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Chromatic Coloring of Distance Graphs VI

Yegnanarayanan Venkataraman and J George Barnabas

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Corresponding author: Dr. Yegnanarayanan Venkataraman

Abstract The class 3 or class 4 characterization of prime distance graphs is one of the enormously difficult open problems in graph coloring research. The following results are proved here. a) The operation of pasting a copy of a bipartite graph of any order at each vertex of a prime distance graph of class l for $2 \leq l \leq 4$ is again a prime distance graph. b) The operation of pasting any number of copies of one crown of a prime distance graph of any order at each vertex of a prime distance graph of class l for $2 \leq l \leq 4$ is again a prime distance graph. c) The torus grid graph $T_{n,m}$ is a prime distance graph of Class 2 or Class 3. d) The generalized friendship graph $F_{n,m}$ with $n \geq 2$ and $m \geq 2$ is a prime distance graph of Class 2 or Class 3 depending on the underlying cycle. e) The torch graph O_n for $n \geq 3$ is a prime distance graph of class 3 for any n.

1 Introduction

By a distance graph $G(Z, D)$, we mean a graph whose vertex set is $V(G) = Z$ and the edge set is obtained by introducing an edge between any two elements $u, v \in V(G)$ if $|u - v| \in D$. If $D = P$, the set of primes, then we call $G(Z, P)$ a prime distance graph.

Given a graph G with vertex set V and edge set E and a set X of colors, an ordinary coloring is an assignment of elements of X to the elements of V in any order. A coloring $f : V \to X$ is called a proper coloring of G if $f(u) \neq f(v)$, $\forall\ (u, v) \in E(G)$. A proper coloring f of G is called a proper k-coloring if $|\{f(u) : u \in V(G)\}| = k$. The least k for which G has a proper k-coloring is called the chromatic number of G and is denoted by $\chi(G)$.

Eggleton, Erdos, and Skilton attempted the task of finding the chromatic number of a prime distance graph $G(Z, D)$ where $D \subseteq P$. They found that $\chi(G(Z, P)) = 4$. Voigt and Walther in [1]. Kemnitz and Kolberg in [2], Yegnanarayanan [3, 4, 5, 6, 7, 8, 9], and Eggleton et al. [10, 11] have contributed more to this topic.

We call D of Class i if and only if $\chi(G(Z, D)) = i$. The empty set is the only set that belongs to Class 1, and a set with only one element belongs to Class 2. It is already known from the results available in the literature that if $|D| \geq 2$ then $D \in$ Class 2 if and only if $2 \notin$ D. Similarly, if $2 \in D$ and $3 \notin D$, then $D \in$ Class 3 and if $|D| = 3$ with $\{2, 3, 5\} \in$ D then we have $\chi(G(Z, D)) = 4$ [7, 8]. Also, if $D = \{2, 3, p\}$ with $p \neq 5$, then $\chi(G(Z, D)) = 3$.

It is known that a bipartite graph is a prime distance graph [12]. So, if $D \subset P$ then $\chi(G(Z, D)) = 2$ if and only if G is a bipartite graph. Hence, it follows that "All trees are prime distance graphs". It remains open to find a characterization of graphs with $\chi(G(Z, D)) = 3$ or 4 when $D \subset P$. By a prime distance labeling of a graph G, we mean a bijection $f : V(G) \to Z$ with the property that $|f(u) - f(v)|$ is a prime whenever $(u, v) \in E(G)$. In view of this, one can state that G is a prime distance graph if and only if G has a prime distance labeling. In

this paper, we demonstrate the existence of several new collections of families of graphs with $\chi(G(Z,D)) = 3$ or 4 by using the equivalence of the definition of a prime distance graph and that of a prime distance labeling.

In this paper, we obtain several interesting general results for graphs with $\chi(G(Z,D)) = 3$ or 4. We have obtained prime distance labeling under the assumption that the twin prime conjecture holds. Furthermore, the equivalence between the concepts of prime distance graph and prime distance labeling enables us to obtain new general families of graphs in Classes 3 and 4.

The much-adored Green-Tao Theorem establishes the existence of arbitrarily long arithmetic progressions among the primes. That is, the prime members contain infinitely many arithmetic progressions of length k for all k. In view of this, one can find a prime q such that $q-(r-1)k$, $q-(r-2)k, \cdots, q-k, q, q+k, q+(s-2)k, q+(s-1)k$, is an arithmetic progression of $(r+s-1)$ primes. We use this to exhibit a prime distance labeling for G^* of Theorem 2.1.

