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FOURIER TRANSFORM OF THE OPERATOR $\oplus_m^k$ IN TEMPERED DISTRIBUTION SPACES

S. Bupasiri

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Corresponding Author: S. Bupasiri

Abstract This paper develops a unified framework for the analysis of iterated differential operators by combining the operator-theoretic and convolutional perspectives. Focusing on the class of diamond operators, we derive explicit formulas for their fundamental solutions in the sense of Schwartz distributions. Our approach not only extends the classical methods of Schwartz [20], Gelfand and Shilov [6], but also consolidates more recent advances on diamond operator within a single distributional setting. The resulting formulas provide a transparent description of singularity structures and highlight connections between iterated convolution kernels and PDE theory. Beyond their intrinsic theoretical interest, these results yield tools for addressing inversion problems and the study of propagation phenomena in higher-order partial differential equations.

1 Introduction

The study of fundamental solutions of differential operators plays a central role in analysis and partial differential equations (PDEs). Classical operators such as the Laplace, Helmholtz, and Klein–Gordon operators provide the foundation of potential theory and mathematical physics, where explicit constructions of fundamental solutions yield significant insights for both theory and applications, ranging from harmonic analysis to scattering theory [5, 7, 21].

A rigorous framework for such studies is provided by the theory of generalized functions. In this setting, distributions such as the Dirac delta $\delta(x-\xi)$ model point sources and impulsive phenomena, acting on test functions $\phi(x) \in \mathcal{D}$ (the space of compactly supported smooth functions) via

$$\int_{\mathbb{R}^n} \delta(x-\xi)\, f(x)\, dx = f(\xi), \tag{1.1}$$

for sufficiently smooth f. This distributional viewpoint underpins the construction of fundamental solutions for higher-order operators.

Within this framework, Kananthai et al. [14] studied weak and fundamental solutions of iterated differential operators. A notable example is the operator

$$\begin{aligned}\oplus^k &= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k \\ &= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k\end{aligned}$$

$$= \diamondsuit^k L^k, \tag{1.2}$$

where $p+q=n$, and

$$L^k = \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \tag{1.3}$$

$$= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right) + i\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right) - i\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k \tag{1.4}$$

$$= L_1^k L_2^k. \tag{1.5}$$

The diamond operator

$$\diamondsuit^k = \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \tag{1.6}$$

$$= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right) - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right]^k \tag{1.7}$$

$$= \triangle^k \Box^k \tag{1.8}$$

connects naturally to the ultra-hyperbolic operator $\Box^k$ and the Laplacian $\triangle^k$ [8].

Following the framework in [14], we first recall the fundamental solutions of the operators L_1^k, L_2^k, and L^k introduced in (1.3) - (1.5). For the equation

$$L_1^k u(x) = \delta, \qquad x \in \mathbb{R}^n,$$

the fundamental solution is given by

$$u(x) = (-1)^k (-i)^{\frac{q}{2}} S_{2k}(x),$$

where

$$S_{2k}(x) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)}\left[x_1^2 + x_2^2 + \cdots + x_p^2 - i(x_{p+1}^2 + \cdots + x_{p+q}^2)\right]^{\frac{2k-n}{2}}.$$

Similarly, for

$$L_2^k u(x) = \delta,$$

the corresponding fundamental solution is

$$u(x) = (-1)^k (i)^{\frac{q}{2}} T_{2k}(x),$$

with

$$T_{2k}(x) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)}\left[x_1^2 + x_2^2 + \cdots + x_p^2 + i(x_{p+1}^2 + \cdots + x_{p+q}^2)\right]^{\frac{2k-n}{2}}.$$

Finally, for the combined operator L^k defined in (1.3), the fundamental solution is obtained by the convolution of the two kernels:

$$u(x) = S_{2k}(x) * T_{2k}(x).$$

The derivation of these results can be found in detail in [14]. These formulas form the basis for our subsequent analysis, as they provide explicit representations of the fundamental solutions for higher-order operators in $\mathbb{R}^n$.

