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Distributional and renormalized solutions for Neumann problems with boundary conditions

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Abstract In this paper, we investigate a class of Neumann quasilinear elliptic equations, whose prototype is given by

$$\begin{cases} -\sum_{i=1}^{N} D^i\left(\frac{b_0|D^iu|^{p_i-2}D^iu\,\omega_i}{(1+|u|)^{\lambda}}\right)+|u|^{p_0-2}u\,\omega_0=f(x,u) & \text{in } \Omega,\\ \sum_{i=1}^{N}\left(\frac{b_0|D^iu|^{p_i-2}D^iu\,\omega_i}{(1+|u|)^{\lambda}}\right).n_i=0 & \text{on } \partial\Omega, \end{cases}$$

where the right-hand side $f(x,s)$ verify only some growth condition in $L^1(\Omega)$. We establish the existence of solution in the sense of distributional in the case of $f(x,u)$ in $L^\infty(\Omega)$. Moreover, we conclude the existence of renormalized solutions when $f(x,u)$ belongs to $L^1(\Omega)$, in the framework of anisotropic weighted Sobolev spaces $W^{1,\vec{p}}(\Omega,\vec{\omega})$.

1 Introduction

We consider a bounded open domain $\Omega\subset\mathbb{R}^N$ with $N\geq 2$ and a Lipschitz boundary $\partial\Omega$. The study of quasilinear elliptic equations with nonstandard data has been the subject of extensive research over the past decades. Alvino et al. in [4] have studied a degenerate quasilinear elliptic problem of the form

$$\begin{cases} -\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right)=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f\in L^m(\Omega)$ for $m\geq 1$, and $0<\theta<1$. They have proved the existence of solutions and derived some regularity results. For more results concerning degenerate elliptic equations, we refer to [3, 22, 28, 31], while further related studies are provided in [12, 15, 19, 37].

The Neumann case has also been studied extensively. Betta, Guibé, and Mercaldo in [11] have considered the nonlinear elliptic problem

$$\begin{cases} -\Delta_p u-\mathrm{div}\big(c(x)|u|^{p-2}u\big)=f & \text{in } \Omega,\\ \big(|\nabla u|^{p-2}\nabla u+c(x)|u|^{p-2}u\big)\cdot\vec{n}=0 & \text{on } \partial\Omega, \end{cases}$$

where $f\in L^1(\Omega)$ and $c(x)$ is a measurable function belonging to a suitable Lebesgue space. They have proved the existence and regularity of renormalized solutions. For more details and results, see [5, 14, 29].

Betta et al. [10] established existence results for solutions to nonlinear elliptic Neumann problems of the form

$$\begin{cases} \lambda|u|^{p-2}u - \Delta_p u - \operatorname{div}\left(c(x)|u|^{p-2}u\right) + b(x)|\nabla u|^{p-2}\nabla u = f & \text{in } \Omega, \\ \left(|\nabla u|^{p-2}\nabla u + c(x)|u|^{p-2}u\right)\cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda > 0$, and the datum f belongs either to the dual space of the classical Sobolev space $W^{1,p}(\Omega)$ or to $L^1(\Omega)$. The coefficients b and c are assumed to satisfy suitable integrability conditions in appropriate Lebesgue or Lorentz spaces.

Recently, Bouzelmate et al. [17] investigated the existence of entropy solutions for quasilinear elliptic equations of the form

$$\begin{cases} Au + g(x,u,\nabla u) + |u|^{p_0-2}u\,\omega_0 = f(x) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,u,\nabla u).n_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $f \in L^1(\Omega)$, within the framework of anisotropic weighted Sobolev spaces. We also refer the reader to [8, 13, 20, 26, 30, 27, 35]. For further details on anisotropic weighted Sobolev spaces, we refer the reader to [7, 16, 18].

In this paper, we focus on the study of the quasilinear anisotropic elliptic equation with a Neumann boundary conditions

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + |u|^{p_0-2}u\,\omega_0 = f(x,u) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,u,\nabla u)\,n_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\vec{n} = (n_1, n_2, \ldots, n_N)$ denotes the outward unit normal on $\partial\Omega$. The term on the right-hand side $f(x,s)$ is assumed to satisfy suitable growth conditions in $L^1(\Omega)$.

Our main goal is to establish the existence of renormalized solutions for the degenerated elliptic problem (1.4) in the framework of anisotropic weighted Sobolev spaces $W^{1,\vec{p}}(\Omega,\vec{\omega})$.

This paper is organized as follows : In section 2, we recall basic definitions and lemmas related to anisotropic weighted Sobolev spaces. The section 3 introduces the fundamental assumptions required for the existence theory. In section 4, we address the case where $f(x,s) \in L^\infty(\Omega)$, establishing the existence of solutions in the sense of distributional. Finally, Section 5 is devoted to proving the existence of renormalized solutions for the degenerated elliptic problem (1.4) in the case case $f(x,s) \in L^1(\Omega)$.

2 Notation and preliminary results.

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, with Lipschitz boundary $\partial\Omega$.
Let $p_1, \ldots, p_N$ be N exponents that satisfying $1 < p_i < \infty$ for $i = 1, \ldots, N$. We denote

$$\vec{p} = (p_1, \ldots, p_N) \qquad \text{and} \qquad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \ldots, N,$$

and set

$$\underline{p} = \min\{p_1, p_2, \ldots, p_N\} \qquad \text{and} \qquad p_m = \max\{p_1, p_2, \ldots, p_N\}.$$

Let us define $\vec{\omega}(x) = (\omega_0(x), \omega_1(x), \ldots, \omega_N(x))$ as a vector of weight functions, such that $\omega_0(x)$ and $\omega_i(x)$ are positive and measurable functions $a.e.$ in Ω, with $\omega_0 \leq \omega_i$ for $i = 1, \ldots, N$. Also, we assume that

$$\omega_0 \in L^1(\Omega) \qquad \text{with} \qquad \omega_0^* = \omega_0^{-\frac{1}{\underline{p}-1}} \in L^1(\Omega), \tag{2.1}$$

and

$$\omega_i \in L^1(\Omega) \qquad \text{with} \qquad \omega_i^* = \omega_i^{-\frac{1}{p_i-1}} \in L^1(\Omega) \qquad \text{for} \quad i = 1, \ldots, N. \tag{2.2}$$

The weighted Lebesgue space $L^{p_i}(\Omega,\omega_i)$ is defined as

$$L^{p_i}(\Omega,\omega_i)=\left\{u \text{ measurable function} \quad \text{such that} \quad \int_\Omega |u(x)|^{p_i}\omega_i(x)\,dx<\infty\right\},$$

equipped with the norm

$$\|u\|_{p_i,\omega_i}=\left(\int_\Omega |u(x)|^{p_i}\omega_i(x)dx\right)^{\frac{1}{p_i}} \qquad \text{for} \quad i=1,\dots,N. \tag{2.3}$$

The space $(L^{p_i}(\Omega,\omega_i),\|u\|_{p_i,\omega_i})$ is a separable and reflexive Banach space (cf [27]). Moreover, we define the Hölder's type inequality by

$$\int_\Omega uv\,dx\le \|u\|_{p_i,\omega_i}\|v\|_{p_i',\omega_i^*}, \tag{2.4}$$

and the Young's inequality by

$$\int_\Omega uv\,dx\le \frac{1}{p_i}\int_\Omega |u|^{p_i}\omega_i\,dx+\frac{1}{p_i'}\int_\Omega |v|^{p_i'}\omega_i^*\,dx \tag{2.5}$$

for any $u\in L^{p_i}(\Omega,\omega_i)$ and $v\in L^{p_i'}(\Omega,\omega_i^*)$, with $\dfrac{1}{p_i}+\dfrac{1}{p_i'}=1$.
The anisotropic weighted Sobolev space $W^{1,\vec{p}}(\Omega,\vec{\omega})$ is defined by

$$W^{1,\vec{p}}(\Omega,\vec{\omega})=\left\{u\in L^{\underline{p}}(\Omega,\omega_0) \text{ and } D^iu\in L^{p_i}(\Omega,\omega_i) \quad \text{for} \quad i=1,\dots,N\right\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,\vec{p},\vec{\omega}}=\|u\|_{\underline{p},\omega_0}+\sum_{i=1}^N \|D^iu\|_{p_i,\omega_i}. \tag{2.6}$$

Thus, under the assumptions (2.1) and (2.2), we have $C_0^\infty(\Omega)$ is a subspace of $W^{1,\vec{p}}(\Omega,\vec{\omega})$.

