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# Entropy solutions for some quasilinear and non-coercive Neumann $p(x)-$elliptic equations

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**Abstract**. In this paper, we consider the following quasilinear and non-coercive elliptic problem

$$\begin{cases} -\text{div}\Big(\dfrac{a(x,|\nabla u|)\nabla u}{(1+|u|)^{\lambda(x)}}\Big) + |u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ \dfrac{a(x,|\nabla u|)\nabla u}{(1+|u|)^{\lambda(x)}} \cdot \eta = g & \text{on } \partial\Omega, \end{cases}$$

we study the existence of entropy solutions for our elliptic problem in the framework of Sobolev spaces with variable exponents, and we conclude some regularity results.

## 1 Introduction

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary $\partial\Omega$. The study of quasilinear elliptic problems with degenerate coercivity has drawn a considerable attention in recent decades, due to its rich mathematical structure and applications in nonlinear physics and engineering. Boccardo et al. [12] have studied the quasilinear elliptic Dirichlet problem involving a general operator $A(x,u)\nabla u$ with the data $f$ is assumed to be in $L^m(\Omega)$ for $m \geq 1$. Later, Alvino et al. [2] have considered the degenerated elliptic problem of the type

$$\begin{cases} -\text{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

They have established the existence of entropy solutions under some assumptions, for more details we refer the reader to [3, 11, 13, 19, 15, 23, 24]. Similar problems involving Neumann and Fourier-type boundary conditions have been considered in [7, 6, 9, 10, 5, 22, 21], and the existence of entropy solutions with variable exponent have been studied in [8].
Furthermore, many questions remain open in the context of degenerated quasilinear elliptic equations involving variable exponent and nonhomogeneous Neumann boundary conditions. In particular, the degenerated elliptic equation of the form

$$\begin{cases} -\text{div}\Big(\dfrac{a(x,|\nabla u|)\nabla u}{(1+|u|)^{\lambda(x)}}\Big) + |u|^{p(x)-2}u = f(x,u) & \text{in } \Omega \\ \dfrac{a(x,|\nabla u|)\nabla u}{(1+|u|)^{\lambda(x)}} \cdot \eta = g & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In this paper, inspired by the results of [5] and [8], we study the existence of entropy solutions for such degenerate quasilinear elliptic problems in the framework of variable exponent Sobolev

spaces, where $Au = -\text{div}\Big(\dfrac{a(x,|\nabla u|)\nabla u}{(1+|u|)^{\lambda(x)}}\Big)$ is a Leray-Lions operator acted from $W^{1,p(\cdot)}(\Omega)$ into its dual. The Carathéodory function $f(x,s)$ verifies only some growth conditions. We also derive some regularity results of the solution.
The motivation for studying this problem in the framework of variable exponent Sobolev spaces lies in their ability to model electro-rheological fluids, image restoration and nonlinear elasticity, for more applications we refer the reader to [14, 25, 26, 31, 32].

This paper is organized as follows: in Section 2, we introduce some definitions and basic properties concerning the variable exponent Lebesgue and Sobolev spaces. In section 3, we present the assumptions under which our problem has at least one entropy solution. Section 4 is devoted to proof the existence of entropy solutions for our quasilinear elliptic problem (3.1), and some regularity results were concluded.

## 2 Preliminaries

Let $\Omega$ be an open bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with boundary $\partial\Omega$. We set

$$C_+(\overline{\Omega}) = \{\text{continuous function } p(\cdot) : \bar{\Omega} \longmapsto \mathbb{R}^N \quad \text{such that} \quad 1 < p_- \leq p_+ < \infty\},$$

where

$$p_- = \inf\{p(x)/\ x \in \bar{\Omega}\} \qquad \text{and} \qquad p_+ = \sup\{p(x)/\ x \in \bar{\Omega}\}.$$

We define the convex modular as follows.

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx. \tag{2.1}$$

The Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \Big\{u : \Omega \longmapsto \mathbb{R} \quad \text{measurable} \ / \ \int_\Omega |u(x)|^{p(x)}\, dx < \infty\Big\},$$

endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0, \quad \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\, dx \leq 1\Big\}.$$

For $p(\cdot) \in C_+(\overline{\Omega})$ the space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}(\Omega)})$ is a separable, reflexive Banach space, and its dual space is denoted by $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(\cdot)} + \dfrac{1}{p'(\cdot)} = 1$.
According to [17, 30], the Hölder type inequality associated to Lebesgue space with variable exponent is given as follows

$$\left|\int_\Omega uv\, dx\right| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{L^{p(\cdot)}(\Omega)}\|u\|_{L^{p'(\cdot)}(\Omega)} \quad \text{for any} \quad u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad v \in L^{p'(\cdot)}(\Omega), \tag{2.2}$$

We define the Sobolev space with variable exponent by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}, \tag{2.3}$$

endower with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \tag{2.4}$$

The space $\big(W^{1,p(\cdot)}(\Omega), \|u\|_{1,p(\cdot)}\big)$ is a reflexive and separable Banach space.
The modular function related to the norm of $W^{1,p(x)}(\Omega)$ is defined by

$$\rho_{1,p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx + \int_\Omega |\nabla u|^{p(x)}\, dx. \tag{2.5}$$

The following lemmas are essential for the study of variable exponents Sobolev spaces.

