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SOME RESULTS ON TOPOLOGICAL BCK-ALGEBRAS

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Abstract. In this paper, we investigate the properties of topological BCK-algebras and compare the linear topology with the topology induced by a system of ideals in a BCK-algebra. We show that systems of ideals generate zero-dimensional linear topologies and explore the conditions under which these structures preserve the BCK-algebraic operations continuously. Furthermore, we demonstrate how prefilters can induce such topologies and define an order topology that satisfies the open condition for quotient maps.

1 Introduction

In 1966, Y. Imai and K. Iséki introduced in [5] a new notion, called a BCK-algebra. This notion originated from two different ways: one of them is based on set theory; another is from classical and nonclassical propositional calculi. The BCK-operator $*$ is an analogue of the set-theoretical difference. As is well known, there is a close relationship between the notions of the set difference in set theory and the implication functor in logical systems. Then the following problems arise from this relationship. What are the most essential and fundamental common properties? Can we formulate a new general algebra from this viewpoint? How can we find an axiom system to establish a consistent general theory of algebras? To answer these problems, Y. Imai and K. Iséki introduced a notion of a new class of general algebras called a BCK-algebra. This name is taken from the BCK-system of C. A. Meredith. Since then, many researchers studied several notions and properties of BCK-algebras. The interaction between algebraic structures and topology often leads to rich theories, as seen in topological groups, rings, and lattices. This paper explores this synergy in the context of BCK-algebras. Building on earlier work, where the topology induced by a system of ideals was introduced [7] and the concept of Cauchy sequences was developed [8], this paper aims to further develop the theory of topological BCK-algebras. In particular, we study the relationships between ideals and topologies on BCK-algebras. We establish a method for constructing a topology from a single ideal, thereby turning a BCK-algebra into a topological algebra. We show that the supremum of a family of such topologies preserves the structure of topological BCK-algebras. We prove that systems of ideals generate linear, zero-dimensional topologies. Furthermore, we demonstrate how prefilters can induce topological structures that respect BCK-operations, and we define an order topology that satisfies the openness condition for canonical quotient maps. These results provide a foundational framework for topological BCK-algebras, analogous to established theories in other algebraic structures.

2 Preliminaries

An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCK-algebra* if it satisfies the following axioms: for any $x, y, z \in X$,
(1) $((x * y) * (x * z)) * (z * y) = 0$;
(2) $(x * (x * y)) * y = 0$;

(3) $x * x = 0$;
(4) $x * y = y * x = 0 \Rightarrow x = y$;
(5) $0 * x = 0$.
In BCK-algebra X, if we define $\leqslant$ by $x \leqslant y$ if and only if $x * y = 0$, then $\leqslant$ is a partial order and the following conclusions hold:
(6) $(y * x) * (z * x) \leqslant (y * z)$,
(7) $x * (x * (x * y)) = x * y$,
(8) $(x * y) * z = (x * z) * y$,
(9) $x * 0 = x$,
(10) $x * y \leqslant x$,
(11) $x \leqslant y$ implies $x * z \leqslant y * z$(See [6]).

Definition 2.1. [6] Let X be a BCK-algebra. An *ideal* is a nonempty set $I \subseteq X$ such that for each $x, y \in X$,
(a) $0 \in I$,
(b) $x * y \in I\ ,\ y \in I \Rightarrow x \in I$.

Proposition 2.2. *[6] Let I be an ideal in a BCK-algebra $(X, *, 0)$. Then*
(i) *if $x \leqslant y$ and $y \in I$, then $x \in I$,*
(ii) *the relation*

$$x \equiv^I y \ \Leftrightarrow\ x * y,\ \ y * x \in I$$

is a congruence relation on X.
(iii) *if $\frac{x}{I} = \{y \in A : x \equiv^I y\}$ and $\frac{X}{I} = \{\frac{x}{I} : x \in X\}$, then $\frac{X}{I}$ is a BCK-algebra under the binary operation*

$$\frac{x}{I} * \frac{y}{I} = \frac{x * y}{I}.$$

In this case, $\frac{X}{I}$ is said to be a quotient BCK-algebra.
(iv) *the mapping $\pi_I : X \hookrightarrow \frac{X}{I}$ by $\pi_I(x) = x/I$ is an epimorphism and for each $S \subseteq X$,*

$$(\pi_I^{-1} \circ \pi_I)(S) = \bigcup_{x \in S} \frac{x}{I}.$$

π_I is also called the canonical epimorphism.