The Torus grid graph, $T_{n,m}$, is the graph built from $C_n \times C_m$ where $\times$ is the cartesian product and C_n, C_m are cycle graphs. That is, $T_{n,m}$ consists of n copies of C_m represented as cycles and arranged from innermost cycle to the outer most cycle as $C_m^{(1)}$, $C_m^{(2)}$, $C_m^{(3)}$, $\cdots$ $C_m^{(n)}$ and m copies of C_n represented by the n cycles and arranged clockwise as $C_n^{(1)}$, $C_n^{(2)}$, $C_n^{(3)}$, $\cdots$ $C_n^{(m)}$. The generalized friendship graph $F_{n,m}$ is defined as m copies of the n-cycle C_n, with each of the n^{th} vertices identified. A fan graph f_n is defined as the graph $K_1 + P_n$, where K_1 is the singleton graph and P_n is the path on n vertices. A Spider graph[14] is a tree that has at most one vertex of degree greater than 2. A triangular book graph is the complete tripartite graph $K_{1,1,n}$. Torch graph O_n is a graph obtained by combining a triangular book graph with a fan graph f_3 containing one leaf in the central vertex. The prime distance labeling of the graphs involving the Torus grid graph, Generalized friendship graph, and torch graph is given as Theorem 2.2, Theorem 2.3, Theorem 2.4.

Let $P_n = u_1, u_2, u_3, \cdots, u_n$ be the path on n vertices; $C_n = u_1u_2u_3\cdots u_nu_1$ be the cycle on n vertices. The proofs of all our results here are conditional on the assumption that the Goldbach Conjecture and the Twin Prime Conjecture are true.

2 Prime Distance Labeling of some families of Graphs

The Green-Tao Theorem. [13] For any positive integer k, there exists a prime arithmetic progression of length k.

Goldbach Conjecture. [12] Every even number greater than 2 is the sum of two primes.

Theorem 2.1. *let G be any prime distance graph of class l, $2 \le l \le 4$. The graph G^* defined by by pasting complete bipartite graph H on each vertex of G is again a prime distance graph of class l, $2 \le l \le 4$*

Proof. Let $V(G) = \{v_t : 1 \le t \le n\}$. As G is a prime distance graph, there exists a prime distance labeling $f : V(G) \to Z$. Let $V(H) = A \cup B$ with $A = \{u_i : 1 \le i \le r\}$ and $B = \{w_j : 1 \le j \le s\}$. As H is a complete bipartite graph with k vertices, all edges between A and B are present. Define $g : V(H) \to Z$ as $g(v) = \begin{cases} (i-1)k & \text{for} \quad 1 \le i \le 4 \quad \text{if} \quad v \in A \\ q+(j-1)k & \text{for} \quad 1 \le j \le s \quad \text{if} \quad v \in B \end{cases}$ where q is a prime. By Green-Tao Theorem one can find an arithmetic progression of rs primes $q, q+k, q+2k, \cdots, q+(s-1)k$; $q-k, q, q+k, \cdots, q+(s-2)k$; $q-2k, q-k, q, \cdots, q+(s-3)k$; $\cdots$; $q-(r-1)k, q-(r-2)k, q+(r-3)k, \cdots, q+(s-r)k$. These rs primes will form the edge labels of the rs edges of H. Therefore, g is a prime distance labeling of H and H is a prime distance graph.

Let H_t be the complete bipartite graph pasted on the vertex v_t, $1 \le t \le n$ of G. For $1 \le t \le n$, Let $V(H_t) = A_t \cup B_t$ where $A_t = \{u_t^i : 1 \le i \le r\}$ and $B_t = \{w_t^j : 1 \le j \le s\}$. Then it follows that $V(G^*) = \{v_t, u_t^i, w_t^j : 1 \le x \le n, 1 \le i \le r, 1 \le j \le s\}$. Upon considering the validity of Goldbach's conjecture, find an even number $e_1 = p_1 + q_1$, where p_1 and q_1 are primes, with $e_1 - p_1$ is larger than $f(v_t)$ for $1 \le t \le n$ and q_1 larger than $r+s-1$. Then we can find an arithmetic progression of primes with q_1 terms. So, considering $f(v_t) \le f(v_{t+1})$ we define $h : V(G^*) \to Z$ by $h(v) = f(v)$ for all $v \in V(G)$; $h(w_1^j) = q_1 + (j-1)k$ for $1 \le j \le s$; $h(u_1^i) = (i-1)k$