**Lemma 2.1** ([16]). *Let $p(\cdot) \in C_+(\overline{\Omega})$, and let $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p_+ < \infty$, then the following properties hold true:*

*i)* $\|u\|_{L^{p(\cdot)}(\Omega)} \geq 1 \Rightarrow \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)}$;

*ii)* $\|u\|_{L^{p(\cdot)}(\Omega)} \leq 1 \Rightarrow \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}$;

*iii)* $\|u\|_{L^{p(\cdot)}(\Omega)} \leq 1$ *(resp.* $= 1; \geq 1$*)* $\Leftrightarrow \rho_{p(\cdot)}(u) \leq 1$ *(resp.* $= 1; \geq 1$*);*

*iv)* $\|u_n\|_{L^{p(\cdot)}(\Omega)} \longrightarrow 0$ *(resp.* $\to \infty$*)* $\Leftrightarrow \rho_{p(\cdot)}(u_n) \longrightarrow 0 \quad (resp. \quad \to \infty)$;

*v)* $\rho_{p(\cdot)}(\dfrac{u}{\|u\|_{L^{p(\cdot)}(\Omega)}}) = 1.$

**Lemma 2.2.** *(see [28]) If $u \in W^{1,p(\cdot)}(\Omega)$, then the following properties hold true:*

*i)* $\|u\|_{1,p(\cdot)} \geq 1 \Rightarrow \|u\|^{p_-}_{1,p(\cdot)} \leq \rho_{1,p(\cdot)}(u) \leq \|u\|^{p_+}_{1,p(\cdot)}$;

*ii)* $\|u\|_{1,p(\cdot)} \leq 1 \Rightarrow \|u\|^{p_+}_{1,p(\cdot)} \leq \rho_{1,p(\cdot)}(u) \leq \|u\|^{p_-}_{1,p(\cdot)}$;

*iii)* $\|u\|_{1,p(\cdot)} \leq 1$ *(respectively* $= 1; \geq 1$*)* $\Longleftrightarrow \rho_{1,p(\cdot)}(u) \leq 1$ *(respectively* $= 1; \geq 1$*).*

We set

$$p^{\partial}(x) = \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)} & \text{if} \quad p(x) < N, \\ \infty & \text{if} \quad p(x) \geq N. \end{cases}$$

**Lemma 2.3.** *(see [17], [29]) Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, then the embeddings are compacts hold true*

- *If $p(x) < N$, then $W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ for any $q(x) \in [p(x), p^\star(x)[$,*
- *If $p(x) = N$, then $W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ for any $q(x) \in [p(x), \infty[$,*
- *If $p(x) > N$, then $W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\infty}(\Omega) \cap C(\overline{\Omega})$.*

For any $u \in W^{1,p(\cdot)}(\Omega)$, we denote by $\tau(u)$ the trace of $u$ on $\partial\Omega$ in the usual sense. In the sequel, we will identify $u$ and $\tau(u)$ on the boundary $\partial\Omega$.

**Definition 2.4.** Let $k > 0$, we consider the truncation function $T_k(\cdot) : I\!R \longmapsto I\!R$, given by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$$\mathcal{T}^{1,p(\cdot)}(\Omega) := \{u : \Omega \mapsto I\!R \text{ measurable, such that } T_k(u) \in W^{1,p(\cdot)}(\Omega) \text{ for any } k > 0\}.$$

We recall the following results.

**Proposition 2.5.** *(see [5, 21, 22]) Let $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$, then there exists a unique measurable function $v : \Omega \longmapsto \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \qquad \textit{for all} \quad k > 0.$$

*The function $v$ is denoted by $\nabla u$. Moreover, if $u \in W^{1,p(\cdot)}(\Omega)$ then $v \in (L^{p(\cdot)}(\Omega))^N$ and $v = \nabla u$ in the usual sense.*

**Definition 2.6.** We define $\mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$ such that there exists a sequence $(u_n)_n \subset W^{1,p(\cdot)}(\Omega)$ satisfies the following conditions:

C1) $u_n \longrightarrow u$ a.e. in $\Omega$.

C2) $\nabla T_k(u_n) \longrightarrow \nabla T_k(u)$ strongly in $(L^1(\Omega))^N$ for any $k > 0$.