Lemma 2.1. *(see. [7]) Assuming that $\omega_0\le\omega_i$, and (2.1) − (2.2) hold true. Thus, the following continuous and compact embedding are concluded*

(i) *The embedding $L^{p_i}(\Omega,\omega_i)\hookrightarrow L^{\underline{p}}(\Omega,\omega_0)$ is continuous.*

(ii) *The embedding $W^{1,\vec{p}}(\Omega,\vec{\omega})\hookrightarrow W^{1,\underline{p}}(\Omega,\omega_0)$ is continuous.*

(iii) *The embedding $L^{p_i}(\Omega,\omega_i)\hookrightarrow L^1(\Omega)$ is continuous for $i=1,\dots,N$.*

(iv) *The embedding $W^{1,\vec{p}}(\Omega,\vec{\omega})\hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega,\omega_0)$ is compact.*

(v) *The embedding $W^{1,\vec{p}}(\Omega,\vec{\omega})\hookrightarrow W^{1,1}(\Omega)$ is continuous.*

(vi) *The embedding $W^{1,\vec{p}}(\Omega,\vec{\omega})\hookrightarrow\hookrightarrow L^q(\Omega)$ is compact for any $1\le q<\dfrac{N}{N-1}$.*

Definition 2.2. Let $k>0$, we consider the truncation function $T_k(\cdot):\mathbb{R}\longmapsto\mathbb{R}$ defined by

$$T_k(s)=\begin{cases} s & \text{if} \quad |s|\le k,\\ k\dfrac{s}{|s|} & \text{if} \quad |s|>k.\end{cases}$$

We define

$$\mathcal{T}^{1,\vec{p}}(\Omega,\vec{\omega}):=\left\{u:\Omega\longmapsto\mathbb{R} \text{ is measurable and } T_k(u)\in W^{1,\vec{p}}(\Omega,\vec{\omega}) \quad \text{for any} \quad k>0\right\}.$$

Proposition 2.3. *Let $u\in\mathcal{T}^{1,\vec{p}}(\Omega,\vec{\omega})$. For any $i\in\{1,\dots,N\}$, there exists a unique measurable function $v_i:\Omega\longmapsto\mathbb{R}$ satisfying*

$$\forall k>0 \qquad D^iT_k(u)=v_i\cdot\chi_{\{|u|<k\}} \quad \text{a.e. in } \Omega,$$

where χ_E represents the characteristic function of a measurable set E. The functions v_i are called as the weak partial derivatives of u and are still denoted by D^iu. Moreover, if $u\in W^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is $v_i=D^iu$.

The proof of Proposition 2.3 is based on the usual techniques developed in [9]. For more details on the anisotropic Sobolev spaces, we refer the reader to [6, 8, 23, 24].

As in [5], for the case of constant exponent, we define the set $\mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega})$ as a subset of $\mathcal{T}^{1,\vec{p}}(\Omega,\vec{\omega})$ that satisfies the following conditions: there exists a sequence $(u_n)_n$ in $W^{1,\vec{p}}(\Omega,\vec{\omega})$ such that

(a) $u_n \to u$ a.e. in Ω,

(b) $D^i T_k(u_n) \to D^i T_k(u)$ strongly in $L^1(\Omega)$ for all $k>0$, and for each $i=1,2,\dots,N$,

(c) there exists a measurable function ν on $\partial\Omega$, such that $u_n \to \nu$ a.e. in $\partial\Omega$.

The function ν represents the trace of the function u in the generalized sense as in [5]. The proof is based on the continuous embedding $W^{1,\vec{p}}(\Omega,\vec{\omega}) \hookrightarrow W^{1,1}(\Omega)$ and follows the same argument as in [5] for classical Sobolev spaces. For more information, we refer the reader to [25] and [33].

Let $u \in W^{1,\vec{p}}(\Omega,\vec{\omega})$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$. Also, for any $u \in \mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega})$, we denote either $tr(u)$ or simply u as the trace of u on $\partial\Omega$. The operator $tr(\cdot)$ satisfying the following properties

(i) if $u \in \mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega})$, then $\tau(T_k(u)) = T_k(tr(u))$ for any $k>0$,

(ii) if $\varphi \in W^{1,\vec{p}}(\Omega,\vec{\omega})$, then for any $u \in \mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega})$, we have $u-\varphi \in \mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega})$ and $tr(u-\varphi) = tr(u) - tr(\varphi)$.

Indeed, if $u \in W^{1,\vec{p}}(\Omega,\vec{\omega})$, we have $tr(u)$ coincides with $\tau(u)$, and it's clear that

$$W^{1,\vec{p}}(\Omega,\vec{\omega}) \subset \mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega}) \subset \mathcal{T}^{1,\vec{p}}(\Omega,\vec{\omega}).$$

Lemma 2.4. *(see [1])* *Let $g \in L^{p_i}(\Omega,\omega_i)$ and $(g_n)_n$ be a sequence in $L^{p_i}(\Omega,\omega_i)$ with $\|g_n\|_{p_i,\omega_i} \le C$ for $1<p_i<\infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^{p_i}(\Omega,\omega_i)$.*

Lemma 2.5. *(see [34], Theorem 13.47) Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that*

(i) *$u_n \to u$ a.e. in Ω,*

(ii) *$u_n \ge 0$ and $u \ge 0$ a.e. in Ω,*

(iii) $\displaystyle\int_\Omega u_n\,dx \to \int_\Omega u\,dx,$

then $u_n \to u$ strongly in $L^1(\Omega)$.

3 Essential assumptions

Let $p_0 > \underline{p}$. We consider the Leray-Lions operator $A : W^{1,\vec{p}}(\Omega,\vec{\omega}) \longmapsto \left(W^{1,\vec{p}}(\Omega,\vec{\omega})\right)'$, defined by

$$Au = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u), \tag{3.1}$$

where each $a_i(x,s,\xi) : \Omega\times\mathbb{R}\times\mathbb{R}^N \longmapsto \mathbb{R}^N$ are Carathéodory functions that satisfying conditions

$$a_i(x,s,\xi)\,\xi_i \ge b(|s|)\,|\xi_i|^{p_i}\,\omega_i \quad \text{with} \quad b(|s|) \ge \frac{b_0}{(1+|s|)^\lambda}, \tag{3.2}$$

for almost every $x\in\Omega$ and all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $b:\mathbb{R}^+ \longmapsto \mathbb{R}^+$ is a decreasing function such that $b \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, with $0 \le \lambda < \underline{p}^+ - 1$ and $b_0 > 0$.

$$|a_i(x,s,\xi)| \le \beta\left(K_i(x) + |s|^{\frac{p}{p_i'}}\omega_0^{\frac{1}{p_i'}}\omega_i^{\frac{1}{p_i}} + |\xi_i|^{p_i-1}\omega_i\right) \qquad i=1,\dots,N, \tag{3.3}$$

where $\beta > 0$ and $K_i \in L^{p_i'}(\Omega, \omega_i^*)$ is a nonnegative function,

$$\big(a_i(x,s,\xi) - a_i(x,s,\xi')\big)(\xi_i - \xi_i') > 0 \quad \text{for} \quad \xi_i \neq \xi_i', \tag{3.4}$$

In view of (3.2) and the continuity of the function $a_i(x,s,\cdot)$ with respect to ξ, we deduce that

$$a_i(x,s,0) = 0.$$

The right-hand side $f(x,s) : \Omega \times \mathbb{R} \longmapsto \mathbb{R}$ is a Carathéodory function that satisfying the growth condition

$$|f(x,s)| \leq f_0(x) + c(x)\,|s|^{\gamma}\omega_0, \tag{3.5}$$

where the nonnegative function $f_0(x)$ belongs to $L^1(\Omega)$, and $c(x) \in L^r(\Omega, \omega_0)$ with $r > \dfrac{p_0 - 1}{p_0 - 1 - \gamma}$, and the exponent γ satisfies $0 \leq \gamma < p_0 - 1$.

Now, we state a key Lemma that play a fundamental role in proving our main existence result.

Lemma 3.1. *(see [7]) Assume that $k > 0$, and the conditions (2.1), (2.2), and (3.2) – (3.4) are satisfied. Let $(u_n)_n$ be a bounded sequence in $W^{1,\vec{p}}(\Omega,\vec{\omega})$ such that $u_n \rightharpoonup u$ weakly in $W^{1,\vec{p}}(\Omega,\vec{\omega})$, and*

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega} \left(a_i(x, T_k(u_n), \nabla u_n) - a_i(x, T_k(u_n), \nabla u)\right)\left(D^i u_n - D^i u\right) dx \\ &\quad + \int_{\Omega} \left(|u_n|^{\underline{p}-2}u_n - |u|^{\underline{p}-2}u\right)(u_n - u)\omega_0\, dx \longrightarrow 0 \quad \textit{as} \quad n \to \infty. \end{aligned} \tag{3.6}$$

Then, we have $u_n \longrightarrow u$ strongly in $W^{1,\vec{p}}(\Omega,\vec{\omega})$ for a subsequence.

