j$ path in H, where $e_i = c_m$ for some m. Since $x \in C_i$ and $y \in C_j$, $x \sim e_1 \sim e_2 \sim \cdots \sim e_{n-1} \sim y$ is an $x - y$ path in $G_{N_D}(R)$. Suppose $x \sim d_1 \sim d_2 \sim \cdots d_{k-1} \sim y$ is the shortest $x - y$ path in $G_{N_D}(R)$, where $k < n$. Since it is the shortest $x - y$path, each d_i belongs to distinct C_i. Since $x \in C_i, y \in C_j, c_i \sim c_1 \sim c_2 \sim \cdots \sim c_{k-1} \sim c_j$ is a $c_i - c_j$ path in H of length $k < n$, we get a contradiction. Hence $d_{G_{N_D}(R)}(x, y) = d_H(c_i, c_j)$. □

Now, we make an important assumption that $V(H) = \{c_1, c_2, \dots c_k\}$ and let the vertices of G_i be $\{v_i^1, v_i^2, \dots, v_i^{n_i}\}$. Further, we derive the distances between any two vertices of $G_{N_D}(R)$.

Lemma 4.2. *If $i = j$ and G_i is complete, then $d_{G_{N_D}(R)}(v_i^p, v_j^q)$ is 1 and if G_i is totally disconnected then the distance between v_i^p and v_j^q is 2. If $i \neq j$ then the distance between v_i^p and v_j^q in $G_{N_D}(R)$ is $d_H(c_i, c_j)$, where $1 \leq p \leq n_i$ and $1 \leq q \leq n_j$.*

Proof. If $i = j$ and G_i is totally disconnected, then since H is connected, there exists $c_k \in H$ such that c_i is adjacent to c_k in H and so every element of C_k is adjacent to every element of C_i. Thus $v_i^p \sim x \sim v_j^q$, where $x \in C_k$. Thus $d_{G_{N_D}(R)}(v_i^p, v_j^q) = 2$. Suppose $i \neq j$, since $G_{N_D}(R)$ is connected, there exists a $c_i - c_j$ path in H. Let $c_i \sim d_1 \sim d_2 \sim \cdots d_{k-1} \sim c_j$ be the shortest $c_i - c_j$ path in H. Then $v_i^p \sim v_{d_1}^{p_1} \sim \cdots \sim v_{d_{k-1}}^{p_{k-1}} \sim v_j^q$ is the shortest $v_i^p - v_j^q$ path in $G_{N_D}(R)$. Hence $d_H(c_i, c_j) = d_{G_{N_D}(R)}(v_i^p, v_j^q)$. □

Theorem 4.3. *With the above discussed notations, the Wiener index of the nil clean divisor graph $G_{N_D}(R)$ is*

$$W(G_{N_D}(R)) = \sum_{c_i^2 \in NC(R)} \binom{|\ C_i\ |}{2} + \sum_{c_i^2 \notin NC(R)} 2\binom{|\ C_i\ |}{2} + \sum_{1 \leq i < j \leq k} |\ C_i\ |\ |\ C_j\ |\ \ d_H(c_i, c_j).$$

Proof. From above discussions, $G_{N_D}(R) = H[G_1, G_2, \ldots G_k]$ where $G_i^{'s}$ are the subgraphs of $G_{N_D}(R)$ induced by C_i, whose nil clean divisor is not singleton. We already have G_i is complete if and only if $c_i^2 \in NC(R)$ and G_i is totally disconnected if and only if $c_i^2 \notin NC(R)$. If G_i is complete then the sum of the distances between any two vertices of this kind is $\sum_{c_i^2 \in NC(R)} \binom{|C_i|}{2}$. If G_i is totally disconnected then the distance between any two vertices in G_i is 2. Thus the sum of the distances between any two vertices of this type is $\sum_{c_i^2 \notin NC(R)} 2\binom{|C_i|}{2}$. We have, if $x \in C_i$ and $y \in C_j$ then $d_{G_{N_D}(R)}(x, y) = d_H(c_i, c_j)$. In this case, the sum of the distances between any two vertices is $\sum_{1 \leq i < j \leq k} |\ C_i\ |\ |\ C_j\ |\ \ d_H(c_i, c_j)$. Hence the wiener index of the nil clean divisor graph becomes $W(G_{N_D}(R)) = \sum_{c_i^2 \in NC(R)} \binom{|C_i|}{2} + \sum_{c_i^2 \notin NC(R)} 2\binom{|C_i|}{2} + \sum_{1 \leq i < j \leq k} |\ C_i\ |\ |\ C_j\ |\ \ d_H(c_i, c_j)$. □

Remark 4.4. Using the above theorem, we will find the Wiener index of the nil clean divisor graph of $(\mathbb{Z}_{14}, \oplus, \odot)$, where nil clean elements are $\{0, 1, 7, 8\}$. Here the equiv nil clean divisor classes are $\{1, 8\}, \{2, 9\}, \{3, 10\}, \{4, 11\}, \{5, 12\}, \{6, 13\}, \{7\}$ with representatives $1, 2, 3, 4, 5, 6, 7$ respectively. Here $C(1), C(6), C(7)$ are complete and all other are totally disconnected.