Bupasiri further generalized the partial differential operator $\circledast^k$ in relation to the wave equation, thereby broadening its analytical framework and potential applications [2]. Moreover,

recent studies have introduced new classes of differential equations, such as the bi-Laplacian Kirchhoff-type equation with a singular term [22] and the Laplace equation subject to nonlinear oblique boundary conditions [19]. These contributions highlight ongoing advancements in the theory of nonlinear partial differential equations and generalized operators.

More recently, the introduction of a mass parameter $m \geq 0$ led to the generalized operator

$$\begin{aligned} \oplus_m^k &= \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} + m^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k \\ &= (\Box + m^2)^k (\triangle + m^2)^k (L_1 + m^2)^k (L_2 + m^2)^k, \end{aligned} \tag{1.9}$$

which extends the class of higher-order elliptic operators and links PDE theory with fractional and iterated operators [1].

The purpose of this paper is to construct and analyze the fundamental solution F_{2k} associated with $\oplus_m^k$. Using Fourier transform techniques, we derive an explicit polynomial form for the Fourier transform of F_{2k} and establish convolution identities and inversion formulas. The central result, Theorem 3.3, confirms that

$$\oplus_m^k F_{2k} = \delta$$

in the sense of tempered distributions, demonstrating that F_{2k} serves as a true fundamental solution.

The remainder of this paper is organized as follows. Section 2 reviews the necessary preliminaries on tempered distributions and convolution. Section 3 derives the fundamental solution and its Fourier transform for F_{2k}. Section 4 discusses applications of the obtained results. Finally, Section 5 summarizes the main findings and outlines possible directions for future research.

2 Preliminary Notes

In this section, we discuss several properties of the ultra-hyperbolic kernel of Marcel Riesz and the fundamental solutions of the corresponding partial differential operators, which will be used in the subsequent analysis.

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n - dimensional space $\mathbb{R}^n$,

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.1}$$

where $p + q = n$. Define $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ which designates the interior of the forward cone and $\overline{\Gamma}_+$ designates its closure and the following functions introduce by Nozaki ([17], p.72) that

$$R_\alpha^H(x) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

$R_\alpha^H(x)$ is called the *ultra-hyperbolic kernel of Marcel Riesz*. Here α is a complex parameter and n the dimension of the space. The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)} \tag{2.3}$$

and p is the number of positive terms of

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad p + q = n$$

and let supp $R_\alpha^H(x) \subset \overline{\Gamma}_+$. Now $R_\alpha^H(x)$ is an ordinary function if Re $\alpha \geq n$ and is a distribution of α if Re $\alpha < n$.

Now, if $p = 1$ then (2.2) reduces to the function $M_\alpha(u)$ say, and defined by

$$M_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{2.4}$$

where $u = x_1^2 - x_2^2 - \cdots - x_n^2$ and $H_n(\alpha) = \pi^{\frac{(n-1)}{2}} 2^{\alpha-1} \Gamma(\frac{\alpha-n+2}{2})$. The function $M_\alpha(u)$ is called the hyperbolic kernel of Marcel Riesz.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and define

$$v = x_1^2 + x_2^2 + \cdots + x_p^2 + x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2. \tag{2.5}$$

Then the elliptic kernel of Marcel Riesz is given by

$$R_\alpha^e(v) = \frac{v^{\frac{\alpha-n}{2}}}{H_n(\alpha)} \tag{2.6}$$

where

$$H_n(\alpha) = \frac{\pi^{1/2} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}. \tag{2.7}$$

Definition 2.3. Let $f(x) \in L^1(\mathbb{R}^n)$ (the space of integrable function in $\mathbb{R}^n$). The Fourier transform of $f(x)$ is defined as

$$\widehat{f(\xi)} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx, \tag{2.8}$$

where $\xi = (\xi_1, \ldots, \xi_n)$ and $x = (x_1, \ldots, x_n)$ belong to $\mathbb{R}^n$, $\xi \cdot x = \sum_{j=1}^n \xi_j x_j$ denotes the usual Euclidean inner product, and $dx = dx_1 \cdots dx_n$. The inverse of the Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \widehat{f(\xi)} d\xi. \tag{2.9}$$

If f is a distribution with compact supports by ([25], Theorem $7.4 - 3$), Equation (2.9) can be written as

$$\widehat{f}(\xi) = \mathcal{F} f(x) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i\xi \cdot x} \right\rangle. \tag{2.10}$$

Lemma 2.4 ([23]). *(The convolution $W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)$) The functions $W_{2k}(x,m) * Y_{2k}(x,m)$ and $M_{2k}(x,m) * N_{2k}(x,m)$ are tempered distributions. Consequently, the full convolution*

$$W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)$$

exists and is also a tempered distribution.