for $1 \leq i \leq r$. Now, considering the validity of Goldbach's conjecture, find even numbers $e_t = p_t + q_t$ with $e_t - p_t$ larger than max$\{h(w_{t-1}^j), h(u_{t-1}^i)$ for $2 \leq t \leq n, 1 \leq i \leq r, 1 \leq j \leq s\}$ and q_t larger than $r+s-1$. Further define $h(w_t^j) = q_t + (j-1)k$ for $2 \leq j \leq s$; $h(u_t^i) = (i-1)k$ for $2 \leq i \leq r$. The choice of selection of q_t ensures the distinct labels among the vertices of $G^* \backslash G$. $h(v_x) \leq h(v_{x+1})$ define $h : V(G^*) \to Z$ by $h(v) = f(v)$ for all $v \in V(G)$; $h(w_x^j) = q_x + (j-1)k$ for $1 \leq j \leq s$; $h(u_x^i) = (i-1)k$ for $1 \leq i \leq r$. Then we can easily see that q_t, $q_t + k$, $q_t + 2k, \cdots, q_t + (s-1)k$; $q_t - k$, q_t, $q_t + k, \cdots, q_t + (s-2)k$; $q_t - 2k$, $q_t - k$, $q_t, \cdots, q_t + (s-3)k$; $\cdots$; $q_t - (r-1)k$, $q_t - (r-2)k$, $q_t + (r-3)k, \cdots, q_t + (s-r)k$ where $1 \leq t \leq n$ are the edge labels of $G^* \backslash G$. Therefore, h is a prime distance labeling for $G^* \backslash G$ and hence G^* is a prime distance graph. Moreover as each H_t has no odd cycles, the graph G^* will be of the same Class l as that of G for $2 \leq l \leq 4$ □

Theorem 2.2. *The torus grid graph $T_{n,m}$ is a prime distance graph of Class 2 or Class 3*

Proof. Let $V(T_{n,m}) = \{v_j^i : 1 \leq j \leq n, i \leq i \leq m\}$. Define $f : V(T_{n.m}) \to Z$ as follows.

Case 1: $n \equiv 0 \pmod 2$, $m \equiv 0 \pmod 2$

$$f(v_j^1) = \begin{cases} x + (j-1)p_1 & 1 \leq j \leq s \\ (x+2) + (2s-j)p_1 & s+1 \leq j \leq 2s \end{cases}$$

$$f(v_j^i) = \begin{cases} f(v_j^1) + (i-1)q_1 & 1 \leq i \leq r \\ f(v_j^1) + q_2 + (2r-i)q_1 & r+1 \leq i \leq 2r \end{cases}$$

Case 2: $n \equiv 0 \pmod 2$, $m \equiv 1 \pmod 2$

$$f(v_j^1) = \begin{cases} x + (j-1)p_1 & 1 \leq j \leq s \\ (x+2) + (2s-j)p_1 & s+1 \leq j \leq 2s \end{cases}$$

$$f(v_j^i) = \begin{cases} f(v_j^1) + (i-1)q_1 & 1 \leq i \leq r+1 \\ f(v_j^1) + q_2 + (2r+1-i)q_1 & r+2 \leq i \leq 2r+1 \end{cases}$$

Case 3: $n \equiv 1 \pmod 2$, $m \equiv 0 \pmod 2$

$$f(v_j^1) = \begin{cases} x + (j-1)p_1 & 1 \leq j \leq s \\ (x+2) + (2s-j)p_1 & s+1 \leq j \leq 2s+1 \end{cases}$$

$$f(v_j^i) = \begin{cases} f(v_j^1) + (i-1)q_1 & 1 \leq i \leq r \\ f(v_j^1) + q_2 + (2r-i)q_1 & r+1 \leq i \leq 2r \end{cases}$$

Case 4: $n \equiv 1 \pmod 2$, $m \equiv 1 \pmod 2$

$$f(v_j^1) = \begin{cases} x + (j-1)p_1 & 1 \leq j \leq s \\ (x+2) + (2s+1-j)p_1 & s+1 \leq j \leq 2s+1 \end{cases}$$

$$f(v_j^i) = \begin{cases} f(v_j^1) + (i-1)q_1 & 1 \leq i \leq r+1 \\ f(v_j^1) + q_2 + (2r+1-i)q_1 & r+2 \leq i \leq 2r+1 \end{cases}$$

Note that p_1 and p_2 are some randomly chosen twin prime pairs with $p_2 = p_1 + 2$ and (q_1, q_2) also such a twin prime pair with $q_1 > f(v_j^1)$ for $1 \leq j \leq n$. We then see that the edge labels in each of the four cases above are all primes, as shown below. That is
if $n \equiv 0 \pmod 2$ and for $1 \leq i \leq m$
$|f(v_j^i) - f(v_{j+1}^i)| = p_1$ for $1 \leq j \leq s-1$ and $s+1 \leq j \leq 2s-1$; $|f(v_s^i) - f(v_{s+1}^i)| = 2 = |f(v_{2s}^i) - f(v_1^i)|$
if $n \equiv 1 \pmod 2$ and for $1 \leq i \leq m$
$|f(v_j^i) - f(v_{j+1}^i)| = p_1$ for $1 \leq j \leq s-1$ and $s+1 \leq j \leq 2s$; $|f(v_s^i) - f(v_{s+1}^i)| = p_2$ and