C3) There exists a measurable function $v$ on $\partial\Omega$, such that $u_n \longrightarrow v$ a.e. in $\partial\Omega$.

The function $v$ is the trace of $u$ in the generalized sense.

In the sequel, the trace of $u \in \mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. Moreover, if $u \in W^{1,p(\cdot)}(\Omega)$ then $tr(u)$ coincides with $\tau(u)$ in the usual sense.
Furthermore, for any $u \in \mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ we have $\tau(T_k(u)) = T_k(tr(u))$ for every $k > 0$, and if $\varphi \in W^{1,p(\cdot)}(\Omega)$ then $(u-\varphi) \in \mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ and $tr(u-\varphi) = tr(u) - \tau(\varphi)$.

## 3 Essential assumptions

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we consider the non-coercive quasilinear elliptic equation

$$\begin{cases} -\mathrm{div}\Big(\dfrac{a(x,|\nabla u|)\nabla u}{(1+|u|)^{\lambda(x)}}\Big) + |u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ \dfrac{a(x,|\nabla u|)\nabla u}{(1+|u|)^{\lambda(x)}}.\eta = g & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\eta$ is the outer unit normal vector on $\partial\Omega$. We assume that the exponents $p(\cdot) \in C_+(\overline{\Omega})$ and $0 \leq \lambda(x) < \min(1, p(x)-1)$. Moreover, the data $g(x)$ belongs to $L^1(\partial\Omega)$.
The Carathéodory function $a(x,|s|) : \Omega \times I\!R^N \longmapsto I\!R^N$ satisfies the following conditions:

$$|a(x,|s|)s| \leq a_1(x) + a_2|s|^{p(x)-1} \qquad \text{with} \quad a_1(x) \in L^{p'(\cdot)}(\Omega) \quad \text{and} \quad a_2 > 0 \tag{3.2}$$

and

$$\min\Big\{a(x,|s|), a(x,|s|) + s\frac{\partial a(x,|s|)}{\partial s}\Big\} \geq a_3|s|^{p(x)-2} \qquad \text{with} \quad a_3 > 0 \tag{3.3}$$

and

$$t^2 a(x,t) \leq p(x)\int_0^t sa(x,s)ds. \tag{3.4}$$

The Carathéodory function $f(x,s)$ satisfies only the growth condition

$$|f(x,s)| \leq f_0(x) + c(x)|s|^{\gamma(x)}, \tag{3.5}$$

where $f_0(x)$ is a measurable function in $L^1(\Omega)$ and $c(x) \in L^{m(x)}(\Omega)$ with $0 \leq \gamma(x) < p(x) - 1$ and $m(x) = \dfrac{p(x)-1}{p(x)-\gamma(x)-1}$ for a.e. $x \in \Omega$.
Thanks to the following two lemmas (see [8]), the operator $a(x,|\xi|)\xi$ is strictly monotone and pseudo-monotone.

**Lemma 3.1.** *Assume that (3.2) − (3.4) hold true. Then there exists a positive constant $a_3$ such that : for all $u, v \in \mathbb{R}^N$ with $(u,v) \neq (0,0)$, we have:*

$$\begin{aligned} &(a(x,|u|)u - a(x,|v|)v)(u-v) \geq a_3 4^{2-p_+}|u-v|^{p(x)} && \text{if} \quad p(x) \geq 2, \\ &(a(x,|u|)u - a(x,|v|)v)(u-v) \geq a_3(|u|+|v|)^{p(x)-2}|u-v|^2 && \text{if} \quad 1 < p(x) < 2. \end{aligned} \tag{3.6}$$

**Lemma 3.2.** *Assume that (3.2) − (3.4) hold true. Let $(u_n)_n$ be a sequence in $W^{1,p(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p(\cdot)}(\Omega)$ and*

$$\begin{aligned} &\int_\Omega (a(x,|\nabla u_n|)\nabla u_n - a(x,|\nabla u|)\nabla u)\cdot(\nabla u_n - \nabla u)\,dx \\ &\quad + \int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)\,dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty, \end{aligned} \tag{3.7}$$

*then $u_n \to u$ strongly in $W^{1,p(\cdot)}(\Omega)$.*

Now, we introduce the notion of entropy solution for the problem (3.1).