Lemma 2.5 ([1]). *Consider the equation*

$$\oplus_m^k F_{2k}(x,m) = \delta, \tag{2.11}$$

where $\oplus_m^k$ denotes the k-fold iterated operator defined in (1.9), δ is the Dirac delta distribution, $x \in \mathbb{R}^n$, $k \in \mathbb{N}_0$, and $m \geq 0$. Then

$$F_{2k}(x,m) = W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m) \tag{2.12}$$

is a fundamental solution of $\oplus_m^k$, where $*$ denotes convolution on $\mathbb{R}^n$. Moreover, in the special case $m = 0$, one has

$$F_{2k}(x,0) = (-1)^k R_{2k}^e(x) * R_{2k}^H(x) * S_{2k}(x) * T_{2k}(x).$$

Lemma 2.6. *For each $j = 1, \ldots, n$ and $\xi \in \mathbb{R}^n$,*

$$\frac{\partial^2}{\partial x_j^2} \left(e^{-i\xi \cdot x}\right) = -\xi_j^2 e^{-i\xi \cdot x}.$$

Proof. Direct differentiation gives

$$\frac{\partial}{\partial x_j}\left(e^{-i\xi\cdot x}\right) = -i\xi_j\, e^{-i\xi\cdot x},$$

and differentiating again yields

$$\frac{\partial^2}{\partial x_j^2}\left(e^{-i\xi\cdot x}\right) = (-i\xi_j)^2 e^{-i\xi\cdot x} = -\xi_j^2\, e^{-i\xi\cdot x}.$$

□

Proposition 2.7. *Let $\triangle = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ denote the Laplacian in $\mathbb{R}^n$. Then, for all $\xi \in \mathbb{R}^n$,*

$$\triangle\, e^{-i\xi\cdot x} = -|\xi|^2\, e^{-i\xi\cdot x}, \qquad |\xi|^2 = \sum_{j=1}^n \xi_j^2.$$

Proof. By Lemma 2.6,

$$\frac{\partial^2}{\partial x_j^2} e^{-i\xi\cdot x} = -\xi_j^2 e^{-i\xi\cdot x}, \quad j = 1, \ldots, n.$$

Summing over j gives

$$\triangle\, e^{-i\xi\cdot x} = -\sum_{j=1}^n \xi_j^2\, e^{-i\xi\cdot x} = -|\xi|^2 e^{-i\xi\cdot x}.$$

□

3 Main Results

In what follows, we establish a key result characterizing how the iterated convolution operator $\oplus_m^k$ acts on its associated fundamental solution.

Theorem 3.1. *Let k and r be positive integers with $0 < r < k$. Then the following identities hold:*

$$\begin{aligned}&\oplus_m^k \left(W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)\right)\\ &\quad = W_{2(k-r)}(x,m) * Y_{2(k-r)}(x,m) * M_{2(k-r)}(x,m) * N_{2(k-r)}(x,m). \end{aligned} \tag{3.1}$$

Furthermore, for all integers $t \geq k$, we have

$$\oplus_m^t \left(W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)\right) = \oplus_m^{t-k}\delta. \tag{3.2}$$

Proof. Let $0 < r < k$. By Lemma 2.5, we have

$$\oplus_m^k \left(W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)\right) = \delta.$$

Applying the decomposition $\oplus_m^k = \oplus_m^{k-r}\oplus_m^r$, we obtain

$$\oplus_m^{k-r}\left(\oplus_m^r \left(W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)\right)\right) = \delta.$$

Equivalently,

$$\oplus_m^{k-r}\delta * \oplus_m^r \left(W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)\right) = \delta.$$