$|f(v^i_{2s+1}) - f(v^i_1)| = 2$
if $m \equiv 0 \pmod 2$ and for $1 \leq j \leq n$
$|f(v^i_j) - f(v^{i+1}_j)| = q_1$ for $1 \leq i \leq r-1$ and $r+1 \leq i \leq 2r-1$; $|f(v^r_j) - f(v^{r+1}_j)| = q_2 = |f(v^{2r}_j) - f(v^1_j)|$
if $m \equiv 1 \pmod 2$ and for $1 \leq j \leq n$
$|f(v^i_j) - f(v^{i+1}_j)| = q_1$ for $1 \leq i \leq r$ and $r+1 \leq i \leq 2r$; $|f(v^r_j) - f(v^{r+1}_j)| = 2$ and $|f(v^{2r+1}_j) - f(v^1_j)| = q_2$
Hence f is a prime distance labeling for $T_{n,m}$ and $T_{n,m}$ is a prime distance graph.

□

Theorem 2.3. *The generalized friendship graph $F_{n,m}$ with $n \geq 2$ and $m \geq 2$ is a prime distance graph of Class 2 or Class 3, depending on the underlying cycle.*

Proof. Define $f : V(F_{n,m}) \to Z$ as follows
$f(V_0) = x$; $f(V^i_{n-1}) = x + p^i_2$, $1 \leq i \leq m$;

If $n = 2s$, then $f(V^i_j) = \begin{cases} x + j(p^i_1) & 1 \leq j \leq s-1 \\ x + p^i_2 + (2s-j-1)p^i_1 & s \leq j \leq 2s-2 \end{cases}$

If $n = 2s+1$, then $f(V^i_j) = \begin{cases} x + j(p^i_1) & 1 \leq j \leq s \\ x + p^i_2 + (2s-j)p^i_1 & s+1 \leq j \leq 2s-1 \end{cases}$

where (p^i_1, p^i_2) is a twin prime pair for each i. Here the choice of p^1_1 is random, but for $i > 1$, choose p^i_1 to be larger than $f(v^{i-1}_j)$ for all j. Now we can easily see that $|f(v^i_j) - f(v^i_{j+1})| = p^i_1$ for $0 \leq j \leq s-2$ and $s \leq j \leq 2s-2$; $|f(v^i_0) - f(v^i_{n-1})| = p^i_2$ = $|f(v^i_{s-1}) - f(v^i_s)|$ are all primes for even n. Also $|f(v^i_j) - f(v^i_{j+1})| = p^i_1$ for $0 \leq j \leq s-1$ and $s+1 \leq j \leq 2s-1$; $|f(v^i_s) - f(v^i_{s+1})| = 2$; $|f(v^i_{2s}) - f(v^i_{2s+1})| = p^i_2$ are all primes for odd n. When n is even, $F_{n,m}$ is of Class 2 because it does not have any odd cycle; otherwise, $F_{n,m}$ is of Class 3. Hence, $F(n,m)$ is a prime distance graph of Class 2 or Class 3, depending on the parity of the underlying cycle. □

Theorem 2.4. *The torch graph O_n for $n \geq 3$ is a prime distance graph of class 3 for any n.*

Proof. Define $f : V(O_n) \to Z$ as $f(v_{n+3}) = x$; $f(v_{n+1}) = x + 2$; $f(v_i) = x + p^i_2$ with $p^i_2 < p^{i+1}_2$ for $2 \leq i \leq n-3$; $f(v_1) = p^1_2$ with $p^1_2 > p^{n-2}_2$; $f(v_{n+2})$=$p^1_2 + 2$; $f(v_n) = p^1_2 + 5$; $f(v_{n+4}) = p^1_2 + 7$;$f(v_{n-1}) = f(v_n) + p$, where p is any prime greater than are equal to 3 and $p^i_2 = p^i_1 + 2$ with (p^i_1, p^i_2) for $1 \leq i \leq n-2$ is a twin prime pair.
Now we can easily see that $|f(v_{n+3}) - f(v_i)| = p^i_1$, $2 \leq i \leq n-2$; $|f(v_{n+1}) - f(v_i)| = p^i_2$, $2 \leq i \leq n-2$; $|f(v_{n+3}) - f(v_1)| = p^1_1$; $|f(v_{n+1}) - f(v_1)| = p^1_2$; $|f(v_1) - f(v_{n+2})| = 2$; $|f(v_1) - f(v_n)| = 5$; $|f(v_1) - f(v_{n+4})| = 7$;$|f(v_{n+2}) - f(v_n)| = 3$; $|f(v_n) - f(v_{n+4})| = 2$; $|f(v_n) - f(v_{n-1})| = p$ are all primes Also, for any n, O_n is a class 3 graph, since one can find $n-1$ C_3's embedded in O_n. Hence, O_n is a prime distance graph in Class 3. □