Recall that a set X with a family $\mathcal{U}$ of its subsets is called a *topological space*, denoted by $(X,\mathcal{U})$, if $X, \emptyset \in \mathcal{U}$, the intersection of any finite numbers of members of $\mathcal{U}$ is in $\mathcal{U}$ and the arbitrary union of members of $\mathcal{U}$ is in $\mathcal{U}$. The members of $\mathcal{U}$ are called *open sets* of X and the complement of $X \in \mathcal{U}$, that is $X \setminus U$, is said to be a *closed set*. If B is a subset of X, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_u B$). A subset P of X is said to be a *neighborhood* of $x \in X$, if there exists an open set U such that $x \in U \subseteq P$. A subfamily $\{U_\alpha\}$ of $\mathcal{U}$ is said to be a *base* of $\mathcal{U}$ if for each $x \in U \in \mathcal{U}$ there exists an α such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{U}$ is a union of members of $\{U_\alpha\}$. A subfamily $\{U_\beta\}$ of $\mathcal{U}$ is said to be form a *subbase* for $\mathcal{U}$ if the family of finite intersections of members of $\{U_\beta\}$ forms a base of $\mathcal{U}$. Let $\mathcal{U}_x$ denote the totality of all neighborhoods of x in X. Then a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a *fundamental system* of neighborhoods of x, if for each U_x in $\mathcal{U}_x$, there exists a V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$. Let X be a BCK-algebra and $\mathcal{U}$ be a topology on X. Then $\tilde{\tau} = \{U \subseteq X/I : \pi_I^{-1}(U) \in \mathcal{U}\}$ is a topology on X/I and is called the quotient topology on X/I (See [3]).

Definition 2.3. Let X be a BCK-algebra and τ be a topology on X. If for any ideal I, the canonical epimorphism $\pi_I : X \to X/I$ is an open map, then we call (X, τ) satisfies the open condition.

Definition 2.4. A poset $(D, \leq)$ is called an *upward directed* set if for any $x, y \in D$ there exists $z \in D$ such that $x \leq z$ and $y \leq z$.

Definition 2.5. Topology τ on BCK-algebra $(X, *, 0)$ is called *linear topology* if there exists a base β for τ such that for any element B of β containing 0, B is an ideal of X.

Definition 2.6. Let Λ be an upward directed set and $\{I_i : i \in \Lambda\}$ be a family of ideals in a BCK-algebra $(X, *, 0)$. Then $\{I_i : i \in \Lambda\}$ is called a *system of ideals* of X if $i \leq j$ implies $I_j \subseteq I_i$, for any $i, j \in \Lambda$.

Let $\{\tau_\gamma : \gamma \in \Gamma\}$ be a non-empty family of topologies on X and $\tau = \cap_{\gamma\in\Gamma}\tau_\gamma$. Then (X, τ) is a topological space. The topology τ is the greatest lower bound of the family of topologies $\{\tau_\gamma : \gamma \in \Gamma\}$ on X and is denoted by $inf\{\tau_\gamma : \gamma \in \Gamma\}$. It is clear that $\tau \subseteq \tau_\gamma$ for any $\gamma \in \Gamma$. Let $\{\tau_\omega^{'} : \omega \in \Omega\}$ be a family of all topologies on X such that $\tau_\gamma \subseteq \tau_\omega^{'}$ for any $\gamma \in \Gamma$ and any $\omega \in \Omega$. Since the discrete topology belongs to $\{\tau_\omega^{'} : \omega \in \Omega\}$, then this family is non-empty and, hence, we can consider the topology $\tau^{'} = inf\{\tau_\omega : \omega \in \Omega\}$. The topology $\tau^{'}$ is called the least upper bound of family $\{\tau_\gamma : \gamma \in \Gamma\}$ and is denoted by $sup\{\tau_\gamma : \gamma \in \Gamma\}$. Clearly, $\tau_\gamma \subseteq \tau^{'}$ for any $\gamma \in \Gamma$. For each $x \in X$, consider the family $B_x(X, \tau^{'})$ of subsets U of X such that there exist finite sets of topologies $\tau_{\gamma_1}, ..., \tau_{\gamma_n}$ and neighbourhoods $U_1, ..., U_n$ in the topological space $(X, \tau_{\gamma_1}), ..., (X, \tau_{\gamma_n})$ respectively, such that $U = \cap_{i=1}^n U_i$. The family $B_x(X, \tau^{'})$ is a fundamental system of neighbourhoods of the element x in $(X, \tau^{'})$ (See [1]).