Now convolve both sides with the fundamental solution corresponding to $\oplus_m^{k-r}$, namely

$$W_{2(k-r)}(x,m) * Y_{2(k-r)}(x,m) * M_{2(k-r)}(x,m) * N_{2(k-r)}(x,m).$$

Using the associativity of convolution, the left-hand side becomes

$$\oplus_m^{k-r}\left(W_{2(k-r)}(x,m) * Y_{2(k-r)}(x,m) * M_{2(k-r)}(x,m) * N_{2(k-r)}(x,m)\right) * \oplus_m^{r}\left(W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)\right),$$

while the right-hand side becomes

$$W_{2(k-r)}(x,m) * Y_{2(k-r)}(x,m) * M_{2(k-r)}(x,m) * N_{2(k-r)}(x,m) * \delta.$$

By Lemma 2.5, convolution with δ leaves the function unchanged. Therefore, equating both sides implies

$$\oplus_m^{r}\left(W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)\right) = W_{2(k-r)}(x,m) * Y_{2(k-r)}(x,m) * M_{2(k-r)}(x,m) * N_{2(k-r)}(x,m),$$

as required.

For the case $t \geq k$, using again Lemma 2.5, we have

$$\begin{aligned}\oplus_m^{t}\left(W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)\right) &= \oplus_m^{t-k}\left(\oplus_m^{k}\left(W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)\right)\right)\\ &= \oplus_m^{t-k}\delta.\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Consider the linear differential equation*

$$\oplus_m^{k} u(x) = \sum_{r=0}^{t} c_r \oplus_m^{r} \delta, \tag{3.3}$$

where $p+q=n$, $x \in \mathbb{R}^n$, each $c_r \in \mathbb{R}$ is a constant, δ is the Dirac delta distribution, and we define $\oplus_m^0 \delta = \delta$. Then, the solution $u(x)$ depends on the relationship between k and t, as follows:

(1) ***Case** $t = 0 < k$**:** The solution is given by*

$$u(x) = c_0\, W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m), \tag{3.4}$$

which represents the fundamental solution to the operator $\oplus_m^k$. This solution is an ordinary function when $2k \geq n$, and it is a tempered distribution when $2k < n$.

(2) ***Case** $t = 0 < k$ **with** $m = 0$**:** Then the solution is*

$$u(x) = c_0\,(-1)^k R_{2k}^{e}(x) * R_{2k}^{H}(x) * S_{2k}(x) * T_{2k}(x),$$

which corresponds to the fundamental solution of the operator defined by equation (1.2). This is an ordinary function when $2k \geq n$, and a tempered distribution when $2k < n$.

(3) ***Case** $0 < t < k$**:** The solution takes the form*

$$u(x) = \sum_{r=0}^{t} c_r\, W_{2(k-r)}(x,m) * Y_{2(k-r)}(x,m) * M_{2(k-r)}(x,m) * N_{2(k-r)}(x,m),$$

which is an ordinary function when $2(k-r) \geq n$ for all r, and a tempered distribution when $2(k-r) < n$.

(4) ***Case** $t \geq k$**:** Then the solution is given by*

$$u(x) = \sum_{r=k}^{t} c_r \oplus_m^{r-k} \delta,$$

which is a purely singular distribution and cannot be expressed as an ordinary or tempered function.

Proof. (1) For $t = 0$, the equation reduces to

$$\oplus_m^k u(x) = c_0 \delta.$$

By Lemma 2.5, the solution is

$$u(x) = c_0\, W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m). \tag{3.5}$$

When $2k \geq n$, each of the functions

$$W_{2k}(x,m), Y_{2k}(x,m), M_{2k}(x,m), N_{2k}(x,m)$$

is analytic, and their convolution in (3.5) exists as an analytic (ordinary) function by (2.12). On the other hand, for $2k < n$, by Lemmas 2.4, the solution is a tempered distribution.

(2) For the case $t = m = 0$, similarly,

$$\oplus_m^k u(x) = c_0 \delta,$$

and by Lemma 2.5, we have

$$\begin{aligned} u(x) &= c_0\, W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m) \\ &= c_0 (-1)^k R^e_{2k}(x) * R^H_{2k}(x) * S_{2k}(x) * T_{2k}(x). \end{aligned}$$

When $2k \geq n$, all components are analytic and their convolution defines an ordinary function by (2.12). For $2k < n$, the solution is a tempered distribution by Lemmas 2.4.