Theorem 2.5. *Let G be any prime distance graph of Class l for $2 \leq l \leq 4$. A graph G^* obtained from G by pasting a cycle at each vertex of G is again a prime distance graph of the same Class l, provided l = 3 or 4. If G is of Class 2, then G^* is of Class 2 or 3, depending on the parity of the length of the underlying cycle in G^*.*

Proof. Let $V(G) = \{v_i : 1 \leq i \leq n\}$ and $V(G^*) = \{v^{i,j} : 1 \leq i \leq n, 1 \leq j \leq t_i\}$. Here t_i is the length of the cycle pasted on the i^{th} vertex of G. Let f be a prime distance labeling of G. Define $g : V(G^*) \to Z^+$ as follows: $g(v) = f(v)$ whenever $v \in V(G)$. Set $v^{i,1} = v_i$ $1 \leq i \leq n$ and $g(v^{i,1}) = f(v_i)$.

For $t_1 = 2s_1$, set $g(v^{1,j}) = \begin{cases} f(v_1) + p^{1,1} + 3(j-2): & 2 \leq j \leq s_1 \\ f(v_1) + p^{1,2} + 3(2s_1 - j): & s_1 + 1 \leq j \leq 2s_1 \end{cases}$

For $2 \leq i \leq n$ and $t_i = 2s_i$, set $g(v^{i,j}) = \begin{cases} f(v_i) + p^{1,1} + 3(j-2): & 2 \leq j \leq s_i \\ f(v_i) + p^{1,2} + 3(2s_i - j): & s_i + 1 \leq j \leq 2s_i \end{cases}$

For $t_1 = 2s_1 + 1$, set

$$g(v^{1,j}) = \begin{cases} f(v_1) + p^{1,1} + 3(j-2): & 2 \leq j \leq s_1 + 1 \\ f(v_1) + p^{1,2} + 3(2s_1 + 1 - j): & s_1 + 2 \leq j \leq 2s_1 + 1 \end{cases}$$

For $2 \leq i \leq n$ and $t_i = 2s_i + 1$, set

$$g(v^{i,j}) = \begin{cases} f(v_i) + p^{i,1} + 3(j-2): & 2 \leq j \leq s_i + 1 \\ f(v_i) + p^{i,2} + 3(2s_i + 1 - j): & s_i + 2 \leq j \leq 2s_i + 1 \end{cases}$$