3 Main results

Proposition 3.1. *Let X be a BCK-algebra and I be an ideal of X. The collection $\tau_I = \{U \subseteq X : \forall x \in U, x/I \subseteq U\}$ is a topology on X such that (X, τ_I) is a topological BCK-algebra.*

Proof. It is straightforward to verify that τ_I satisfies the axioms of a topology. The continuity of the operation $*$ follows from the properties of the congruence classes. □

Remark 3.2. Let X be a BCK-algebra. If $I_1 = \{0\}$ and $I_2 = X$, then τ_{I_1} and τ_{I_2} are the discrete and indiscrete topologies, respectively. If I is a proper ideal, then τ_I is a non-trivial topology where x/I is the smallest open set containing x. If I_1 and I_2 are distinct ideals, then $\tau_{I_1} \neq \tau_{I_2}$. Therefore, the mapping $I \mapsto \tau_I$ from the set of ideals $\mathcal{I}(X)$ to the set of topological BCK-algebra topologies $\mathcal{T}(X)$ is injective. Hence, $|\mathcal{I}(X)| \leq |\mathcal{T}(X)|$.

Theorem 3.3. *Let X be a BCK-algebra and Ξ be a family of topologies on X such that (X, τ) is a topological BCK-algebra for any $\tau \in \Xi$. Then $(X, \tau^{'})$ is a topological BCK-algebra where $\tau^{'} = sup\{\tau : \tau \in \Xi\}$.*

Proof. Let $\mathcal{B}_a(\tau)$ be the family of all neighbourhoods of the element a in topological space (X, τ). Then

$$\mathcal{B}_a(\tau^{'}) = \{U \subseteq X : U = \cap_{i=1}^n U_i, U_i \in \mathcal{B}_a(\tau_i), \tau_i \in \Xi, 1 \leq i \leq n\}.$$

Let $a, b \in X$ and $U \in \mathcal{B}_{a*b}(\tau^{'})$. Then there exist topologies $\tau_1, \tau_2, ..., \tau_n \in \Xi$ such that $U = \cap_{i=1}^n U_i$ where $U_i \in \mathcal{B}_{a*b}(\tau_i)$. Since (X, τ_i) is a topological BCK-algebra for $i = 1, 2, ..., n$, there exist $V_i \in \mathcal{B}_a(\tau_i)$ and $W_i \in \mathcal{B}_b(\tau_i)$ such that for $V_i * W_i \subseteq U_i$ for $i = 1, 2, ..., n$. Let $V = \cap_{i=1}^n V_i$ and $W = \cap_{i=1}^n W_i$. Then $V \in \mathcal{B}_a(\tau^{'})$ and $W \in \mathcal{B}_b(\tau^{'})$ and we have

$$V * W = \cap_{i=1}^n V_i * \cap_{i=1}^n U_i \subseteq \cap_{i=1}^n (V_i * W_i) \subseteq \cap_{i=1}^n U_i = U.$$

Hence $*$ is continuous. □

Proposition 3.4. *Let (X, τ) be a topological BCK-algebra and I be an ideal of X. Then*
(i) If $\{0\}$ is open (closed), then $\{x\}$ is open (closed), for any $x \in X$.
(ii) If 0 is an isolated point of I, then the subspace topology on I is discrete.
(iii) If 0 is an interior point of I, then I is closed.

Proof. (i) For a fixed $x \in X$, define $f : X \to X \times X$ by $f(y) = (y * x, x * y)$. This map is continuous. If $\{0\}$ is open, then $\{(0, 0)\}$ is open in $X \times X$, so $f^{-1}(\{(0, 0)\}) = \{y \in X : y * x = x * y = 0\} = \{x\}$ is open.
(ii) Let $x \in I$. We show that $\{x\}$ is open in I. Since 0 is an isolated point of I, there exists an open neighbourhood U of 0 such that $U \cap I = \{0\}$. Since $x*x = 0 \in U$ and $*$ is continuous, there exists an open neighbourhood V of x such that $V*V \subseteq U$. Then $(V \cap I)*(V \cap I) \subseteq (U \cap I) = \{0\}$.