4 Existence of solutions in the sense of distribution for $F(x,s) \in L^{\infty}(\Omega)$

We study the quasilinear Neumann elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x, T_n(u), \nabla u) + |u|^{p_0-2}u\,\omega_0 = F(x,u) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x, T_n(u), \nabla u).n_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

in the anisotropic weighted Sobolec space, such that

$$|F(x,s)| \leq C_0, \qquad \text{for a. e. } x \in \Omega \text{ and any } s \in \mathbb{R}. \tag{4.2}$$

Definition 4.1. A measurable function u is a solution in the sense of distributions to the quasilinear anisotropic elliptic equation (4.1), if $u \in W^{1,\vec{p}}(\Omega,\vec{\omega})$ and $|u|^{p_0}\omega_0 \in L^1(\Omega)$, such that u satisfies the following equation

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u), \nabla u) D^i v\, dx + \int_{\Omega} |u|^{p_0-2}uv\omega_0\, dx = \int_{\Omega} F(x,u)v\, dx, \tag{4.3}$$

for any $v \in W^{1,\vec{p}}(\Omega,\vec{\omega}) \cap L^{\infty}(\Omega)$.

Theorem 4.2. *Under the conditions (2.1), (2.2), and (3.2) – (3.4) and (4.2), there exists at least one solution in the sense of distributions for the quasilinear elliptic problem (4.1).*

Proof of theorem 4.2

Step 1: Approximate problems.

In order to prove the existence of solutions in the sense of distribution to the problem (4.1), we introduce the following quasilinear elliptic approximate problem

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, T_n(u_m), \nabla u_m) + |T_m(u_m)|^{p_0-\underline{p}}|u_m|^{\underline{p}-2}u_m\,\omega_0 = F(x,u) & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, T_n(u_m), \nabla u_m).n_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

for and $m \in \mathbb{N}^*$.
We define the operator B_m, acted from $W^{1,\vec{p}}(\Omega,\vec{\omega})$ into its dual $(W^{1,\vec{p}}(\Omega,\vec{\omega}))'$, by

$$\langle B_m u, v\rangle = \sum_{i=1}^{N}\int_\Omega a_i(x, T_n(u), \nabla u) D^i v\,dx + \int_\Omega |T_m(u)|^{p_0-\underline{p}}|u|^{\underline{p}-2}uv\,\omega_0\,dx - \int_\Omega F(x,u)v\,dx,$$

for any $u, v \in W^{1,\vec{p}}(\Omega,\vec{\omega})$.

Lemma 4.3. *The operator B_m is bounded and pseudo-monotone. Furthermore, the operator B_m is coercive in the following sense*

$$\frac{\langle B_m v, v\rangle}{\|v\|_{1,\vec{p},\vec{\omega}}} \to +\infty \quad \textit{if} \quad \|v\|_{1,\vec{p},\vec{\omega}} \to +\infty \quad \textit{for} \quad v \in W^{1,\vec{p}}(\Omega,\vec{\omega}).$$

Proof of Lemma 4.3

In view of (3.3) and the Hölder's type inequality, we have

$$\begin{aligned} |\langle B_m u, v\rangle| &\le \sum_{i=1}^{N}\int_\Omega |a_i(x, T_n(u), \nabla u)|\,|D^i v|\,dx + \int_\Omega |T_m(u)|^{p_0-\underline{p}}|u|^{\underline{p}-1}|v|\,\omega_0\,dx \\ &\quad + \int_\Omega |F(x,u)|\,|v|\,dx \\ &\le \beta\sum_{i=1}^{N}\int_\Omega \left(K_i(x) + |T_n(u)|^{\frac{p}{p_i'}}\omega_0^{\frac{1}{p_i'}}\omega_i^{\frac{1}{p_i}} + |D^i u|^{p_i-1}\omega_i\right)|D^i v|\,dx \\ &\quad + m^{p_0-\underline{p}}\int_\Omega |u|^{\underline{p}-1}|v|\,\omega_0\,dx + C_0\int_\Omega |v|\,dx \\ &\le C_1\sum_{i=1}^{N}(\|K_i(x)\|_{p_i',\omega_i^*} + n^{\frac{p}{p_i'}}\|1\|_{p_i,\omega_0} + \|D^i u\|_{p_i,\omega_i}^{p_i-1})\|D^i v\|_{p_i,\omega_i} \\ &\quad + m^{p_0-\underline{p}}\|u\|_{\underline{p},\omega_0}^{\underline{p}-1}\|v\|_{\underline{p},\omega_0} + C_2\|v\|_{1,\vec{p},\vec{\omega}} \\ &\le C_3\left(1 + \|u\|_{1,\vec{p},\vec{\omega}}^{p_m-1}\right)\|v\|_{1,\vec{p},\vec{\omega}} \qquad \text{for any} \quad u, v \in W^{1,\vec{p}}(\Omega,\vec{\omega}), \end{aligned} \tag{4.5}$$

with C_3 is a positive constant depending on n and m. Hence, the operator B_m is bounded in $W^{1,\vec{p}}(\Omega,\vec{\omega})$.

For the coercivity, we have

$$\begin{aligned}\langle B_m u, u\rangle &= \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D^i u\,dx + \int_\Omega |T_m(u)|^{p_0-\underline{p}}|u|^{\underline{p}}\omega_0\,dx \\ &\quad - \int_\Omega |F(x,u)||u|\,dx \\ &\geq \sum_{i=1}^{N}\int_\Omega b(|T_n(u)|)|D^i u|^{p_i}\omega_i\,dx + \int_\Omega |u|^{\underline{p}}\omega_0\,dx - \int_\Omega \omega_0\,dx - C_0\int_\Omega |u|\,dx \\ &\geq \frac{b_0}{(1+n)^\lambda}\sum_{i=1}^{N}\|D^i u\|^{p_i}_{p_i,\omega_i} + \|u\|^{\underline{p}}_{\underline{p},\omega_0} - C_4 - C_5\|u\|_{1,\vec{p},\vec{\omega}} \\ &\geq C_6\|u\|^{\underline{p}}_{1,\vec{p},\vec{\omega}} - C_5\|u\|_{1,\vec{p},\vec{\omega}},\end{aligned} \tag{4.6}$$

then

$$\frac{\langle B_m u, u\rangle}{\|u\|_{1,\vec{p},\vec{\omega}}} \geq C_6\|u\|^{\underline{p}-1}_{1,\vec{p},\vec{\omega}} - C_5 \longrightarrow +\infty \quad \text{as} \quad \|u\|_{1,\vec{p},\vec{\omega}} \to +\infty.$$

It remains to establish that the operator B_m is pseudo-monotone. To this end, let $(u_k)_k$ be a sequence in $W^{1,\vec{p}}(\Omega,\vec{\omega})$ satisfying the following conditions

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in} \quad W^{1,\vec{p}}(\Omega,\vec{\omega}), \\ B_m u_k \rightharpoonup \chi_m & \text{weakly in} \quad (W^{1,\vec{p}}(\Omega,\vec{\omega}))', \\ \limsup\limits_{k\to\infty}\langle B_m u_k, u_k\rangle \leq \langle \chi_m, u\rangle. \end{cases} \tag{4.7}$$

Our goal is to prove that

$$\chi_m = B_m u \quad \text{and} \quad \langle B_m u_k, u_k\rangle \longrightarrow \langle \chi_m, u\rangle \quad \text{as} \quad k \to +\infty.$$

Since the embedding $W^{1,\vec{p}}(\Omega,\vec{\omega}) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega,\omega_0)$ is compact, then $u_k \to u$ strongly in $L^{\underline{p}}(\Omega,\omega_0)$ for a subsequence still denoted $(u_k)_k$.
The sequence $(u_k)_k$ is bounded in $W^{1,\vec{p}}(\Omega,\vec{\omega})$, and thanks to (3.3), the Carathéodory functions $(a_i(x,T_n(u_k),\nabla u_k))_k$ are uniformly bounded in $L^{p_i'}(\Omega,\omega_i^*)$. Therefore there exists a measurable function $\varphi_i \in L^{p_i'}(\Omega,\omega_i^*)$ such that

$$a_i(x,T_n(u_k),\nabla u_k) \rightharpoonup \varphi_i \quad \text{weakly in } L^{p_i'}(\Omega,\omega_i^*) \quad \text{as } k \to +\infty. \tag{4.8}$$