We choose $p^{1,1} > \max_{1 \leq i \leq n}\{f(v_i)\}$ and $p^{i,1} > g(v^{i-1,s_{i-1}+1})$ as the first members of the respective twin prime pairs $(p^{i,1}, p^{i,2})$, $(p^{i,1}, p^{i,2})$ uniformly for both the choices of parity of t_1 and t_i respectively. Also note that $g(v^{i-1,s_{i-1}+1})$ is the largest label among the labels of the vertices of the cycle pasted at the vertex v_{i-1}. Then the edge labels are, $|g(v^{i,1}) - g(v^{i,2})| = p^{i,1}$; $|g(v^{i,j}) - g(v^{i,j+1})| = 3$, $2 \leq j \leq s_i - 1$ and $s_i + 1 \leq j \leq 2s_i - 1$; $|g(v^{i,s_i}) - g(v^{i,s_i+1})| = 5$; $|g(v^{i,2s_i-1}) - g(v^{i,2s_i})| = p^{i,2}$ for $t_i = 2s_i$ and $|g(v^{i,1}) - g(v^{i,2})| = p^{i,1}$; $|g(v^{i,j}) - g(v^{i,j+1})| = 3$, $2 \leq j \leq s_i$ and $s_i + 2 \leq j \leq 2s_i + 1$; $|g(v^{i,s_i+1}) - g(v^{i,s_i+2})| = 2$; $|g(v^{i,2s_i}) - g(v^{i,2s_i+1})| = p^{i,2}$ for $t_i = 2s_i + 1$. One can see that all these edge labels are primes and G^* is a prime distance graph. If G is of Class 2, then G^* is also of Class 2 when the length of the underlying cycles of G^* is all of even length, and G^* is of Class 3 when the length of even one underlying cycle of G^* is of odd length. If G belongs to Class 3 or Class 4, then the vertex set of the cycle attached to each vertex of G can be accommodated in any two of the l partite sets for $l = 3$ or 4. Thus G^* is of Class 3 or Class 4, respectively. The proof is complete provided we justify the sanctity of the choice of twin prime pairs $(p^{i,1}, p^{i,2})$. It is enough to demonstrate an instance where $p^{i,1} \leq \max\{f(v_i)\}$, $p^{i,1} \leq g(v^{i-1,s_i+1})$ for $2 \leq i \leq n$ results in contradiction. Suppose that $G = K_4$ with $V(K_4) = \{v_1, v_2, v_3, v_4\}$ and $V(G^*) = \{v^{1,j}, 1 \leq j \leq 6, v^{2,j}, 1 \leq j \leq 5\}$; $v^{3,j}, 1 \leq j \leq 6$; $\{v^{4,j}, j \in Z^+$ be any integer$\}$. Let $f(v_1) = 2$, $f(v_2) = 4$, $f(v_3) = 7$, $f(v_4) = 9$. If $(p^{1,1}, p^{1,2}) = (5, 7)$, then $g(v^{1,2}) = f(v_1) + 5 = 2 + 5 = 7 = f(v_3)$, a contradiction. So, let $(p^{1,1}, p^{1,2}) = (11, 13)$. Now, $p^{2,1}$ should be greater than $g(v^{1,4}) = g(v^{1,5}) + 3$. As $g(v^{1,5}) = g(v^{1,6}) + 3 = [g(v^{1,1}) + p^{1,2}] + 3 = (2+13)+3 = 18$. Therefore $g(v^{1,4}) = 18+3 = 21$. We infer $p^{2,1} > 21$. If $p^{2,1} \leq 21$ and $(p^{2,1}, p^{2,2}) = (11, 13)$ then $g(v^{2,2}) = g(v^{2,1}) + p^{2,1} = 4+11 = 15 = g(v^{1,6})$, a contradiction. So let $(p^{1,1}, p^{1,2}) = (29, 31)$. Now, $p^{3,1}$ should be greater than $g(v^{2,4}) = g(v^{2,5}) + 3$. As $g(v^{2,5}) = g(v^{2,6}) + 3 = [g(v^{2,1}) + p^{2,2}] + 3 = (4 + 31) + 3 = 38$. We infer that $p^{3,1} > 38$. If $p^{3,1} \leq 38$ and $(p^{3,1}, p^{3,2}) = (29, 31)$, then $g(v^{3,6}) = \{g(v^{3,1}) + p^{3,2}\} = 7 + 31 = 38 = g(v^{2,4})$, a contradiction. So let $(p^{1,1}, p^{1,2}) = (11, 13)$, $(p^{2,1}, p^{2,2}) = (29, 31)$, and $(p^{3,1}, p^{3,2}) = (41, 43)$. Now, $p^{4,1}$ should be greater than $g(v^{3,4}) = g(v^{3,5}) + 3 = [g(v^{3,6}) + 3] + 3 = [(g(v^{3,1}) + p^{3,2}) + 3] + 3 = [(7 + 43) + 3] + 3 = 56$. We infer that $p^{4,1} > 56$. If $p^{4,1} \leq 56$ and $(p^{4,1}, p^{4,1}) = (41, 43)$, then $g(v^{4,2}) = g(v^{4,1}) + p^{4,1} = 9+41 = 50 = g(v^{3,6})$, a contradiction. □

Theorem 2.6. *Let G be any prime distance graph of Class l for $2 \leq l \leq 4$. A graph G^* obtained from G by pasting any number of cycles at each vertex of G is again a prime distance graph of the same Class l if l = 3 or 4. If G is of Class 2, then G^* is of Class 2 or 3, depending on the parity of the length of the underlying cycles in G^*.*

Proof. Let $V(G) = \{v_i : 1 \leq i \leq n\}$ and $V(G^*) = \{v_k^{i,j} : 1 \leq i \leq n, 1 \leq j \leq r_i, 1 \leq k \leq t^{i,j}\}$. Here r_i is the number of cycle pasted on the i^{th} vertex of G, $t^{i,j}$ denote the length of the j^{th} cycle pasted on the i^{th} vertex of G. Let f be a prime distance labeling of G and $v_k^{i,j}$ represents k^{th} vertex of the j^{th} cycle pasted on the i^{th} vertex of G. Define $g : V(G^*) \to Z^+$ as follows: $g(v) = f(v)$ whenever $v \in V(G)$. Set $v_1^{i,j} = v_i$ $1 \leq i \leq n, 1 \leq j \leq r_i$ and $g(v_1^{i,j}) = f(v_i)$.
If $i = 1$, two cases arise.