If $y \in V \cap I$ then $x * y, y * x \in (V \cap I) * (V \cap I) = \{0\}$. Thus $x \leq y$ and $y \leq x$. Hence $x = y$ and therefore $V \cap I = \{x\}$. Thus $\{x\}$ is open in I.
(iii) If 0 is an interior point of I, then there exists an open neighborhood U of 0 that is contained in I. To show I is closed, let $x \in X \setminus I$. Since $x * x = 0 \in U$, there exists an open neighborhood V of x such that $V * V \subseteq U$. We claim $V \subseteq X \setminus I$. Suppose not; then there exists $y \in V \cap I$. For any $z \in V$, we have $z * y \in V * V \subseteq U \subseteq I$. Since $y \in I$ and I is an ideal, $z \in I$. This implies $V \subseteq I$, contradicting $x \in V$ and $x \notin I$. Hence, $V \subseteq X \setminus I$, so $X \setminus I$ is open. □

Proposition 3.5. *[7] Let $\mathcal{I} = \{I_i : i \in \Lambda\}$ be a system of ideals of BCK-algebra $(X, *, 0)$. Then the set $\beta = \{x/I_i : x \in X,\ i \in \Lambda\}$ is a base for a linear topology $\tau_{\mathcal{I}}$ on X. Moreover, $(X, \tau_{\mathcal{I}})$ is a topological BCK-algebra.*

Remark 3.6. Let $\mathcal{I} = \{I_i : i \in \Lambda\}$ be a system of ideals. Then:

$$\tau_{\mathcal{I}} = \{U \subseteq X : \forall x \in U, \exists I_i \in \mathcal{I} \text{ such that } x/I_i \subseteq U\}.$$

When $\mathcal{I} = \{I\}$, we have $\tau_{\mathcal{I}} = \tau_I$, consistent with Proposition 3.1. By Remark 3.2, there is an injective map from $\mathcal{I}(X)$ to $L\mathcal{T}(X)$, the set of all linear topological BCK-algebra topologies on X, so $|\mathcal{I}(X)| \leq |L\mathcal{T}(X)|$.

Lemma 3.7. *Let $\mathcal{I} = \{I_i : i \in \Lambda\}$ be a family of ideals of a BCK-algebra X. Then $\mathcal{I}$ is a system of ideals of X if and only if $(\mathcal{I}, \subseteq)$ is a downward directed set.*

Proof. The result follows directly from the definitions of a system of ideals and a downward directed set. □

Theorem 3.8. *Let $\mathcal{I} = \{I_i : i \in \Lambda\}$ be a system of ideals of a BCK-algebra X. Then $\tau_{\mathcal{I}} = sup\{\tau_{I_i} : i \in \Lambda\}$.*

Proof. Let $\tau' = sup\{\tau_{I_i} : i \in \Lambda\}$. Since (X, τ_{I_i}) is a topological BCK-algebra for any $i \in \Lambda$, by Theorem 3.3, (X, τ') is a topological BCK-algebra. Suppose that $x \in U \in \tau'$. There exist neighbourhoods $U_{i_1}, U_{i_2}, ..., U_{i_n}$ of x in the topological space $(X, \tau_{i_1}), (X, \tau_{i_2}), ..., (X, \tau_{i_n})$ respectively, such that $U = \cap_{j=1}^{n} U_{i_j}$. By Lemma 3.7, since $(\mathcal{I}, \subseteq)$ is a downward directed set, there exists $I \in \mathcal{I}$ such that $I \subseteq \cap_{j=1}^{n} U_{i_j}$. Thus

$$x \in x/I \subseteq x/ \cap_{j=1}^{n} I_{i_j} = \cap_{j=1}^{n} x/I_{i_j} \subseteq \cap_{j=1}^{n} U_{i_j} = U.$$

Hence $\tau' \subseteq \tau_{\mathcal{I}}$. The converse inclusion $\tau_{\mathcal{I}} \subseteq \tau'$ is obvious because each $\tau_{I_i} \subseteq \tau'$ and thus the base for $\tau_{\mathcal{I}}$ is contained in τ'. □

Remark 3.9. Let (X, υ) be a topological BCK-algebra such that υ is a linear topology on X. By the assumption, a base β for υ exists such that any element of β that contains 0 is an ideal of X. Let S be the set of all elements of β that contain 0 and $\mathcal{Y}$ is the set of all finite intersections of the elements of S. Suppose that $\mathcal{Y} = \{I_i : i \in J\}$. Define a relation $\leq$ on J by $i \leq j$ if and only if $I_j \subseteq I_i$ for any $i, j \in J$. For any $i, j \in J$, the intersection $I_i \cap I_j$ is an element of $\mathcal{Y}$ (say I_k), and $I_k \subseteq I_i$ and $I_k \subseteq I_j$, meaning $i, j \leq k$. It follows that $(J, \leq)$ is an upward directed set and $\mathcal{Y}$ is a system of ideals of X.