We have $|F(x,u_k)| \leq C_0$ and $F(x,u_k) \to F(x,u)$ a.e. in Ω. Using the Lebesgue dominated convergence theorem, we deduce that

$$F(x,u_k) \longrightarrow F(x,u) \quad \text{strongly in } L^{\underline{p}'}(\Omega,\omega_0^*) \quad \text{as } k \to +\infty. \tag{4.9}$$

We have $|T_m(u_k)|^{p_0-\underline{p}} \leq m^{p_0-\underline{p}}$. Since the embedding $W^{1,\vec{p}}(\Omega,\vec{\omega}) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega,\omega_0)$ is compact, we have $u_k \to u$ strongly in $L^{\underline{p}}(\Omega,\omega_0)$ and almost everywhere in Ω. Consequently, we obtain

$$|T_m(u_k)|^{p_0-\underline{p}}|u_k|^{\underline{p}-2}u_k\omega_0 \to |T_m(u)|^{p_0-\underline{p}}|u|^{\underline{p}-2}u\omega_0 \quad \text{strongly in } L^{\underline{p}'}(\Omega,\omega_0^*). \tag{4.10}$$

Thus, for all $v \in W^{1,\vec{p}}(\Omega,\vec{\omega})$ we have

$$\begin{aligned}\langle \chi_m, v\rangle &= \lim_{k\to\infty}\langle B_m u_k, v\rangle \\ &= \lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v\,dx + \lim_{k\to\infty}\int_\Omega |T_m(u_k)|^{p_0-\underline{p}}\,|u_k|^{\underline{p}-2}u_k\,v\,\omega_0\,dx \\ &\quad - \lim_{k\to\infty}\int_\Omega F(x,u_k)v\,dx \\ &= \sum_{i=1}^{N}\int_\Omega \varphi_i D^i v\,dx + \int_\Omega |T_m(u)|^{p_0-\underline{p}}\,|u|^{\underline{p}-2}u\,v\,\omega_0\,dx - \int_\Omega F(x,u)v\,dx.\end{aligned} \tag{4.11}$$

From (4.7) and (4.11), we conclude that

$$\begin{aligned}\limsup_{k\to\infty}\langle B_m u_k, u_k\rangle &= \limsup_{k\to\infty}\left(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx\right.\\ &\quad\left.+\int_\Omega |T_m(u_k)|^{p_0-\underline{p}}\,|u_k|^{\underline{p}}\,\omega_0\,dx-\int_\Omega F(x,u_k)u_k\,dx\right)\\ &\le \sum_{i=1}^{N}\int_\Omega \varphi_i D^i u\,dx+\int_\Omega |T_m(u)|^{p_0-\underline{p}}\,|u|^{\underline{p}}\,\omega_0\,dx-\int_\Omega F(x,u)u\,dx.\end{aligned} \tag{4.12}$$

Thanks to (4.9), we have

$$\int_\Omega F(x,u_k)u_k\,dx\longrightarrow\int_\Omega F(x,u)u\,dx\quad\text{as } k\to+\infty, \tag{4.13}$$

and in view of (4.10), we derive

$$\int_\Omega |T_m(u_k)|^{p_0-\underline{p}}\,|u_k|^{\underline{p}}\,\omega_0\,dx\longrightarrow\int_\Omega |T_m(u)|^{p_0-\underline{p}}\,|u|^{\underline{p}}\,\omega_0\,dx. \tag{4.14}$$

By (4.12) and (4.13) – (4.14), we get

$$\limsup_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx\le\sum_{i=1}^{N}\int_\Omega\varphi_i D^i u\,dx. \tag{4.15}$$

Moreover, according to (3.4), we have

$$\sum_{i=1}^{N}\int_\Omega\left(a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u)\right)\left(D^i u_k-D^i u\right)\,dx\ge 0, \tag{4.16}$$

then

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx &\ge\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u\,dx\\ &\quad+\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u)\left(D^i u_k-D^i u\right)\,dx.\end{aligned}$$

In view of the dominated convergence theorem, we deduce that $T_n(u_k)\longrightarrow T_n(u)$ strongly in $L^{p_i}(\Omega,\omega_i)$, which implies that $a_i(x,T_n(u_k),\nabla u)\longrightarrow a_i(x,T_n(u),\nabla u)$ strongly in $L^{p_i'}(\Omega,\omega_i^*)$, and since $D^i u_k\rightharpoonup D^i u$ weakly in $L^{p_i}(\Omega,\omega_i)$, it follows that

$$\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u)\left(D^i u_k-D^i u\right)\,dx\longrightarrow 0\qquad\text{as}\quad k\to+\infty.$$

Using the weak convergence result in (4.8), we conclude that

$$\liminf_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx\ge\sum_{i=1}^{N}\int_\Omega\varphi_i D^i u\,dx.$$

Taking into account (4.15), it follows that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx=\sum_{i=1}^{N}\int_\Omega\varphi_i D^i u\,dx. \tag{4.17}$$

Therefore, by combining the results from (4.11), (4.13) – (4.14), and (4.17), we deduce that

$$\langle B_m u_k,u_k\rangle\longrightarrow\langle\chi_m,u\rangle\quad\text{as}\quad k\to+\infty. \tag{4.18}$$

On the other hand, using (4.17) we have

$$\lim_{k\to+\infty}\Big(\sum_{i=1}^{N}\int_\Omega \left(a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u)\right)\left(D^iu_k-D^iu\right)\,dx + \int_\Omega \left(|u_k|^{\underline{p}-2}u_k-|u|^{\underline{p}-2}u\right)(u_k-u)\,\omega_0\,dx\Big)=0.$$

In view of (4.7) and Lemma 3.1, we conclude that

$$u_k \longrightarrow u \quad \text{strongly in} \quad W^{1,\vec{p}}(\Omega,\vec{\omega}) \qquad \text{and} \qquad D^iu_k \longrightarrow D^iu \quad \text{a.e. in} \quad \Omega.$$

Consequently, $a_i(x,T_n(u_k),\nabla u_k) \to a_i(x,T_n(u),\nabla u)$ almost everywhere in Ω. Using the growth condition (3.3) together with Lemma 2.4, we deduce the weak convergence

$$a_i(x,T_n(u_k),\nabla u_k) \rightharpoonup a_i(x,T_n(u),\nabla u) \quad \text{weakly in} \quad L^{p_i'}(\Omega,\omega_i^*) \qquad \text{for} \quad i=1,\dots,N.$$

Finally, combining (4.9) – (4.10), we get $\chi_m = B_m u$, which complete the proof of the Lemma 4.3.

In view of Lemma 4.3 (see [32], Theorem 8.2), there exists at least one weak solution $u_m \in W^{1,\vec{p}}(\Omega,\vec{\omega})$ for the approximate problem (4.4), i.e.

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_m),\nabla u_m)D^iv\,dx+\int_\Omega |T_m(u_m)|^{p_0-\underline{p}}|u_m|^{\underline{p}-2}u_m\,v\,\omega_0\,dx\\&=\int_\Omega F(x,u)v\,dx,\end{aligned}\tag{4.19}$$

for any $v\in W^{1,\vec{p}}(\Omega,\vec{\omega})$.

Step 2: A priori estimates.

Taking $u_m\in W^{1,\vec{p}}(\Omega,\vec{\omega})$ as a test function for the approximate problem (4.19), we get

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_m),\nabla u_m)D^iT_m(u_m)\,dx+\int_\Omega |T_m(u_m)|^{p_0-\underline{p}}|u_m|^{\underline{p}}\omega_0\,dx\\&=\int_\Omega F(x,u_m)T_m(u_m)\,dx.\end{aligned}\tag{4.20}$$

According to (3.2), (4.2) and the Young's inequality, we get

$$\begin{aligned}&\frac{b_0}{(1+n)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^iu_m|^{p_i}\omega_i\,dx+\int_\Omega |T_m(u_m)|^{p_0-\underline{p}}|u_m|^{\underline{p}}\omega_0\,dx\\&\le C_1\int_\Omega |F(x,u_m)|^{p_0'}\omega_0^*\,dx+\frac12\int_\Omega |T_m(u_m)|^{p_0}\omega_0\,dx\\&\le C_2+\frac12\int_\Omega |T_m(u_m)|^{p_0-\underline{p}}|u_m|^{\underline{p}}\omega_0\,dx.\end{aligned}\tag{4.21}$$

This implies that

$$\frac{b_0}{(1+n)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^iu_m|^{p_i}\omega_i\,dx+\frac12\int_\Omega |T_m(u_m)|^{p_0-\underline{p}}|u_m|^{\underline{p}}\omega_0\,dx\le C_3.\tag{4.22}$$