For $t^{1,1} = 2s^{1,1}$, set $g(v_k^{1,1}) = \begin{cases} f(v_1) + p_1^{1,1} + 3(k-2): & 2 \leq k \leq s^{1,1} \\ f(v_1) + p_2^{1,1} + 3(2s^{1,1} - k): & s^{1,1} + 1 \leq k \leq 2s^{1,1} \end{cases}$

For $t^{1,1} = 2s^{1,1}+1$, set $g(v_k^{1,1}) = \begin{cases} f(v_1) + p_1^{1,1} + 3(k-2): & 2 \leq k \leq s^{1,1} + 1 \\ f(v_1) + p_2^{1,1} + 3(2s^{1,1} + 1 - k): & s^{1,1} + 2 \leq k \leq 2s^{1,1} + 1 \end{cases}$

If $2 \leq j \leq r_1$, two cases arise,

For $t^{1,j} = 2s^{1,j}$, set $g(v_k^{1,j}) = \begin{cases} f(v_1) + p_1^{1,j} + 3(k-2): & 2 \leq k \leq s^{1,j} \\ f(v_1) + p_2^{1,j} + 3(2s^{1,j} - k): & s^{1,j} + 1 \leq k \leq 2s^{1,j} \end{cases}$

For $t^{1,j} = 2s^{1,j} + 1$, set

$$g(v_k^{1,j}) = \begin{cases} f(v_1) + p_1^{1,j} + 3(k-2): & 2 \leq k \leq s^{1,j} + 1 \\ f(v_1) + p_2^{1,j} + 3(2s^{1,j} + 1 - k): & s^{1,j} + 2 \leq k \leq 2s^{1,j} + 1 \end{cases}$$

If $2 \leq i \leq n$, $2 \leq j \leq r_i$, two cases arise,

For $t^{i,j} = 2s^{i,j}$, set $g(v_k^{i,j}) = \begin{cases} f(v_i) + p_1^{i,j} + 3(k-2): & 2 \leq j \leq s^{i,j} \\ f(v_i) + p_2^{i,j} + 3(2s^{i,j} - k): & s^{i,j}+1 \leq j \leq 2s^{i,j} \end{cases}$

For $t^{i,j} = 2s^{i,j} + 1$, set

$$g(v_k^{i,j}) = \begin{cases} f(v_i) + p_1^{i,j} + 3(k-2): & 2 \leq j \leq s^{i,j}+1 \\ f(v_i) + p_2^{i,j} + 3(2s^{i,j}+1-k): & s^{i,j}+2 \leq j \leq 2s^{i,j}+1 \end{cases}$$

We choose $p_1^{1,1} > \max_{1 \leq i \leq n}\{f(v_i)\}$ as the first member of twin prime pair $(p_1^{1,1}, p_2^{1,1})$ uniformly for both the choices of parity of t_1. Then, we choose $p_1^{1,j} > g(v_{s^{1,j-1}+1}^{1,j-1})$, $2 \leq j \leq r_1$ as the first member of twin prime pair $(p_1^{1,j}, p_2^{1,j})$ uniformly for both the choices of parity of $t^{1,j}$. Also, we choose $p_1^{i,1} > g(v_{s^{i-1,r_{i-1}}+1}^{i-1,r_{i-1}})$, $2 \leq i \leq n$ as the first member of twin prime pair $(p_1^{i,1}, p_2^{i,1})$ uniformly for both the choices of parity of $t^{1,j}$. Further, we choose $p_1^{i,j} > g(v_{s^{i,j-1}+1}^{i,j-1})$, $2 \leq j \leq r_i$, $2 \leq i \leq n$ as the first member of twin prime pair $(p_1^{i,j}, p_2^{i,j})$ uniformly for both the choices of parity of $t^{i,j}$. Also note that $g(v_{s^{i,j}+1}^{i,j})$ is the largest label among the labels of the vertices of the cycle pasted at the vertex v_i. Then the edge labels are $|g(v_1^{i,j}) - g(v_2^{i,j})| = p_1^{1,1}$; $|g(v_k^{i,j}) - g(v_{k+1}^{i,j})| = 3$, $2 \leq k \leq s^{i,j}-1$ and $s^{i,j}+1 \leq k \leq 2s^{i,j}-1$; $|g(v_{s^{i,j}}^{i,j}) - g(v_{s^{i,j}+1}^{i,j})| = 5$; $|g(v_{2s^{i,j}-1}^{i,j}) - g(v_{2s^{i,j}}^{i,j})| = p_2^{i,j}$ for $t_j^i = 2s^{i,j}$ and $|g(v_1^{i,j}) - g(v_2^{i,j})| = p_2^{i,j}$; $|g(v_k^{i,j}) - g(v_{k+1}^{i,j})| = 3$, $2 \leq k \leq s^{i,j}$ and $s^{i,j}+2 \leq k \leq 2s^{i,j+1}$; $|g(v_{s^{i,j+1}}^{i,j}) - g(v_{s^{i,j}+2}^{i,j})| = 2$; $|g(v_{2s^{i,j}}^{i,j}) - g(v_{2s^{i,j}+1}^{i,j})| = p_2^{i,j}$ for $t^{i,j} = 2s^{i,j+1}$. We see that all the edge labels are primes, and G^* is a prime distance graph. If G is of Class 2, then G^* is also of Class 2 when the length of the underlying cycles of G^* is all of even length, and G^* is of Class 3 when the length of even one underlying cycle of G^* is of odd length. If G belongs to Class 3 or Class 4, then the vertex set of the cycle attached to each vertex of G can be accommodated in any two of the l partite sets for l = 3 or 4. Thus G^* is of Class 3 or Class 4, respectively. The proof is complete provided we justify the sanctity of the choice of twin prime pairs $(p_1^{i,j}, p_2^{i,j})$. It is enough to demonstrate an instance where $p_1^{1,1} \leq \max_{1 \leq i \leq n}\{f(v_i)\}$, $p_1^{1,j} \leq g(v_{s^{1,j-1}+1}^{1,j-1})$ for $2 \leq j \leq r_1$, $p_1^{i,1} \leq g(v_{s^{i-1,r_{i-1}}+1}^{i-1,r_{i-1}})$ for $2 \leq i \leq n$, $p_1^{i,j} \leq g(v_{s^{i,j-1}+1}^{i,j-1})$, $2 \leq i \leq n$, $2 \leq j \leq r_i$ results in a contradiction. Suppose that $G = K_4$ with $V(K_4) = \{v_1, v_2, v_3, v_4\}$ and $(G^*) = \{v_k^{i,j} : 1 \leq i \leq n, 1 \leq j \leq r_i, 1 \leq k \leq t^{i,j}\}$. Note that when $r_i = 1$, G^* reduces to the graph G^* of Theorem 2.5 and hence it serves as an instance of demonstration. □