A topological space $(X, \mathcal{U})$ is called zero-dimensional if $\mathcal{U}$ has a base consisting of clopen sets.

Proposition 3.10. *[8] Let $\mathcal{I} = \{I_i : i \in \Lambda\}$ be a system of ideals of a BCK-algebra X. Then $(X, \tau_{\mathcal{I}})$ is a zero-dimensional space.*

Proposition 3.11. *[7] Let $\mathcal{I} = \{I_i : i \in \Lambda\}$ be a system of ideals of a BCK-algebra X. If A is a subset of the topological space $(X, \tau_{\mathcal{I}})$, and $A/I_i = \bigcup_{x \in A} x/I_i$, then the closure of A is given by $\overline{A} = \bigcap_{i \in \Lambda} A/I_i$.*

Theorem 3.12. *Let $S, \mathcal{Y}$ and υ be as given in Remark 3.9. Then $\tau_{\mathcal{Y}} \subseteq \upsilon$. Moreover, if $\cap_{i \in \Lambda} I_i \in \mathcal{Y}$ and (X, υ) is zero-dimensional space, then $\upsilon \subseteq \tau_{\mathcal{Y}}$.*

Proof. Let $x \in X$. Define the maps $r_x, l_x : (X, \upsilon) \to (X, \upsilon)$ by $r_x(a) = x * a$ and $l_x(a) = a * x$, respectively, for any $a \in X$. These maps are continuous. Since $I_i \in \upsilon$ for any i (as it is a finite intersection of base elements), the preimages $r_x^{-1}(I_i)$ and $l_x^{-1}(I_i)$ are open in (X, υ). Since $x/I_i = \{y \in X : x*y \in I_i, y*x \in I_i\} = r_x^{-1}(I_i) \cap l_x^{-1}(I_i)$, we obtain $x/I_i \in \upsilon$. Since sets of the form x/I_i form a base for $\tau_{\mathcal{Y}}$, it follows that $\tau_{\mathcal{Y}} \subseteq \upsilon$. Now, assume (X, υ) is zero-dimensional and $I = \bigcap_{i\in\Lambda} I_i \in \mathcal{Y}$. Let η be a clopen base for υ. Let $x \in U \in \eta$. Since U is closed, by Proposition 3.11, we have $\cap_{i\in\Lambda} x/I_i = \cap_{i\in\Lambda}\{x\}/I_i = \overline{\{x\}} \subseteq \overline{U} = U$. By assumption, $I \in \mathcal{Y}$, and $\bigcap_{i\in\Lambda} x/I_i = x/\left(\bigcap_{i\in\Lambda} I_i\right) = x/I$. Thus, $x/I \subseteq U$. Since x/I is a $\tau_{\mathcal{Y}}$-neighborhood of x, this shows that U is also a $\tau_{\mathcal{Y}}$-neighborhood of x. Because η is a base for υ, we conclude $\upsilon \subseteq \tau_{\mathcal{Y}}$. □

Proposition 3.13. *Let $S, \mathcal{Y}$ and υ be as given in Remark 3.9. Then*
(i) S is a fundamental system of neighbourhoods of 0 for υ.
(ii) If $\mathcal{Y}$ is finite and $I = \cap\mathcal{Y}$, then $I \in S$.

Proof. *(i)* Let η be the given base for υ and let U be an open neighbourhood of 0 in υ. Then there exists $B \in \eta$ such that $0 \in B \subseteq U$. Since υ is a linear topology and $0 \in B \in \eta$, we have $B \in S$ (by definition of S). Hence, S is a fundamental system of neighbourhoods of 0.
(ii) Let $\mathcal{Y}$ be finite and $I = \bigcap\mathcal{Y}$. Then I is a finite intersection of sets containing 0 and open sets, so $0 \in I \in \upsilon$. By part (i), there exists $J \in S$ such that $J \subseteq I$. Since $J \in S \subseteq \mathcal{Y}$, we have $I \subseteq J$. Hence, $I = J \in S$. □

Proposition 3.14. *If I is an ideal of a BCK-algebra X, then $x * I = \{x * y : y \in I\} \subseteq x/I$ for any $x \in X$.*

Proof. Let $y \in I$. We need to show $x * y \in x/I$, i.e., that $(x * y) * x \in I$ and $x * (x * y) \in I$. By axiom (8), $(x*y)*x = (x*x)*y = 0*y = 0 \in I$. By axiom (1) and (5), and since I is an ideal, we have $x * (x * y) \leq y$. Since $y \in I$ and I is an ideal, Proposition 2.2 implies $x * (x * y) \in I$. Therefore, $x * y \in x/I$. □