(3) For $0 < t < k$, the equation is

$$\oplus_m^k u(x) = \sum_{r=1}^{t} c_r \oplus_m^r \delta.$$

Convolve both sides by the fundamental solution

$$W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m),$$

to obtain

$$\begin{aligned} &\oplus_m^k \left(W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)\right) * u(x) \\ &= \sum_{r=1}^{t} c_r \oplus_m^r \left(W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)\right). \end{aligned}$$

By Lemma 2.5 and Theorem 3.1, this implies

$$u(x) = \sum_{r=1}^{t} c_r\, W_{2(k-r)}(x,m) * Y_{2(k-r)}(x,m) * M_{2(k-r)}(x,m) * N_{2(k-r)}(x,m).$$

As in case (1), $u(x)$ is an ordinary function if $2(k-r) \geq n$; otherwise, it is a tempered distribution.

(4) For $t \geq k$ with $k \leq t \leq M$, the equation is

$$\oplus_m^k u(x) = \sum_{r=k}^{M} c_r \oplus_m^r \delta.$$

Convolving both sides by the fundamental solution as before, and using Lemma 2.5 and Theorem 3.1, yields

$$u(x) = \sum_{r=k}^{M} c_r \oplus_m^{r-k} \delta.$$

Since $\oplus_m^{r-k}\delta$ are singular distributions, $u(x)$ is a singular distribution in this case. This completes the proof.

□

Theorem 3.3 (Fourier Transform and Fundamental Solution of $\oplus_m^k$). *Let $n = p + q$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, and $m \geq 0$. Define the iterated operator*

$$\oplus_m^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} + m^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k$$

acting on tempered distributions $\mathcal{S}'(\mathbb{R}^n)$.

Then the Fourier transform of its fundamental solution F_{2k} is

$$\mathcal{F}[F_{2k}](\xi) = \frac{1}{(2\pi)^{n/2}} \frac{1}{\left(\left(\sum_{r=1}^{p} \xi_r^2 + m^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k}.$$

Consequently, F_{2k} satisfies

$$\oplus_m^k F_{2k} = \delta \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

and there exists a constant $C = C(m, p, q, k) > 0$ such that

$$|\mathcal{F}[F_{2k}](\xi)| \leq C(1 + \|\xi\|)^{4k}, \quad \forall \xi \in \mathbb{R}^n,$$

so that $F_{2k} \in \mathcal{S}'(\mathbb{R}^n)$.

Proof. Let $\varphi(x) = e^{-i\xi \cdot x}$. By Lemma 2.6 and Proposition 2.7, each second derivative acts as multiplication by $-\xi_j^2$. Therefore,

$$\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \varphi = -\sum_{r=1}^{p} \xi_r^2 \, \varphi, \quad \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \varphi = -\sum_{j=p+1}^{p+q} \xi_j^2 \, \varphi.$$

Applying $\oplus_m^k$ to φ gives

$$\oplus_m^k \varphi = \left[\Big(\sum_{r=1}^{p} \xi_r^2 + m^2 \Big)^4 - \Big(\sum_{j=p+1}^{p+q} \xi_j^2 \Big)^4 \right]^k \varphi.$$

By definition of the Fourier transform on distributions,

$$\mathcal{F}[\oplus_m^k \delta](\xi) = \frac{1}{(2\pi)^{n/2}} \langle \delta, \oplus_m^k \varphi \rangle = \frac{1}{(2\pi)^{n/2}} \left[\Big(\sum_{r=1}^{p} \xi_r^2 + m^2 \Big)^4 - \Big(\sum_{j=p+1}^{p+q} \xi_j^2 \Big)^4 \right]^k.$$

Using the convolution theorem for tempered distributions and the Fourier transforms of $W_{2k}, Y_{2k}, M_{2k}, N_{2k}$, one obtains the same expression for $\mathcal{F}[F_{2k}]$, proving that F_{2k} is indeed the fundamental solution.