Lemma 2.7. *(Observation 1.3 [11]), Let G be any graph, $f : V(G) \to Z$ an injection, and c any real number. The function f is a prime distance labeling of G if and only if $f + c$ is a prime distance labeling of G.*

Theorem 2.8. *Let G be a prime distance graph on n vertices. Consider the 1-Crown of G denoted G' obtained by pasting a copy of K_2 on each of the n vertices of G. Let $\{H_i\}_{1 \leq i \leq n}$ be a sequence of prime distance graphs. Then the graph G^* obtained from G' by pasting H_i at each pendant vertex u_i for $1 \leq i \leq n$ is a prime distance graph of some class l for $1 \leq i \leq 4$.*

Proof. Let $V(G) = \{v_i : 1 \leq i \leq n\}$ and $V(G') = \{v_i, u_i : 1 \leq i \leq n\}$. As G is a prime distance graph, there exists a prime distance labeling $g : V(G) \to Z$. Also, as $\{H_i\}_{1 \leq i \leq n}$ is a sequence of prime distance graphs. Each H_i has a prime distance labeling $f_i : V(H_i) \to Z$. Clearly $V(G^*) = V(G) \cup (\cup_{i=1}^{n} V(H_i))$. Now define $h : V(G^*) \to Z$ as such that $h(u) = f_i(u) + p_i$ for $u \in V(H_i)$ where p_1 is larger than $g(v_i)$ for all $v_i \in V(G)$ for $1 \leq i \leq n$ and p_i is larger than the $max\{h(u) : u \in V(H_{i-1})\}$ for $2 \leq i \leq n$. The choice of selection of p_i ensures the distinct labels among the vertices of G^*. Further, Lemma 2.7 implies that h produces the edge label for the edges of G^* exactly as in the edges of H_i for $1 \leq i \leq n$. This means h is a prime distance labeling of G^* and G^* is a prime distance graph of some class l for $1 \leq l \leq 4$. □

3 Conclusion

Recent progress has been made towards the uphill task of determining whether prime distance graphs belong to class 3 or class 4, as in [10, 11]. Also, several works, such as the Prime distance graph of Laman graphs, the Laman Torus Graph [15], and a complete characterization of Line Graph [16], are part of our ongoing studies. As our efforts continue, we found a few more general results for such graphs in classes 3 and 4.

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Declaration

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Author information

Yegnanarayanan Venkataraman, Research Track Professor, School of Engineering and Technology-SOET, CMR University, Chagalahatti, Off Hennur–Bagalur Main Road, Bengaluru-562149, Karnataka, India.
E-mail: prof.yegna@gmail.com

J George Barnabas, Department of Mathematics, Kalasalingam Academy of Research and Education, Krishnankoil, India.
E-mail: jgeorgebarnabas@gmail.com

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