**Definition 3.3.** A measurable function $u$ is an entropy solution of the non-coercive quasilinear elliptic problem (3.1) if $u \in \mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$, with $f(x,u) \in L^1(\Omega)$ and $|u|^{p(x)-2}u \in L^1(\Omega)$, such that $u$ satisfies the inequality

$$\begin{aligned}&\int_\Omega \frac{a(x,|\nabla u|)\nabla u}{(1+|u|)^{\lambda(x)}} \cdot \nabla T_k(u-\varphi)\,dx + \int_\Omega |u|^{p(x)-2}uT_k(u-\varphi)\,dx \\ &\leq \int_\Omega f(x,u)T_k(u-\varphi)\,dx + \int_{\partial\Omega} gT_k(u-\varphi)\,d\sigma,\end{aligned} \tag{3.8}$$

for any $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

**Theorem 3.4.** *Assuming that the conditions (3.2) – (3.5) hold true, then the problem quasilinear elliptic (3.1) has at least one entropy solution.*

**Remark 3.5.** Note that the result of the theorem 3.4 remains true even in the case of $N \leq p(x) < \infty$.

## Proof of Theorem 3.4

### Step 1: Approximate problem

We define $f_n(x,u) = T_n(f(x,u))$ and $g_n(x) = T_n(g(x))$, we consider the following approximate problem

$$\begin{cases} -\mathrm{div}\Big(\dfrac{a(x,|\nabla u_n|)\nabla u_n}{(1+|T_n(u_n)|)^{\lambda(x)}}\Big) + |u_n|^{p(x)-2}u_n = f_n(x,u_n) & \text{in} \quad \Omega, \\ \dfrac{a(x,|\nabla u_n|)\nabla u_n}{(1+|T_n(u_n)|)^{\lambda(x)}} \cdot \eta = g_n(x) & \text{on} \quad \partial\Omega. \end{cases} \tag{3.9}$$

We consider the operator $B_n : W^{1,p(\cdot)}(\Omega) \longmapsto (W^{1,p(\cdot)}(\Omega))'$ defined by:

$$\begin{aligned}\langle B_n u, v\rangle \quad &= \int_\Omega \frac{a(x,|\nabla u|)\nabla u}{(1+|T_n(u)|)^{\lambda(x)}} \cdot \nabla v\,dx + \int_\Omega |u|^{p(x)-2}uv\,dx \\ &\quad - \int_\Omega f_n(x,u)v\,dx - \int_{\partial\Omega} g_n(x)v\,d\sigma \qquad \forall u,v \in W^{1,p(\cdot)}(\Omega).\end{aligned} \tag{3.10}$$

**Lemma 3.6.** *The bounded operator $B_n$ defined from $W^{1,p(\cdot)}(\Omega)$ into its dual $(W^{1,p(\cdot)}(\Omega))'$, is pseudo-monotone and coercive in the following sense :*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,p(\cdot)}} \longrightarrow \infty \quad as \quad \|v\|_{1,p(\cdot)} \to \infty \quad for \quad v \in W^{1,p(\cdot)}(\Omega). \tag{3.11}$$

### Proof of Lemma 3.6

Firstly, we show that the operator $B_n$ is bounded. In view of Hölder's inequality and (3.2), we have:

$$\begin{aligned}|\langle B_n u, v\rangle| \quad &\leq \int_\Omega \frac{|a(x,|\nabla u|)||\nabla u|}{(1+|T_n(u)|)^{\lambda(x)}}|\nabla v|\,dx + \int_\Omega |u|^{p(x)-1}|v|\,dx \\ &\quad + \int_\Omega |f_n(x,u)||v|\,dx + \int_{\partial\Omega} |g_n||v|\,d\sigma \\ &\leq 2\|a(x,|\nabla u|)\nabla u\|_{p'(\cdot)}\|\nabla v\|_{p(\cdot)} + 2\||u|^{p(x)-1}\|_{p'(\cdot)}\|v\|_{p(\cdot)} \\ &\quad + n\|v\|_{L^1(\Omega)} + n\|v\|_{L^1(\partial\Omega)} \\ &\leq 2\Big(\|a_1(x) + a_2|\nabla u|^{p(x)-1}\|_{p'(\cdot)}\Big)\|\nabla v\|_{p(\cdot)} + 2\||u|^{p(x)-1}\|_{p'(\cdot)}\|v\|_{p(\cdot)} \\ &\quad + nC_0\|v\|_{1,p(\cdot)} \\ &\leq C_1\Big(\|a_1(x)\|_{p'(\cdot)} + a_2\||\nabla u|^{p(\cdot)-1}\|_{p'(\cdot)} + \||u|^{p(\cdot)-1}\|_{p'(\cdot)} + n\Big)\|v\|_{1,p(\cdot)} \\ &\leq C_2\big(1 + \|u\|_{1,p(\cdot)}^{p_+-1}\big)\|v\|_{1,p(\cdot)}.\end{aligned} \tag{3.12}$$