Proposition 3.15. *[7] Let I be an ideal of a commutative BCK-algebra X and $x \in X$. If the congruence class x/I has a maximum element, then*

$$y = \max(x/I) \;\Leftrightarrow\; y * I = x/I.$$

Theorem 3.16. *In the context of Remark 3.9, if X is a finite commutative BCK-algebra, then $\tau_{\mathcal{Y}}$ is finer than υ if and only if $max(x/I) \in U$ for any $x \in U \in \eta$, where $I = \cap\mathcal{Y}$ and η is a base for υ.*

Proof. Let η and $\beta = \{x/I_i : x \in X, i \in J\}$ be bases for topologies υ and $\tau_{\mathcal{Y}}$ respectively.

Assume $\tau_{\mathcal{Y}}$ is finer than υ. Let $x \in U \in \eta$. Then $U \in \tau_{\mathcal{Y}}$. Since β is a base for $\tau_{\mathcal{Y}}$, there exists some basic open set $y/I_i \in \beta$ such that $x \in y/I_i \subseteq U$. This implies $x/I_i = y/I_i$. Since $I \subseteq I_i$, we have $x/I \subseteq x/I_i = y/I_i \subseteq U$. In particular, since $\max(x/I) \in x/I$, we get $\max(x/I) \in U$. Conversely, let $\max(x/I) \in U$ for any $x \in U \in \eta$. Let $x \in X$ and $U \in \eta$ such that $x \in U$. We need to find a $\tau_{\mathcal{Y}}$-open set V such that $x \in V \subseteq U$. If $U \in S$ (the base of ideals around 0), then $U \in \mathcal{Y}$ and $U = 0/U$ is itself a basic open set in $\tau_{\mathcal{Y}}$, so we can take $V = U$. If $U \notin S$, let $y = \max(x/I)$. By assumption, $y \in U$. Since S is a fundamental system of neighbourhoods of 0 in υ and the operation $*$ is continuous, there exists $I_y \in S$ such that $y * I_y \subseteq U$. Since $I = \bigcap\mathcal{Y} \subseteq I_y$, we have $y * I \subseteq y * I_y \subseteq U$. Similarly, by continuity, there exists $I_x \in S$ such that $x * I_x \subseteq U$, and thus $x * I \subseteq U$. By Propositions 3.14 and 3.15, we have $x/I = y/I = y * I$. Therefore, $x \in x/I = y * I \subseteq U$. So we can take $V = x/I \in \tau_{\mathcal{Y}}$. Hence, $\tau_{\mathcal{Y}}$ is finer than υ. □

Recall that a family Σ of non-empty subsets of a set X is called a prefilter on X, if $X \in \Sigma$ and for elements $A_1, A_2, ..., A_n$ of Σ, there exists $B \in \Sigma$ such that $B \subseteq \cap_{i=1}^n A_i$.

Definition 3.17. Let W be a non-empty subset of a BCK-algebra X. For any $x \in X$ define

$$W(x) = \{y \in X : r_x(y) \in W \text{ and } l_x(y) \in W\}$$

where $r_x(y) = x * y$ and $l_x(y) = y * x$ for any $y \in X$.

Proposition 3.18. *Let W be a non-empty subset of a BCK-algebra X. Then the following properties hold:*
(i) *if* $0 \in W$, *then* $x \in W(x)$,
(ii) $y \in W(x)$ *if and only if* $x \in W(y)$,
(iii) *if* $0 \in W$, *then* $W(0) = W$,
(iv) *if* I *is an ideal of* X, *then* $I(x) = x/I$,
(v) *if* $V \subseteq W \subseteq X$, *then* $V(x) \subseteq W(x)$,
for any $x, y \in X$.

Proof. The proofs follow directly from Definition 3.17 and the axioms of a BCK-algebra. □

Theorem 3.19. *Let* Σ *be a prefilter on a BCK-algebra* X *such that for any* $a, b \in W \in \Sigma$:
(i) $0 \in \cap\Sigma$,
(ii) *if* $(r_a \circ r_b)(x) = 0$, *then* $x \in W$.
Then there exists a topology τ *on* X *such that* (X, τ) *is a topological BCK-algebra.*