Finally, the polynomial growth of the Fourier transform ensures $F_{2k} \in \mathcal{S}'(\mathbb{R}^n)$. □

Theorem 3.4 (Convolution Identity). *Let $n = p + q$ and decompose $\xi = (\xi', \xi'')$ with $\xi' = (\xi_1, \ldots, \xi_p) \in \mathbb{R}^p$ and $\xi'' = (\xi_{p+1}, \ldots, \xi_n) \in \mathbb{R}^q$. Define*

$$A(\xi) := (|\xi'|^2 + m^2)^4 - (|\xi''|^2)^4.$$

For $k, l \in \mathbb{N}_0$, let

$$F_k := W_{2k} * Y_{2k} * M_{2k} * N_{2k}, \quad G_l := W_{2l} * Y_{2l} * M_{2l} * N_{2l},$$

with Fourier transforms in $\mathcal{S}'(\mathbb{R}^n)$ given by

$$\mathcal{F}(F_k)(\xi) = (2\pi)^{-n/2} A(\xi)^{-k}, \quad \mathcal{F}(G_l)(\xi) = (2\pi)^{-n/2} A(\xi)^{-l}.$$

Then, the following convolution identity holds in $\mathcal{S}'(\mathbb{R}^n)$:

$$F_k * G_l = F_{k+l}, \quad \text{and} \quad \mathcal{F}[F_k * G_l](\xi) = (2\pi)^{-n/2} A(\xi)^{-(k+l)}.$$

Proof. By the distributional convolution theorem,

$$\begin{aligned}\mathcal{F}[F_k * G_l] &= (2\pi)^{n/2}\mathcal{F}[F_k]\,\mathcal{F}[G_l] \\ &= (2\pi)^{n/2} \cdot (2\pi)^{-n/2} A(\xi)^{-k} \cdot (2\pi)^{-n/2} A(\xi)^{-l} \\ &= (2\pi)^{-n/2} A(\xi)^{-(k+l)}.\end{aligned}$$

The singularities of $A(\xi)^{-r}$ on $\{A(\xi) = 0\}$ are interpreted in the sense of tempered distributions via analytic continuation or principal value regularization, making the identity valid in $\mathcal{S}'(\mathbb{R}^n)$. □

An Illustrative Example

Example 3.5. To illustrate the distributional nature and singular behavior of fundamental solutions associated with iterated differential operators of diamond type, we consider a simplified one-dimensional model.

Let

$$\oplus := \frac{d^2}{dx^2},$$

and denote its k-fold iteration by

$$\oplus^k := \frac{d^{2k}}{dx^{2k}},$$

acting on the space of tempered distributions $\mathcal{S}'(\mathbb{R})$.

The fundamental solution $E_k \in \mathcal{S}'(\mathbb{R})$ is defined in the distributional sense by

$$\oplus^k E_k = \delta,$$

where δ denotes the Dirac distribution. Taking the Fourier transform yields

$$\mathcal{F}(E_k)(\xi) = C_k|\xi|^{-2k},$$

for a suitable normalization constant C_k. This expression highlights the singular behavior at the origin $\xi = 0$, which is characteristic of iterated elliptic operators in the framework of tempered distributions.

Since E_k is a distribution and cannot be visualized pointwise, we introduce a regularized kernel

$$E_{k,\varepsilon} := E_k * \rho_\varepsilon, \qquad \rho_\varepsilon(x) = \frac{1}{\sqrt{2\pi}\varepsilon} \exp\left(-\frac{x^2}{2\varepsilon^2}\right),$$

where ρ_ε is a Gaussian mollifier. This regularization preserves the qualitative structure of the singularity while allowing a graphical illustration.

Although this one-dimensional operator does not capture the full mixed-type structure or higher-order composition present in the general operator $\oplus_m^k$, it serves as a representative toy model for visualizing the behavior of regularized fundamental solutions and for illustrating the role of convolution-based regularization in the distributional setting.