For the coercivity, thanks to (3.3) we obtain

$$\begin{aligned}\langle B_n u, u\rangle &= \int_\Omega \frac{a(x,|\nabla u|)|\nabla u|^2}{(1+|T_n(u)|)^{\lambda(x)}}\,dx + \int_\Omega |u|^{p(x)}\,dx - \int_{\partial\Omega} g_n u\,d\sigma - \int_\Omega f_n(x,u)u\,dx \\ &\geq \int_\Omega a_3\frac{|\nabla u|^{p(x)}}{(1+n)^{\lambda_+}}\,dx + \int_\Omega |u|^{p(x)}\,dx - \int_{\partial\Omega}|g_n||u|\,d\sigma - \int_\Omega |f_n(x,u)||u|\,dx \\ &\geq \frac{a_3}{(1+n)^{\lambda_+}}\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx - n\|u\|_{L^1(\partial\Omega)} - n\|u\|_{L^1(\Omega)} \\ &\geq min\left\{\frac{a_3}{(1+n)^{\lambda_+}},1\right\}\rho_{1,p(\cdot)}(u) - nC_3\|u\|_{1,p(\cdot)} \\ &\geq min\left\{\frac{a_3}{(1+n)^{\lambda_+}},1\right\}(\|u\|^{p_-}_{1,p(\cdot)} - 1) - nC_3\|u\|_{1,p(\cdot)} \quad \text{for any} \quad u\in W^{1,p(\cdot)}(\Omega).\end{aligned} \tag{3.13}$$

It follows that

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,p(\cdot)}} \longrightarrow \infty \quad \text{as} \quad \|v\|_{1,p(\cdot)}\to\infty \quad \text{for} \quad v\in W^{1,p(\cdot)}(\Omega).$$

Now, it remains to show that $B_n$ is pseudo-monotone : assuming that

$$\begin{cases} u_k \rightharpoonup u \quad \text{weakly in } W^{1,p(\cdot)}(\Omega), \\ B_n u_k \rightharpoonup \chi_n \quad \text{weakly in } (W^{1,p(\cdot)}(\Omega))', \\ \limsup\limits_{k\to\infty}\langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle \quad \text{in } W^{1,p(\cdot)}(\Omega). \end{cases} \tag{3.14}$$

We will prove that $B_n u = \chi_n$ and $\langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle$ as $k\to\infty$.

In view of (3.2), we have $\left(\frac{a(x,|\nabla u_k|)\nabla u_k}{(1+|T_n(u_k)|)^{\lambda(x)}}\right)_k$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, then there exists a measurable function $\varphi_n$ in $(L^{p'(\cdot)}(\Omega))^N$ such that

$$a(x,|\nabla u_k|)\nabla u_k \rightharpoonup \varphi_n \quad \text{weakly in} \quad (L^{p'(\cdot)}(\Omega))^N \quad \text{as} \quad k\to\infty. \tag{3.15}$$

Moreover, in view of (3.14) and the compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega)$ we have $u_k \longrightarrow u$ strongly in $L^{p(\cdot)}(\Omega)$, then

$$|u_k|^{p(x)-2}u_k \longrightarrow |u|^{p(x)-2}u \quad \text{strongly in} \quad L^{p'(\cdot)}(\Omega) \quad \text{as} \quad k\to\infty. \tag{3.16}$$

Furthermore, we have $|f_n(x,u_k)|\leq n$ and $f_n(x,u_k)\to f_n(x,u)$ almost everywhere in $\Omega$, in view of Lebesgue's dominated convergence theorem, we get

$$f_n(x,u_k) \longrightarrow f_n(x,u) \quad \text{strongly in} \quad L^{p'(\cdot)}(\Omega) \quad \text{as} \quad k\to\infty. \tag{3.17}$$

Since $g_n\in L^\infty(\partial\Omega)\subseteq L^{p'(\cdot)}(\partial\Omega)$, then we have:

$$\int_{\partial\Omega} g_n(x)u_k\,d\sigma \longrightarrow \int_{\partial\Omega} g_n(x)u\,d\sigma \quad \text{as} \quad k\to\infty. \tag{3.18}$$

Thus, for any $v\in W^{1,p(\cdot)}(\Omega)$ we have

$$\begin{aligned}\lim_{k\to\infty}\langle B_n u_k, v\rangle &= \lim_{k\to\infty}\int_\Omega \frac{a(x,|\nabla u_k|)\nabla u_k}{(1+|T_n(u_k)|)^{\lambda(x)}}\nabla v\,dx + \lim_{k\to\infty}\int_\Omega |u_k|^{p(x)-2}u_k v\,dx \\ &- \lim_{k\to\infty}\int_\Omega f_n(x,u_k)v\,dx - \lim_{k\to\infty}\int_{\partial\Omega} g_n v\,d\sigma \\ &= \int_\Omega \varphi_n\nabla v\,dx + \int_\Omega |u|^{p(x)-2}uv\,dx - \int_\Omega f_n(x,u)v\,dx - \int_{\partial\Omega} g_n v\,d\sigma.\end{aligned} \tag{3.19}$$