Proof. Let $\tau = \{U \subseteq X : \forall x \in U \ \exists W \in \Sigma \ \ s.t.\ W(x) \subseteq U\}$. We claim that
(a) τ is a topology on X,
(b) Σ is a fundamental system of neighbourhoods of 0,
(c) If $W \in \Sigma$, then W is an ideal,
(d) the prefilter Σ is a system of ideals,
(e) The set $\xi = \{W(x) : x \in X, W \in \Sigma\}$ is a base for τ,
(f) (X, τ) is a topological BCK-algebra.
(a) Clearly $\emptyset, X \in \tau$. Let $U, V \in \tau$ and $x \in U \cap V$. Then there exist $W_1, W_2 \in \Sigma$ such that $W_1(x) \subseteq U$ and $W_2(x) \subseteq V$. Since $W_1, W_2 \in \Sigma$ and Σ is a prefilter, there is $W \in \Sigma$ such that $W \subseteq W_1 \cap W_2$. Hence $W(x) \subseteq (W_1 \cap W_2)(x) \subseteq W_1(x) \cap W_2(x) \subseteq U \cap V$. Therefore $U \cap V \in \tau$. Let $\{U_\alpha\}_{\alpha \in K}$ be a family of elements of τ and $x \in \cup_{\alpha \in K} U_\alpha$. Then $x \in U_\alpha$ for some $\alpha \in K$. There exists $W \in \Sigma$ such that $W(x) \subseteq U_\alpha \subseteq \cup_{\alpha \in K} U_\alpha$. Hence $\cup_{\alpha \in K} U_\alpha \in \tau$.
(b) Let $W \in \Sigma$. First, we show that $W \in \tau$. Let $x \in W$. We prove that $W(x) \subseteq W$. Let $y \in W(x)$. Then $x * y \in W$. By (3) we have,

$$(r_{x*y} \circ r_x)(y) = r_{x*y}(x * y) = (x * y) * (x * y) = 0.$$

By condition (ii), since $x * y \in W$ and $x \in W$, this implies $y \in W$. So $W(x) \subseteq W$, thus $W \in \tau$ and is an open neighbourhood of 0 (since $0 \in W$ by (i)). Now, let U be any open neighbourhood of 0. Then there exists some $V \in \Sigma$ such that $V(0) \subseteq U$. By (i) and Proposition 3.18 part (iii), $V(0) = V$, so $V \subseteq U$. Hence, Σ is a fundamental system of neighbourhoods of 0.
(c) Let $W \in \Sigma$. By (i), $0 \in W$. Let $x * y \in W$ and $y \in W$. We need to show $x \in W$. Consider:

$$(r_{x*y} \circ r_y)(x) = (x * y) * (y * x).$$

Thus, $(r_{x*y} \circ r_y)(x) = 0$. Since $x * y \in W$ and $y \in W$, condition (ii) implies that $x \in W$. Therefore, W is an ideal.
(d) Since Σ is a prefilter consisting of ideals (by (c)) and for any $W_1, W_2 \in \Sigma$, there exists $W_3 \in \Sigma$ with $W_3 \subseteq W_1 \cap W_2$, the family $(\Sigma, \subseteq)$ is downward directed. By Lemma 3.7, Σ is a system of ideals.
(e) Let $W(x) \in \xi$. First, we show $W(x) \in \tau$. Let $y \in W(x)$. We prove $W(y) \subseteq W(x)$. Let $z \in W(y)$. Then $y * z, z * y \in W$. Since $y \in W(x)$, $x * y, y * x \in W$. Using BCK-algebra axioms (1) and (8), one can show $x * z, z * x \in W$ (since W is an ideal). Thus $z \in W(x)$, so $W(y) \subseteq W(x)$. Hence $W(x) \in \tau$. Now, let $x \in U \in \tau$. Then there exists $W \in \Sigma$ such that $W(x) \subseteq U$, and $x \in W(x)$ by (i) and Proposition 3.18 part (i). So ξ is a base for τ.
(f) Let $x, y \in X$ and let U be an open neighbourhood of $x * y$. Then there exists $W \in \Sigma$ such that $W(x * y) \subseteq U$. Since W is an ideal, by Proposition 3.18 part (iv), $W(x * y) = (x * y)/W$. In the quotient algebra, we have $x/W * y/W = (x * y)/W$. The sets $x/W = W(x)$ and $y/W = W(y)$ are open neighbourhoods of x and y, respectively (by (e)). Their product under $*$ is $W(x) * W(y) = (x * y)/W = W(x * y) \subseteq U$. Therefore, $*$ is continuous.