$$W(G_{N_D}(R)) = \sum_{c_i^2 \in NC(R)} \binom{|C_i|}{2} + \sum_{c_i^2 \notin NC(R)} 2\binom{|C_i|}{2} + \sum_{1 \leq i < j \leq k} |\ C_i\ |\ |\ C_j\ |\ \ d_H(c_i, c_j).$$

$$= \binom{2}{2} + \binom{2}{2} + 2[\binom{2}{2} + \binom{2}{2} + \binom{2}{2} + \binom{2}{2}] + \sum_{1 \leq i < j \leq k} |\ C_i\ |\ |\ C_j\ |\ \ d_H(c_i, c_j).$$

After some computations, $W(G_{N_D}(\mathbb{Z}_{14})) = 134$.

References

[1] Abhyendra Prasad, Om Prakash and Ravindra Kumar,*Topological indices and Linear codes of generalized Franklin Graphs*, Palestine Journal of Mathematics, **15(1)**, 1018-1029, (2026).

[2] Ajay Sharma, Dhiren Kumar Basnet, *Nil clean divisor graph*, arXiv:1903.02287, (2019).

[3] Andrey A.Dobrynin, Roger Entringer, Ivan Gutman, *Wiener index of trees, theory and applications*, Acta Appln. Math., **66** , 211–249, (2001).

[4] L.Babai,*Spectra of Cayley graphs*, J. Comb. Theory Ser., B, **27(2)**, 180–189, (1979).

[5] I.Beck, *Coloring of commutative rings*, J.Algebra, **16(1)**, 208–226, (1988).

[6] S.Banerjee,A. Adhikari, *On spectra and spectral radius of signless Laplacian of power graphs of some finite groups*,Asian - Eur. J. Math., **14(6)**,(2020).

[7] S. Banerjee, A. Adhikari, *Signless Laplacian spectrum of power graphs of finite cyclic groups*, AICCE Int.J. Graphs Comb.,**17(1)**, 356–366,(2020).

[8] D.M.Cardoso, M.A. de Freitas, E.A.Martins, M. Robbiano, *Spectra of graphs obtained by a generalization of the join graph operation*,Discrete Math., **313**, 733–741, (2013).

[9] Dhiren kumar Basnet, Jayanta Bhattacharya, *Nil clean graph of rings*, Algebra Colloquium, **24(03)**, 481–492, (2017).

[10] H.Hosaya, *Topological index, newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons*, Bulletin of the Chemical Society of Japan, **44(9)**, 2332–2339, (1971).

[11] Kexiang Xua, Muhuo Liub, Kinkar ch.Dasd, Ivan Gutman, Beris Furtuale, *A survery on graphs extremal with respect to distance based topological indices*, MATCH Commun.Math.Comput.Chem.,**71**, 461–508,(2014).

[12] Mohammad Habibi, Ece Yetkin Celikel, Cihat Abdioglu, *Clean graph of a ring*, Journal of Algebra and Its Applications, **2150156**,(2021).

[13] Pavel Pal Jyotirmoy Jana and Sujit Kumar Sardar,*Square Element Graph of a Near-ring*, Palestine Journal of Mathematics, **15(1)**, 60-70, (2026).

[14] A.J.Schwenk, *Computing the characteristic polynomial of a graph* in, R.Bary, F.Harary(Eds.), Graphs combinatorics in , Lecture Notes in Mathematics, Springer-Verlag, Berlin, **406** , 153–172, (1974).

[15] K.Selvakumar, P.Gangaeswari, G. Arunkumar, *The Wiener Index of the zero-divisor graph of a finite commutative ring with unity*, Discrete Appl.Math., **311** , 72–84, (2022).

[16] Shahul Hameed K, Ramakrishnan K.O. and Biju K, *Spectra and Energy of conjugate skew gain graphs*, Palestine Journal of Mathematics, **15(1)**, 288-300, (2026).

[17] Sonja Nikolic, Nenad Trinajstic, *The Wiener index , Development and applications*, Croatica Chemica Acta **68(1)**, 105–129,(1995).

[18] M.Torktaz, A.R.Ashrafi, *Spectral properties of the commuting graphs of certain groups*, AKCE Int. J.Graphs Comb., **16(3)**, 300–309, (2019).

[19] D.B.West, *Introduction to graph theory*,Prentice Hall, Inc., Upper Saddle River, (1996).

[20] H.Wiener, *Structural determination of paraffin boing points*, J.Am.chem.suc., **69(1)**, 17–20, (1947).

[21] Xiaoying Wu, Huiqing Liu, *On the Wiener index of graphs*, Acta appl.Math., **110(2)**, 535-544, (2010).

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