By combining (3.14) and (3.19), we conclude that

$$\begin{aligned}\limsup_{k\longrightarrow\infty}\langle B_n u_k, u_k\rangle &\leq \limsup_{k\longrightarrow\infty}\Big(\int_\Omega \frac{a(x,|\nabla u_k|)|\nabla u_k|^2}{(1+|T_n(u_k)|)^{\lambda(x)}}\,dx+\int_\Omega |u_k|^{p(x)}\,dx\\ &\quad -\int_\Omega f_n(x,u_k)u_k\,dx-\int_{\partial\Omega} g_n u_k\,d\sigma\Big)\\ &\leq \int_\Omega \varphi_n\nabla u\,dx+\int_\Omega |u|^{p(x)}\,dx-\int_\Omega f_n(x,u)u\,dx-\int_{\partial\Omega} g_n u\,d\sigma.\end{aligned} \tag{3.20}$$

In view of (3.16), (3.17) and (3.18) we have

$$\limsup_{k\longrightarrow\infty}\int_\Omega \frac{a(x,|\nabla u_k|)|\nabla u_k|^2}{(1+|T_n(u_k)|)^{\lambda(x)}}\,dx\leq\int_\Omega \varphi_n\nabla u\,dx. \tag{3.21}$$

On the one hand, according to Lemma 3.1, we have

$$(a(x,|\nabla u_k|)\nabla u_k-a(x,|\nabla u|)\nabla u)(\nabla u_k-\nabla u)\geq 0 \qquad \text{almost everywere in} \quad \Omega. \tag{3.22}$$

Then

$$\int_\Omega\Big(\frac{a(x,|\nabla u_k|)\nabla u_k}{(1+|T_n(u_k)|)^{\lambda(x)}}-\frac{a(x,|\nabla u|)\nabla u}{(1+|T_n(u_k)|)^{\lambda(x)}}\Big)(\nabla u_k-\nabla u)\,dx\geq 0. \tag{3.23}$$

It follows that

$$\int_\Omega \frac{a(x,|\nabla u_k|)|\nabla u_k|^2}{(1+|T_n(u_k)|)^{\lambda(x)}}\,dx\geq\int_\Omega \frac{a(x,|\nabla u_k|)\nabla u_k\nabla u}{(1+|T_n(u_k)|)^{\lambda(x)}}\,dx+\int_\Omega \frac{a(x,|\nabla u|)\nabla u}{(1+|T_n(u_k)|)^{\lambda(x)}}(\nabla u_k-\nabla u)\,dx. \tag{3.24}$$

We have $\dfrac{a(x,|\nabla u|)\nabla u}{(1+|T_n(u_k)|)^{\lambda(x)}}\to\dfrac{a(x,|\nabla u|)\nabla u}{(1+|T_n(u)|)^{\lambda(x)}}$ strongly in $(L^{p'(\cdot)}(\Omega))^N$, and since $\nabla u_k-\nabla u\rightharpoonup 0$ weakly in $(L^{p(\cdot)}(\Omega))^N$, then

$$\int_\Omega \frac{a(x,|\nabla u|)\nabla u}{(1+|T_n(u_k)|)^{\lambda(x)}}(\nabla u_k-\nabla u)\,dx\longrightarrow 0 \qquad \text{as} \quad k\to\infty.$$

Thanks to (3.21) we obtain

$$\liminf_{k\longrightarrow\infty}\int_\Omega \frac{a(x,|\nabla u_k|)|\nabla u_k|^2}{(1+|T_n(u_k)|)^{\lambda(x)}}\,dx\geq\int_\Omega \varphi_n\nabla u\,dx. \tag{3.25}$$

In view of (3.21) and (3.25), we conclude that

$$\lim_{k\to\infty}\int_\Omega \frac{a(x,|\nabla u_k|)|\nabla u_k|^2}{(1+|T_n(u_k)|)^{\lambda(x)}}\,dx=\int_\Omega \varphi_n\nabla u\,dx. \tag{3.26}$$

By combining (3.16) − (3.18) and (3.26), we deduce that

$$\langle B_n u_k, u_k\rangle\longrightarrow\langle\chi_n, u\rangle \qquad \text{as} \quad k\to\infty. \tag{3.27}$$

On the other hand, we have $u_k\longrightarrow u$ strongly in $L^{p(\cdot)}(\Omega)$, and in view of (3.23) and (3.26) we obtain

$$\begin{aligned}\lim_{k\longrightarrow\infty}\Big(&\int_\Omega (a(x,|\nabla u_k|)\nabla u_k-a(x,|\nabla u|)\nabla u))(\nabla u_k-\nabla u)\,dx\\ &+\int_\Omega (|u_k|^{p(x)-2}u_k-|u|^{p(x)-2}u)(u_k-u)\,dx\Big)\longrightarrow 0 \qquad \text{as} \quad k\to\infty.\end{aligned} \tag{3.28}$$