□

Example 3.20. Let $X = \{0, a, b, c\}$. Define a binary operation $*$ on X as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	c	0

Then $(X, *)$ is a BCK-algebra. Let $\Sigma = \{W_1, W_2\}$, where $W_1 = \{0, a\}$ and $W_2 = \{0, a, b, c\}$. One can easily show that Σ is a prefilter and satisfies the conditions of Theorem 3.19. Then

$$\tau = \{U \subseteq X : \forall x \in U,\ \exists W \in \Sigma\ s.t.\ W(x) \subseteq U\}$$

is a topology on X. Routine calculation shows that $W_1(0) = \{0, a\} = W_1(a), W_1(b) = \{b\}, W_1(c) = \{c\}$ and $W_2(x) = \{0, a, b, c\}$ for any $x \in X$. Hence $\{\{0, a\}, \{b\}, \{c\}, \{0, a, b, c\}\}$ is a base of τ. Therefore, (X, τ) is a topological BCK-algebra.

Proposition 3.21. *[7] Let I be an ideal of a BCK-algebra X and $x, y \in X$. If $x/I \leq y/I$, then for each $b \in y/I$ there exists $a \in x/I$ such that $a \leq b$.*

Let X be a BCK-algebra. for any $x \in X$ we define $T(x) = \{y \in X : y \leq x\}$. Then
(i) $0 \in T(x)$, for any $x \in X$,
(ii) $y \in T(x)$ implies $T(y) \subseteq T(x)$,
(iii) $T(0) = \{0\}$.

Theorem 3.22. *Let X be a BCK-algebra. Then there exists a topology ξ on X such that (X, ξ) satisfies the open condition.*

Proof. Let $\xi = \{U \subseteq X : \forall x \in U :\ T(x) \subseteq U\}$. It is straightforward to verify that ξ is a topology and $\gamma = \{T(x) : x \in X\}$ is a base for it. We show that for any ideal I, the canonical map $\pi_I : (X, \xi) \to (X/I, \tilde{\tau})$ is open, where $\tilde{\tau}$ is the quotient topology. First, we show that for any $x \in X$, the image $\pi_I(T(x))$ is open in X/I. Consider $T(\pi_I(x)) = \{y/I \in X/I : y/I \leq x/I\}$. We claim $\pi_I^{-1}(T(\pi_I(x))) \in \xi$. Let $y \in \pi_I^{-1}(T(\pi_I(x)))$, so $y/I \leq x/I$. Let $z \in T(y)$, meaning $z \leq y$. Then $z/I \leq y/I \leq x/I$, so $z/I \in T(\pi_I(x))$, i.e., $z \in \pi_I^{-1}(T(\pi_I(x)))$. This shows $T(y) \subseteq \pi_I^{-1}(T(\pi_I(x)))$ for any y in this preimage. Therefore, $\pi_I^{-1}(T(\pi_I(x))) \in \xi$, so $T(\pi_I(x))$ is open in X/I. Now, let U be an arbitrary open set in (X, ξ). To show $\pi_I(U)$ is open, let $x/I \in \pi_I(U)$. Then there exists $a \in U$ with $a/I = x/I$. Since U is open and $a \in U$, we have $T(a) \subseteq U$. We will show $\pi_I(T(a)) = T(\pi_I(a)) = T(x/I)$. Let $y/I \in \pi_I(T(a))$. Then $y/I = \pi_I(z)$ for some $z \leq a$, so $z/I \leq a/I = x/I$, hence $y/I = z/I \in T(x/I)$. Conversely, let $y/I \in T(x/I)$, so $y/I \leq x/I = a/I$. By Proposition 3.21, there exists $z \in y/I$ such that $z \leq a$, i.e., $z \in T(a)$. Then $\pi_I(z) = y/I$, so $y/I \in \pi_I(T(a))$. Therefore, $\pi_I(T(a)) = T(x/I)$. Since $T(a) \subseteq U$, we have $\pi_I(T(a)) \subseteq \pi_I(U)$. Thus, $x/I \in T(x/I) = \pi_I(T(a)) \subseteq \pi_I(U)$. This shows that $\pi_I(U)$ contains a neighbourhood $T(x/I)$ of each of its points, so it is open. Hence, π_I is an open map. □

Example 3.23. In Example 3.20, one can easily verify that $T(0) = \{0\}, T(c) = X, T(a) = \{0, a\}$ and $T(b) = \{0, a, b\}$. Using the previous theorem we obtain $\xi = \{\emptyset, \{0\}, \{0, a\}, \{0, a, b\}, X\}$. By considering $I = \{0, a\}$ we have $X/I = \{0/I = a/I = I, b/I, c/I\} = \{\{0, a\}, \{b\}, \{c\}\}$ and the canonical epimorphism π_I is an open map.

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