4 Applications

The fundamental solution $F_{2k}(x, m)$ of the iterated operator $\oplus_m^k$ admits several applications:

(i) **Fourier Multiplier:** For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{F}[\oplus_m^k \varphi](\xi) = \Big((\sum_{r=1}^{p} \xi_r^2 + m^2)^4 - (\sum_{j=p+1}^{p+q} \xi_j^2)^4\Big)^k \mathcal{F}[\varphi](\xi),$$

allowing spectral analysis and estimates in Sobolev or weighted spaces.

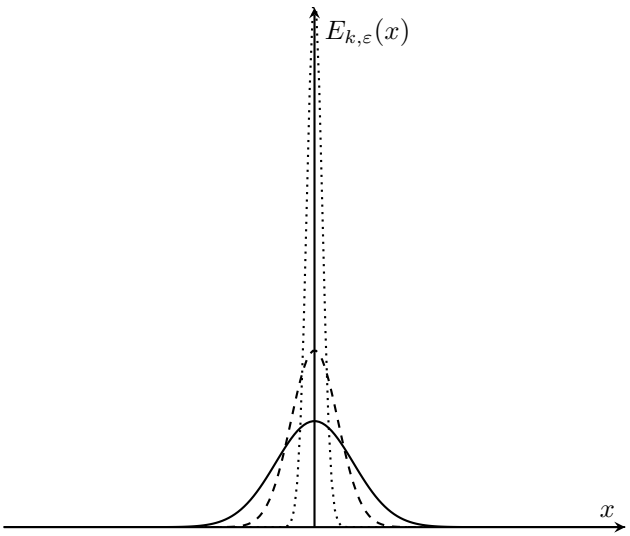


Figure 1. Regularized kernels $E_{k,\varepsilon}$ associated with the iterated one-dimensional operator $\oplus^k$, obtained via convolution with Gaussian mollifiers of decreasing width ε. The increasing concentration near the origin illustrates the underlying singular behavior of the fundamental solution in the sense of tempered distributions.

(ii) **Generalized Klein–Gordon Operator:** When $p = 1$ and $q = n - 1$, $\oplus_m^k$ generalizes the Klein–Gordon operator with higher iterations, describing propagation in anisotropic media with distinct "time-like" and "space-like" directions.

(iii) **Green's Function Construction:** For a domain $\Omega \subset \mathbb{R}^n$ with appropriate boundary conditions, F_{2k} can be used to construct Green's functions solving

$$\oplus_m^k u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = g,$$

via integral representations, provided F_{2k} is extended suitably to respect boundaries.

(iv) **Regularity Estimates:** The polynomial growth of $\mathcal{F}[F_{2k}]$,

$$|\mathcal{F}[F_{2k}](\xi)| \leq C(1 + \|\xi\|)^{4k},$$

allows one to deduce smoothness and decay properties of $u = F_{2k} * f$ in terms of f, useful in PDE theory and distributional analysis.

5 Conclusion

The iterated operator $\oplus_m^k$ together with its fundamental solution F_{2k} provides a rigorous framework for analyzing a broad class of partial differential equations. The distribution F_{2k} serves as a convolution kernel, yielding explicit solution formulas, while its Fourier transform provides a precise spectral characterization of the operator.

This dual perspective enables several important outcomes:

(i) Derivation of regularity and decay estimates for solutions $u = F_{2k} * f$, linking the smoothness of f to that of u.

(ii) Construction of Green's functions for boundary value problems, allowing integral representations of solutions under suitable boundary conditions.

(iii) Analysis of wave propagation and field behavior in anisotropic media, especially when the operator models distinct "time-like" and "space-like" directions.

(iv) Utilization of the Fourier multiplier property of F_{2k} for spectral analysis and estimates in Sobolev, weighted, or distributional spaces.

(v) Extension to convolution identities and iterated operators, providing algebraic tools for decomposing higher-order PDEs into simpler components.

In summary, $\oplus_m^k$ and its fundamental solution F_{2k} not only offer theoretical insights in distribution theory and Fourier analysis but also furnish practical tools for applications in mathematical physics, partial differential equations, and related areas of analysis.

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Author information

S. Bupasiri, Faculty of Education, Sakon Nakhon Rajabhat University, Thailand.
E-mail: sudprathai@gmail.com

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