In view of Lemma 3.2, we deduce that

$$u_k\longrightarrow u \quad \text{strongly in} \quad W^{1,p(\cdot)}(\Omega) \quad \text{and} \quad \nabla u_k\longrightarrow\nabla u \quad \text{a.e. in} \quad \Omega. \tag{3.29}$$

Moreover, we have $\left(\frac{a(x,|\nabla u_k|)\nabla u_k}{(1+|T_n(u_k)|)^{\lambda(x)}}\right)_n$ is uniformly bounded in $(L^{p'(\cdot)}(\Omega))^N$. and since $\frac{a(x,|\nabla u_k|)\nabla u_k}{(1+|T_n(u_k)|)^{\lambda(x)}} \to \frac{a(x,|\nabla u|)\nabla u}{(1+|T_n(u)|)^{\lambda(x)}}$ almost everywhere in $\Omega$, we conclude that

$$\frac{a(x,|\nabla u_k|)\nabla u_k}{(1+|T_n(u_k)|)^{\lambda(x)}} \rightharpoonup \frac{a(x,|\nabla u|)\nabla u}{(1+|T_n(u)|)^{\lambda(x)}} \qquad \text{weakly in} \quad (L^{p'(\cdot)}(\Omega))^N. \tag{3.30}$$

Having in mind (3.16) – (3.18), we deduce that $\chi_n = B_n u$ which completes the proof of Lemma 3.6.
According to Lemma 3.6 (cf. [20], Theorem 8.2), there exists at least one weak solution $u_n \in W^{1,p(\cdot)}(\Omega)$ for the approximate problem (3.9) i.e. :

$$\begin{aligned}&\int_\Omega \frac{a(x,|\nabla u_n|)\nabla u_n}{(1+|T_n(u_n)|)^{\lambda(x)}}\nabla v\,dx + \int_\Omega |u_n|^{p(x)-2}u_n v\,dx\\ &= \int_\Omega f_n(x,u_n)v\,dx + \int_{\partial\Omega} g_n v\,d\sigma \qquad \text{for any } v \in W^{1,p(\cdot)}(\Omega).\end{aligned} \tag{3.31}$$

**Step 2: A priori estimate**

Let $n \geq k \geq 1$, by choosing $T_k(u_n)$ as a test function for the approximate problem (3.31), we have

$$\begin{aligned}&\int_\Omega \frac{a(x,|\nabla u_n|)\nabla u_n}{(1+|T_n(u_n)|)^{\lambda(x)}}\cdot\nabla T_k(u_n)\,dx + \int_\Omega |u_n|^{p(x)-2}u_n T_k(u_n)\,dx\\ &= \int_\Omega f_n(x,u_n)T_k(u_n)\,dx + \int_{\partial\Omega} g_n T_k(u_n)\,d\sigma.\end{aligned} \tag{3.32}$$

In view of (3.3) and (3.5), we obtain

$$\begin{aligned}&a_3\int_{\{|u_n|\leq k\}} \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\lambda(x)}}\,dx + \int_\Omega |u_n|^{p(x)-1}|T_k(u_n)|\,dx\\ &\leq \int_\Omega |f_n(x,u_n)||T_k(u_n)|\,dx + \int_{\partial\Omega}|g_n||T_k(u_n)|\,d\sigma\\ &\leq k\|f_0\|_{L^1(\Omega)} + \int_\Omega |c(x)||u_n|^{\gamma(x)}|T_k(u_n)|\,dx + k\|g\|_{L^1(\partial\Omega)}\\ &\leq C_0 k + \int_\Omega |c(x)||u_n|^{\gamma(x)}|T_k(u_n)|\,dx.\end{aligned} \tag{3.33}$$

We have $0 \leq \gamma(x) < p(x) - 1$, and in view of Young's inequality we get

$$\int_\Omega c(x)|u_n|^{\gamma(x)}|T_k(u_n)|\,dx \leq C_1\int_\Omega |c(x)|^{\frac{p(x)-1}{p(x)-\gamma(x)-1}}|T_k(u_n)|\,dx + \frac{1}{2}\int_\Omega |u_n|^{p(x)-1}|T_k(u_n)|\,dx, \tag{3.34}$$

with $c(x) \in L^{\frac{p(x)-1}{p(x)-1-\gamma(x)}}(\Omega)$. It follows that

$$a_3\int_\Omega \frac{|\nabla T_k(u_n)|^{p(x)}}{(1+|T_k(u_n)|)^{\lambda(x)}}\,dx + \frac{1}{2}\int_\Omega |u_n|^{p(x)-1}|T_k(u_n)|\,dx \leq C_2 k. \tag{3.35}$$

Then, we obtain

$$\frac{a_3}{(1+k)^{\lambda_+}}\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx + \frac{1}{2}\int_\Omega |T_k(u_n)|^{p(x)}\,dx \leq C_2 k. \tag{3.36}$$