Moreover, since $p_0>\underline{p}$ we have

$$\begin{aligned}\|u_m\|_{1,\vec{p},\vec{\omega}}&=\sum_{i=1}^{N}\int_\Omega |D^iu_m|^{\underline{p}}\omega_i\,dx+\int_\Omega |u_m|^{\underline{p}}\omega_0\,dx\\&\le\sum_{i=1}^{N}\int_\Omega |D^iu_m|^{\underline{p}}\omega_i\,dx+\int_\Omega |T_m(u_m)|^{p_0-\underline{p}}|u_m|^{\underline{p}}\omega_0\,dx+\int_\Omega \omega_0\,dx\\&\le C_4,\end{aligned}\tag{4.23}$$

where C_4 is a constant that does not depend on m. Hence, the sequence $(u_m)_m$ is uniformly bounded in $W^{1,\vec{p}}(\Omega,\vec{\omega})$, and there exists a subsequence still denoted $(u_m)_m$ such that

$$\begin{cases} u_m \rightharpoonup u & \text{weakly in } W^{1,\vec{p}}(\Omega,\vec{\omega}), \\ u_m \to u & \text{strongly in } L^{\underline{p}}(\Omega,\omega_0) \quad \text{and a.e in } \Omega, \\ u_m \rightharpoonup u & \text{weakly in } L^1(\partial\Omega). \end{cases} \tag{4.24}$$

Due to (4.22), we have

$$\int_\Omega |T_m(u_m)|^{p_0}\,\omega_0\,dx \le \int_\Omega |T_m(u_m)|^{p_0-\underline{p}}|u_m|^{\underline{p}}\,\omega_0\,dx \le C_5, \tag{4.25}$$

hence, $(T_m(u_m))_m$ is uniformly bounded in $L^{p_0}(\Omega,\omega_0)$, and since $T_m(u_m)\to u$ a.e. in Ω, we obtain

$$T_m(u_m) \rightharpoonup u \quad \text{weakly in} \quad L^{p_0}(\Omega,\omega_0). \tag{4.26}$$

Step 3 : The convergence almost everywhere of the gradient.

By taking $u_m - u \in W^{1,\vec{p}}(\Omega,\vec{\omega})$ as a test function for the approximate problem (4.19), we have

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega \Big(a_i(x,T_n(u_m),\nabla u_m) - a_i(x,T_n(u_m),\nabla u)\Big)(D^i u_m - D^i u)\,dx \\ &\quad + \int_\Omega |T_m(u_m)|^{p_0-\underline{p}}\Big(|u_m|^{\underline{p}-2}u_m - |u|^{\underline{p}-2}u\Big)\,(u_m-u)\,\omega_0\,dx \\ &\le \int_\Omega |T_m(u_m)|^{p_0-\underline{p}}|u|^{\underline{p}-1}|u_m-u|\,\omega_0\,dx + \int_\Omega |F(x,u_m)|\,|u_m-u|\,dx \\ &\quad - \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_m),\nabla u)(D^i u_m - D^i u)\,dx. \end{aligned} \tag{4.27}$$

Due to (4.25), we have

$$\begin{aligned} \int_\Omega (|T_m(u_m)|^{p_0-\underline{p}}|u_m|)^{\frac{p_0}{p_0-\underline{p}+1}}\omega_0\,dx &= \int_\Omega |T_m(u_m)|^{p_0-\underline{p}}|u_m|^{\underline{p}}\left(\frac{|T_m(u_m)|}{|u_m|}\right)^{\frac{(p_0-\underline{p})(\underline{p}-1)}{p_0-\underline{p}+1}}\omega_0\,dx \\ &\le \int_\Omega |T_m(u_m)|^{p_0-\underline{p}}|u_m|^{\underline{p}}\omega_0\,dx \le C_1. \end{aligned}$$

Moreover, in view of Young's inequality we have

$$\int_\Omega (|T_m(u_m)|^{p_0-\underline{p}}|u|)^{\frac{p_0}{p_0-\underline{p}+1}}\omega_0\,dx \le \int_\Omega |T_m(u_m)|^{p_0}\omega_0\,dx + \int_\Omega |u|^{p_0}\omega_0\,dx \le 2C_2,$$

Thus, the sequence $(|T_m(u_m)|^{p_0-\underline{p}}(u_m-u))_m$ is uniformly bounded in $L^{\frac{p_0}{p_0-\underline{p}+1}}(\Omega,\omega_0)$. Having in mind that $|T_m(u_m)|^{p_0-\underline{p}}(u_m-u)\to 0$ almost everywhere in Ω, in view of Lemma 2.4 we conclude that $|T_m(u_m)|^{p_0-\underline{p}}(u_m-u)\rightharpoonup 0$ weakly in $L^{\frac{p_0}{p_0-\underline{p}+1}}(\Omega,\omega_0)$, and since $|u|^{\underline{p}-2}u \in L^{\frac{p_0}{\underline{p}-1}}(\Omega,\omega_0)$, it follows that

$$\int_\Omega |T_m(u_m)|^{p_0-\underline{p}}|u|^{\underline{p}-2}u(u_m-u)\omega_0\,dx \longrightarrow 0 \quad \text{as } m\to\infty. \tag{4.28}$$

Also, we have $F(x,u_m)\rightharpoonup F(x,u)$ weak$-*$ in $L^\infty(\Omega)$, and since $u_m\to u$ strongly in $L^1(\Omega)$, we obtain

$$\int_\Omega |F(x,u_m)|\,|u_m-u|\,dx \longrightarrow 0 \qquad \text{as} \quad m\to\infty. \tag{4.29}$$

Furthermore, we have $a_i(x,T_n(u_m),\nabla u)\to a_i(x,T_n(u),\nabla u)$ strongly in $L^{p_i'}(\Omega,\omega_i^*)$, and since $D^i u_m \rightharpoonup D^i u$ weakly in $L^{p_i}(\Omega,\omega_i)$, it follows that

$$\sum_{i=1}^N \int_\Omega a_i(x,T_n(u_m),\nabla u)(D^i u_m - D^i u)\,dx \longrightarrow 0 \qquad \text{as} \quad m\to\infty. \tag{4.30}$$

Combining (4.27) and (4.29)-(4.30), and since $u_m \to u$ strongly in $L^{\underline{p}}(\Omega, \omega_0)$ we obtain

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} \left(a_i(x, T_n(u_m), \nabla u_m) - a_i(x, T_n(u_m), \nabla u)\right) (D^i u_m - D^i u)\, dx \\ + \int_{\Omega} \left(|u_m|^{\underline{p}-2} u_m - |u|^{\underline{p}-2} u\right) (u_m - u)\omega_0\, dx \longrightarrow 0 \quad \text{as} \quad m \to \infty.\end{aligned} \tag{4.31}$$

In view of Lemma 3.1 we obtain

$$\begin{cases} u_m \to u & \text{strongly in} \quad W^{1,\vec{p}}(\Omega, \vec{\omega}), \\ D^i u_m \to D^i u & \text{a.e in } \Omega \quad \text{for } i = 1, \cdots, N. \end{cases} \tag{4.32}$$

Hence, we have $a_i(x, T_n(u_m), \nabla u_m) \longrightarrow a_i(x, T_n(u), \nabla u)$ a.e. in Ω, and we conclude that

$$a_i(x, T_n(u_m), \nabla u_m) \rightharpoonup a_i(x, T_n(u), \nabla u) \quad \text{weakly in} \quad L^{p_i'}(\Omega, \omega_i^*) \quad \text{for} \quad i = 1, \cdots, N, \tag{4.33}$$

and

$$F(x, u_m) \longrightarrow F(x, u) \quad \text{strongly in} \quad L^{\underline{p}'}(\Omega, \omega_0^*). \tag{4.34}$$

Step 4 : Passage to the limit.

By taking $v \in W^{1,\vec{p}}(\Omega, \vec{\omega}) \cap L^{\infty}(\Omega)$ as a test function for the approximate problem (4.19), we get

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_m), \nabla u_m) D^i v\, dx + \int_{\Omega} |T_m(u_m)|^{p_0 - \underline{p}} |u_m|^{\underline{p}-2} u_m v\, \omega_0\, dx \\ = \int_{\Omega} F(x, u_m) v\, dx.\end{aligned} \tag{4.35}$$

By combining (4.25) $-$ (4.26) and (4.34), then passing to the limit as $m \to \infty$, we deduce that

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u), \nabla u) D^i v\, dx + \int_{\Omega} |u|^{p_0 - 2} uv\omega_0\, dx = \int_{\Omega} F(x, u) v\, dx, \tag{4.36}$$

which conclude the proof of the Theorem 4.2.

5 Existence of renormalized solution for $f(x, u) \in L^1(\Omega)$

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, with Lipschitz boundary $\partial\Omega$. We consider the degenerated quasilinear elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u) + |u|^{p_0 - 2} u\, \omega_0 = f(x, u) & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, u, \nabla u).n_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Definition 5.1. A measurable function u is called a renormalized solution for the quasilinear degenerated elliptic problem (5.1), if $u \in \mathcal{T}_{tr}^{1,\vec{p}}(\Omega, \vec{\omega})$ and $|u|^{p_0 - 2} u\, \omega_0 \in L^1(\Omega)$ such that

$$\lim_{h \to \infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{|u| \leq h\}} a_i(x, u, \nabla u) D^i u\, dx = 0, \tag{5.2}$$

and u satisfies the following equality

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla u) \left(S'(u)\varphi D^i u + S(u) D^i \varphi\right)\, dx + \int_{\Omega} |u|^{p_0 - 2} u S(u) \varphi \omega_0\, dx \\ = \int_{\Omega} f(x, u) S(u) \varphi\, dx,\end{aligned} \tag{5.3}$$

for all $\varphi \in W^{1,\vec{p}}(\Omega,\vec{\omega}) \cap L^{\infty}(\Omega)$ and any $S(\cdot) \in W^{1,\infty}(I\!R)$ with compact support.

The aim of this paper is to prove the existence of renormalized solutions for the quasilinear degenerate elliptic problem (5.1).

Theorem 5.2. *Assuming that (3.2) – (3.4) are satisfied. Then, the quasilinear degenerate elliptic problem (5.1) has at least one renormalized solution u.*

Remark 5.3. The uniqueness of the renormalized solution can be obtained using some additional assumptions on the Carathéodory function $a_i(x,s,\xi)$ and $f(x,s)$. For further details and related results, we refer the reader to [23, 24].

Proof of theorem 5.2

Step 1: Approximate problems.

We consider the approximate problem for our quasilinear degenerated elliptic equation

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,T_n(u_n),\nabla u_n) + |u_n|^{p_0-2}u_n\,\omega_0 = f_n(x,u_n) & \text{in} \quad \Omega, \\ \sum_{i=1}^{N} a_i(x,T_n(u_n),\nabla u_n).n_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5.4}$$

with $f_n(x,s) = T_n(f(x,s))$. In view of Theorem 4.2, there exists at least one solution in the sense of distributions $u_n \in W^{1,\vec{p}}(\Omega,\vec{\omega})$ for the approximate problem (5.4), that is

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)D^i v\,dx + \int_{\Omega}|u_n|^{p_0-2}u_n v\omega_0\,dx = \int_{\Omega} f_n(x,u_n)v\,dx, \tag{5.5}$$

for any $v \in W^{1,\vec{p}}(\Omega,\vec{\omega}) \cap L^{\infty}(\Omega)$, with $|u_n| \in L^{p_0}(\Omega,\omega_0)$.

Step 2: Weak convergence of truncations.

Let $k \geq 1$, by taking $T_k(u_n)$ as a test function for the approximate problem (5.4), we have

$$\begin{aligned} &\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,dx + \int_{\Omega}|u_n|^{p_0-1}|T_k(u_n)|\omega_0\,dx \\ &= \int_{\Omega} f_n(x,u_n)T_k(u_n)\,dx \end{aligned} \tag{5.6}$$

According to (3.2) and (3.5), we get

$$\begin{aligned} &\frac{b_0}{(1+k)^{\lambda}}\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}|D^i u_n|^{p_i}\omega_i\,dx + \int_{\Omega}|u_n|^{p_0-1}|T_k(u_n)|\omega_0\,dx \\ &\leq k\|f_0\|_{L^1(\Omega)} + k\int_{\Omega}|c(x)|^{\frac{p_0-1}{p_0-1-\gamma}}\omega_0\,dx + \frac{1}{2}\int_{\Omega}|u_n|^{p_0-1}|T_k(u_n)|\omega_0\,dx. \end{aligned} \tag{5.7}$$

Therefore

$$\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}|D^i u_n|^{p_i}\omega_i\,dx \leq C_1 k(1+k)^{\lambda}, \tag{5.8}$$

and

$$\int_{\Omega}|T_k(u_n)|^{\underline{p}}\omega_0\,dx \leq \int_{\Omega}(|T_k(u_n)|^{p_0}+1)\omega_0\,dx + C_2 \leq \int_{\Omega}|u_n|^{p_0-1}|T_k(u_n)|\omega_0\,dx + C_3 \leq C_4 k, \tag{5.9}$$

with C_1 and C_4 being two positive constants independent of k and n. Furthermore, we have

$$\begin{aligned}\|T_k(u_n)\|_{1,\vec{p},\vec{\omega}} &= \|T_k(u_n)\|_{\underline{p},\omega_0} + \sum_{i=1}^{N} \|D^i T_k(u_n)\|_{p_i,\omega_i} \\ &= \left(\int_\Omega |T_k(u_n)|^{\underline{p}} \omega_0 \, dx\right)^{\frac{1}{\underline{p}}} + \sum_{i=1}^{N} \left(\int_\Omega |D^i T_k(u_n)|^{p_i} \omega_i \, dx\right)^{\frac{1}{p_i}} \\ &\le C_5 k + \sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i} \omega_i \, dx + N \\ &\le C_6 k^{\lambda+1}.\end{aligned} \tag{5.10}$$

Thus, the sequence $(T_k(u_n))_{n\in\mathbb{N}^*}$ is uniformly bounded in $W^{1,\vec{p}}(\Omega,\vec{\omega})$, and there exists a subsequence still denoted $(T_k(u_n))_{n\in\mathbb{N}^*}$ and a measurable function $v_k \in W^{1,\vec{p}}(\Omega,\vec{\omega})$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k & \text{weakly in } \ W^{1,\vec{p}}(\Omega,\vec{\omega}), \\ T_k(u_n) \to v_k & \text{strongly in } \ L^{\underline{p}}(\Omega,\omega_0) \ \text{ and a.e in } \ \Omega. \end{cases} \tag{5.11}$$

Due to (5.9), it follows that

$$\begin{aligned} k \cdot \ \text{meas}\left(\{|u_n| > k\}\right) &= \int_{\{|u_n|>k\}} |T_k(u_n)| \, dx \le \int_\Omega |T_k(u_n)| \, dx \\ &\le C_7 \, \|T_k(u_n)\|_{\underline{p},\omega_0} \\ &\le C_8 k^{\frac{1}{\underline{p}}}. \end{aligned} \tag{5.12}$$

Then

$$\text{meas}\left(\{|u_n| > k\}\right) \le \frac{C_8 k^{\frac{1}{\underline{p}}}}{k} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{5.13}$$

Now, we will show that the sequence $(u_n)_n$ is a Cauchy sequence in measure.
For each $\rho > 0$ and $k > 0$, we have

$$\text{meas}\{|u_n - u_m| > \rho\} \le \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \rho\}.$$

For $\varepsilon > 0$, using (5.13) we can choose $k_0(\varepsilon) > 0$ large enough such that

$$\text{meas}\{|u_n| > k\} \le \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \le \frac{\varepsilon}{3} \qquad \text{for any} \quad k \ge k_0(\varepsilon). \tag{5.14}$$

On the other hand, thanks to (5.11) we have $T_k(u_n) \to \nu_k$ strongly in $L^{\underline{p}}(\Omega,\omega_0)$ and a.e in Ω, then the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. It follows that : for any $k > 0$, and $\rho, \varepsilon > 0$, there exists $n_0 = n_0(k,\rho,\varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \rho\} \le \frac{\varepsilon}{3} \quad \text{for any} \quad n, m \ge n_0(k,\rho,\varepsilon). \tag{5.15}$$

Thanks to (5.14) and (5.15), we conclude that : for all $\rho, \varepsilon > 0$, there exists $n_0 = n_0(\rho,\varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \rho\} \le \varepsilon \quad \text{for any} \quad n, m \ge n_0(\rho,\varepsilon).$$

Thus the sequence $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere in Ω to some measurable function u. Hence, thanks to (5.11) we conclude that

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } \ W^{1,\vec{p}}(\Omega,\vec{\omega}), \\ T_k(u_n) \to T_k(u) & \text{strongly in } \ L^{\underline{p}}(\Omega,\omega_0) \ \text{ and a.e in } \ \Omega. \end{cases} \tag{5.16}$$

Step 4: Some regularity results.

Firstly, we will establish the following regularity result:

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx=0. \tag{5.17}$$

By taking $\dfrac{T_h(u_n)}{h}\in W^{1,\vec{p}}(\Omega,\vec{\omega})$ as a test function for the approximate problem (5.4), we have

$$\begin{aligned}
&\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx+\frac{1}{h}\int_\Omega|u_n|^{p_0-1}|T_h(u_n)|\omega_0\,dx\\
&\le\frac{1}{h}\int_\Omega|f_0(x)|\,|T_h(u_n)|\,dx+\frac{1}{h}\int_\Omega|c(x)|\,|u_n|^{\gamma}\,|T_h(u_n)|\omega_0\,dx\\
&\le\frac{1}{h}\int_\Omega|f_0(x)|\,|T_h(u_n)|\,dx+\frac{1}{h}\int_\Omega|c(x)|^{\frac{p_0-1}{p_0-1-\gamma}}|T_h(u_n)|\omega_0\,dx\\
&\quad+\frac{1}{2h}\int_\Omega|u_n|^{p_0-1}\,|T_h(u_n)|\omega_0\,dx.
\end{aligned} \tag{5.18}$$

According to (5.13), we have $\mathrm{meas}\{|u_n|>h\}\to 0$ as $h\to\infty$, then $\dfrac{|T_h(u_n)|}{h}\rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$. It follows that

$$\frac{1}{h}\int_\Omega|f_0(x)|\,|T_h(u_n)|\,dx\longrightarrow 0\quad\text{as}\quad h\to\infty. \tag{5.19}$$

Similarly, we have $|c(x)|^{\frac{p_0-1}{p_0-1-\gamma}}\omega_0\in L^1(\Omega)$, then

$$\frac{1}{h}\int_\Omega|c(x)|^{\frac{p_0-1}{p_0-1-\gamma}}|T_h(u_n)|\omega_0\,dx\longrightarrow 0\quad\text{as}\quad h\to\infty. \tag{5.20}$$

By combining (5.18) and (5.19) $-$ (5.20), we get

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx=0, \tag{5.21}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}}|u_n|^{p_0-1}\omega_0\,dx=0. \tag{5.22}$$

Now, we will show that

$$|u_n|^{p_0-1}\omega_0\longrightarrow|u|^{p_0-1}\omega_0\quad\text{strongly in}\quad L^1(\Omega).$$

Indeed, for any measurable subset $E\subset\Omega$ it's clear that

$$\int_E|u_n|^{p_0-1}\omega_0\,dx\le\int_{\{|u_n|>h\}}|u_n|^{p_0-1}\omega_0\,dx+\int_E|T_h(u_n)|^{p_0-1}\omega_0\,dx. \tag{5.23}$$

On the one hand, thanks to (5.22) we have : for any $\varepsilon>0$, there exists $h_0(\varepsilon)>0$ such that

$$\int_{\{|u_n|>h\}}|u_n|^{p_0-1}\omega_0\,dx\le\frac{\varepsilon}{2}\qquad\text{for any}\quad h\ge h_0(\varepsilon). \tag{5.24}$$

On the other hand, there exists a positive constant small enough $\vartheta(\varepsilon,h)>0$ such that : for any $E\subset\Omega$ with $\mathrm{meas}(E)\le\vartheta(\varepsilon,h)$, we have

$$\int_E|T_h(u_n)|^{p_0-1}\omega_0\,dx\le\frac{\varepsilon}{2}. \tag{5.25}$$

By combining (5.23) and (5.24) – (5.25), we obtain : for any $\varepsilon > 0$, there exists a positive constant $\vartheta(\varepsilon) > 0$ such that

$$\int_E |u_n|^{p_0-1}\omega_0\, dx \le \varepsilon \quad \text{for any} \quad E \subset \Omega \quad \text{with} \quad \text{meas}(E) \le \vartheta(\varepsilon). \tag{5.26}$$

Consequently, the sequence $(|u_n|^{p_0-2}u_n\omega_0)_n$ is uniformly equi-integrable.
By applying Vitali's theorem and since $|u_n|^{p_0-1}\omega_0 \longrightarrow |u|^{p_0-1}\omega_0$ a.e. in Ω, we deduce that

$$|u_n|^{p_0-1}\omega_0 \longrightarrow |u|^{p_0-1}\omega_0 \quad \text{strongly in} \quad L^1(\Omega). \tag{5.27}$$

Moreover, we will show that $(f_n(x,u_n))_n$ converges strongly to $f(x,u)$ in $L^1(\Omega)$.
We have $f_n(x,u_n) \to f(x,u)$ almost everywhere in Ω. In view of the Vitali's theorem, we will show that the sequence $(f_n(x,u_n))_n$ is uniformly equi-integrable.
For any measurable subset $E \subset \Omega$ we have

$$\begin{aligned}\int_E |f_n(x,u_n)|\, dx &\le \int_E |f_0(x)|\, dx + \int_E c(x)|T_h(u_n)|^\gamma\omega_0\, dx + \int_{\{|u_n|>h\}} c(x)|u_n|^\gamma\omega_0\, dx \\ &\le \int_E |f_0(x)|\, dx + \int_E c(x)|T_h(u_n)|^\gamma\omega_0\, dx + \int_{\{|u_n|>h\}} |c(x)|^{\frac{p_0-1}{p_0-1-\gamma}}\omega_0\, dx \\ &\quad + \int_{\{|u_n|>h\}} |u_n|^{p_0-1}\omega_0\, dx.\end{aligned} \tag{5.28}$$

Thus, for any $\varepsilon > 0$ and $h > 0$ there exists a positive constant $\mu(\varepsilon,h) > 0$ such that

$$\int_E |f_0(x)|\, dx + \int_E c(x)|T_h(u_n)|^\gamma\omega_0\, dx + \int_{\{|u_n|>h\}} |c(x)|^{\frac{p_0-1}{p_0-1-\gamma}}\omega_0\, dx \le \frac{\varepsilon}{2}, \tag{5.29}$$

By combining (5.24), (5.28) and (5.29) we conclude that : For any $\varepsilon > 0$ there exists $\mu(\varepsilon) > 0$ such that

$$\int_E |f_n(x,u_n)|\, dx \le \varepsilon \quad \text{for any} \quad \text{meas}(E) \ge \mu(\varepsilon).$$

Thanks to Vilali's theorem, we conclude that

$$f_n(x,u_n) \longrightarrow f(x,u) \qquad \text{strongly in} \quad L^1(\Omega). \tag{5.30}$$

Step 5 : The strong convergence of the gradients.

In this sequel, we denote by $\varepsilon_i(n)$ for $i = 1, 2, \ldots$ some various functions of real number which converge to 0 as n tends to infinity. Similarly, we define $\varepsilon_i(h)$ and $\varepsilon_i(n,h)$.
Let $1 \le k \le h$, we define $S_h(s) = 1 - \dfrac{|T_{2h}(s) - T_h(s)|}{h}$.
Choosing $(T_k(u_n) - T_k(u))S_h(u_n) \in W^{1,\vec{p}}(\Omega,\vec{\omega})$ as a test function for the approximate problem (5.4), we obtain

$$\begin{aligned}&\sum_{i=1}^N \int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\left(D^iT_k(u_n) - D^iT_k(u)\right)S_h(u_n)\, dx \\ &\quad - \sum_{i=1}^N \int_{\{|u_n|>k\}} a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u)S_h(u_n)\, dx \\ &\quad - \frac{1}{h}\sum_{i=1}^N \int_{\{h<|u_n|\le 2h\}} a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_{2h}(u_n)|T_k(u_n) - T_k(u)|\, dx \\ &\quad + \int_\Omega |u_n|^{p_0-2}u_n(T_k(u_n) - T_k(u))S_h(u_n)\omega_0\, dx \\ &= \int_\Omega f_n(x,u_n)(T_k(u_n) - T_k(u))S_h(u_n)\, dx.\end{aligned} \tag{5.31}$$

On the set $\{|u_n| \leq k\}$, we have $S_h(u_n) = 1$, and having in mind that $T_k(u_n) - T_k(u)$ has the same sign as u_n on the set $\{|u_n| > k\}$, we derive

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) \left(D^i T_k(u_n) - D^i T_k(u)\right) \, dx \\
&\leq \int_{\Omega} |f_0(x)| \, |T_k(u_n) - T_k(u)| \, dx + \int_{\Omega} |c(x)|^{\frac{p_0-1}{p_0-1-\gamma}} |T_k(u_n) - T_k(u)| \omega_0 \, dx \\
&+ \int_{\Omega} |u_n|^{p_0-1} |T_k(u_n) - T_k(u)| \omega_0 \, dx \\
&+ \frac{2k}{h} \sum_{i=1}^{N} \int_{\{h<|u_n|\leq 2h\}} a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n)) D^i T_{2h}(u_n) \, dx \\
&+ \sum_{i=1}^{N} \int_{\{k\leq|u_n|\leq 2h\}} |a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n))| \, |D^i T_k(u)| \, dx.
\end{aligned} \tag{5.32}$$

We have $(T_k(u_n) - T_k(u)) \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$ as n goes to infinity, and since $f_0(x) \in L^1(\Omega)$, $|c(x)|^{\frac{p_0-1}{p_0-1-\gamma}} \omega_0 \in L^1(\Omega)$ and $|u_n|^{p_0-1}\omega_0 \to |u|^{p_0-1}\omega_0$ strongly in $L^1(\Omega)$, we deduce that

$$\varepsilon_1(n) = \int_{\Omega} |f_0(x)| \, |T_k(u_n) - T_k(u)| \, dx \longrightarrow 0 \quad \text{as} \quad n \to \infty, \tag{5.33}$$

$$\varepsilon_2(n) = \int_{\Omega} |c(x)|^{\frac{p_0-1}{p_0-1-\gamma}} |T_k(u_n) - T_k(u)| \omega_0 \, dx \longrightarrow 0 \quad \text{as} \quad n \to \infty, \tag{5.34}$$

and

$$\varepsilon_3(n) = \int_{\Omega} |u_n|^{p_0-1} |T_k(u_n) - T_k(u)| \omega_0 \, dx \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{5.35}$$

Moreover, in view of (5.21) we get

$$\varepsilon_4(h) = \frac{2k}{h} \sum_{i=1}^{N} \int_{\{h<|u_n|\leq 2h\}} a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n)) D^i T_{2h}(u_n) \, dx \longrightarrow 0 \quad \text{as} \quad h \to \infty. \tag{5.36}$$

For the last term on the right-hand side of (5.32), we observe that $(a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n)))_n$ is bounded in $L^{p_i'}(\Omega, \omega_i^*)$, then there exists a measurable function $\zeta_i \in L^{p_i'}(\Omega, \omega_i^*)$ such that $a_i(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \zeta_i$ weakly in $L^{p_i'}(\Omega, \omega_i^*)$, it follows that

$$\begin{aligned}
\varepsilon_5(h) \;=\; & \sum_{i=1}^{N} \int_{\{k<|u_n|\leq 2h\}} a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n)) D^i T_k(u) \, dx \\
& \longrightarrow \sum_{i=1}^{N} \int_{\{k<|u|\leq 2h\}} \zeta_i D^i T_k(u) \, dx = 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{5.37}$$

From (5.32) and (5.33) – (5.37), we get

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) \left(D^i T_k(u_n) - D^i T_k(u)\right) \, dx \leq \varepsilon_6(n, h). \tag{5.38}$$

Hence

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega} \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big) \left(D^i T_k(u_n) - D^i T_k(u)\right) \, dx \\
&\leq \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u)) \left(D^i T_k(u_n) - D^i T_k(u)\right) \, dx + \varepsilon_6(n, h).
\end{aligned} \tag{5.39}$$

Concerning the first term on the right-hand side of (5.39), thanks to (5.16) we have $T_k(u_n) \to T_k(u)$ converges strongly in $L^{p_i}(\Omega, \omega_i)$. By applying the Lebesgue's dominated convergence theorem we conclude that $a_i(x, T_k(u_n), \nabla T_k(u))$ strongly to $a_i(x, T_k(u), \nabla T_k(u))$ in $L^{p_i'}(\Omega, \omega_i^*)$, and since $D^i T_k(u_n) \rightharpoonup D^i T_k(u)$ weakly in $L^{p_i}(\Omega, \omega_i)$, we conclude that

$$\varepsilon_7(n) = \sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u)) D^i T_k(u_n) - D^i T_k(u)\, dx \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{5.40}$$

Combining (5.39) and (5.40), we conclude that

$$\begin{aligned} 0 \quad &\leq \sum_{i=1}^{N} \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \Big) \left(D^i T_k(u_n) - D^i T_k(u) \right) dx \\ &+ \int_\Omega \left(|T_k(u_n)|^{\underline{p}-2} T_k(u_n) - |T_k(u)|^{\underline{p}-2} T_k(u) \right) (T_k(u_n) - T_k(u)) \omega_0\, dx \to 0 \ \text{ as } \ n \to \infty. \end{aligned} \tag{5.41}$$

According to Lemma 3.1, we deduce that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in} \quad W^{1,\vec{p}}(\Omega, \vec{\omega}) \quad \text{and} \quad D^i u_n \longrightarrow D^i u \quad \text{a.e. in} \quad \Omega. \tag{5.42}$$

Consequently, we have $a_i(x, T_n(u_n), \nabla u_n) D^i u_n$ tends to $a_i(x, u, \nabla u) D^i u$ almost everywhere in Ω. Thanks to (5.21) and Fatou's lemma we obtain

$$\begin{aligned} &\lim_{h\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{|u|\leq h\}} a_i(x, u, \nabla u) D^i u\, dx \\ &\leq \lim_{h\to\infty} \liminf_{n\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{|u_n|\leq h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n\, dx \\ &\leq \lim_{h\to\infty} \limsup_{n\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{|u_n|\leq h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n\, dx = 0. \end{aligned} \tag{5.43}$$

Step 6: Passage to the limit.

Let $\varphi \in W^{1,\vec{p}}(\Omega, \vec{\omega}) \cap L^\infty(\Omega)$ and $S(\cdot)$ be a smooth function in $W^{1,\infty}(\mathbb{R})$ with $supp(S(\cdot)) \subseteq [-M, M]$ for some $M \geq 0$.
Choosing $S(u_n)\varphi \in W^{1,\vec{p}}(\Omega, \vec{\omega}) \cap L^\infty(\Omega)$ as a test function for the approximate problem (5.4), we get

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \left(D^i u_n S'(u_n)\varphi + S(u_n) D^i \varphi \right) dx + \int_\Omega |u_n|^{p_0 - 2} u_n S(u_n) \varphi \omega_0\, dx \\ &= \int_\Omega f_n(x, u_n) S(u_n) \varphi\, dx. \end{aligned} \tag{5.44}$$

Since $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is uniformly bounded in $L^{p_i'}(\Omega, \omega_i^*)$ and $a_i(x, T_M(u_n), \nabla T_M(u_n))$ tends to $a_i(x, T_M(u), \nabla T_M(u))$ almost everywhere in Ω, we get

$$a_i(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a_i(x, T_M(u), \nabla T_M(u)) \quad \text{weakly in} \quad L^{p_i'}(\Omega, \omega_i^*),$$

and since $D^i T_M(u_n) S'(u_n)\varphi + S(T_M(u_n)) D^i \varphi$ converges strongly to $D^i T_M(u) S'(u)\varphi + S(T_M(u))\, D^i \varphi$ in $L^{p_i}(\Omega, \omega_i)$, we deduce that

$$\begin{aligned} &\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \left(D^i u_n S'(u_n)\varphi + S(u_n) D^i \varphi \right) dx \\ &= \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \left(D^i u S'(u)\varphi + S(u) D^i \varphi \right) dx. \end{aligned} \tag{5.45}$$

Moreover, we have $S(T_M(u_n))\varphi \rightharpoonup S(T_M(u))\varphi$ weak$-*$ in $L^\infty(\Omega)$, and by (5.27), we get

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega |u_n|^{p_0-2}u_n S(T_M(u_n))\varphi\omega_0\,dx &= \int_\Omega |u|^{p_0-2}uS(T_M(u))\varphi\omega_0\,dx \\ &= \int_\Omega |u|^{p_0-2}uS(u)\varphi\omega_0\,dx.\end{aligned} \tag{5.46}$$

Similarly, we have $f_n(x,u_n)\to f(x,u)$ strongly in $L^1(\Omega)$, then

$$\lim_{n\to\infty}\int_\Omega f_n(x,u_n)S(T_M(u_n))\varphi\,dx = \int_\Omega f(x,u)S(T_M(u))\varphi\,dx = \int_\Omega f(x,u)S(u)\varphi\,dx. \tag{5.47}$$

Combining (5.44) – (5.45) and (5.47), we deduce that

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x,u,\nabla u)\left(D^i uS'(u)\varphi + S(u)D^i\varphi\right)\,dx + \int_\Omega |u|^{p_0-2}uS(u)\varphi\omega_0\,dx \\ &= \int_\Omega f(x,u)S(u)\varphi\,dx.\end{aligned} \tag{5.48}$$

which conclude the proof of Theorem 5.2.

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