We conclude that

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx + \int_\Omega |T_k(u_n)|^{p(x)}\,dx \leq C_3 k^{1+\lambda_+} \qquad \text{for any} \quad k \geq 1. \tag{3.37}$$

Hence

$$\|T_k(u_n)\|_{1,p(\cdot)} \leq C_3 k^{\frac{1+\lambda_+}{p_-}} \qquad \text{for any} \quad k \geq 1. \tag{3.38}$$

Then the sequence $(T_k(u_n))_n$ is uniformly bounded in $W^{1,p(\cdot)}(\Omega)$, and there exists a measurable function $\eta_k \in W^{1,p(\cdot)}(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup \eta_k & \text{weakly in } W^{1,p(\cdot)}(\Omega), \\ T_k(u_n) \to \eta_k & \text{strongly in } L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{3.39}$$

Moreover, in view of (3.36) we obtain

$$\begin{aligned} k^{p_-} \text{ meas}\left(\{|u_n| > k\}\right) &= \int_{\{|u_n|>k\}} |T_k(u_n)|^{p_-} \, dx \\ &\leq \int_{\Omega} |T_k(u_n)|^{p(x)} \, dx \\ &\leq 2C_2 k. \end{aligned} \tag{3.40}$$

We deduce that

$$\text{meas}\left(\{|u_n| > k\}\right) \leq \frac{2C_2 k}{k^{p_-}} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{3.41}$$

Now, we will show that the sequence $(u_n)_n$ is a Cauchy sequence in measure. Indeed, we have for every $\delta > 0$,

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \delta\} \leq\ & \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ & + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned}$$

Let $\varepsilon > 0$, in view of (3.41) there exists a positive constant $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{3.42}$$

Moreover, thanks to (3.39) we get

$$T_k(u_n) \to \eta_k \quad \text{strongly in} \quad L^{p(\cdot)}(\Omega) \quad \text{and a.e. in} \quad \Omega.$$

Consequently, the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, thus : for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \qquad \text{for all} \qquad n, m \geq n_0(k, \delta, \varepsilon). \tag{3.43}$$

By combining (3.42) and (3.43), we deduce that: for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon \qquad \text{for any} \qquad n, m \geq n_0.$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function $u$. Thanks to (3.39) we obtain

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W^{1,p(\cdot)}(\Omega), \\ T_k(u_n) \to T_k(u) & \text{strongly in } L^1(\Omega) \text{ and a.e. in } \Omega, \\ T_k(u_n) \to T_k(u) & \text{strongly in } L^1(\partial\Omega) \text{ and a.e. in } \partial\Omega. \end{cases} \tag{3.44}$$

Moreover, in view of (3.35) we get

$$\int_{\{|u_n|>k\}} |u_n|^{p(x)-1} \, dx \leq 2C_2 \qquad \text{for any} \quad k \geq 1. \tag{3.45}$$

Thus, we conclude that

$$\int_{\Omega} |u_n|^{p(x)-1} \, dx \leq C_4, \tag{3.46}$$

with $C_4$ is a positive constant that doesn't depend on $k$ and $n$.

**Step 3: The equi-integrability of $(|u_n|^{p(x)-1})_n$ and $(f_n(x,u_n))_n$.**

Let $n \geq 2h > 0$, by taking $\frac{1}{h}(T_{2h}(u_n) - T_h(u_n))$ as a test function for the approximate problem (3.31), we obtain

$$\begin{aligned} &\frac{1}{h}\int_{\{h<|u_n|\leq 2h\}} \frac{a(x,|\nabla u_n|)|\nabla u_n|^2}{(1+|u_n|)^{\lambda(x)}}\,dx + \frac{1}{h}\int_\Omega |u_n|^{p(x)-1}|T_{2h}(u_n) - T_h(u_n)|\,dx \\ &\leq \frac{1}{h}\int_\Omega |f_n(x,u_n)|\,|T_{2h}(u_n) - T_h(u_n)|\,dx + \frac{1}{h}\int_{\partial\Omega} |g_n|\,|T_{2h}(u_n) - T_h(u_n)|\,d\sigma. \end{aligned} \tag{3.47}$$

In view of Young's inequality, we conclude that

$$\begin{aligned} &\frac{1}{h}\int_{\{h<|u_n|\leq 2h\}} \frac{a(x,|\nabla T_{2h}(u_n)|)|\nabla T_{2h}(u_n)|^2}{(1+|T_{2h}(u_n)|)^{\lambda(x)}}\,dx + \frac{1}{h}\int_\Omega |u_n|^{p(x)-1}|T_{2h}(u_n) - T_h(u_n)|\,dx \\ &\leq \frac{1}{h}\int_\Omega |f_0(x)|\,|T_{2h}(u_n) - T_h(u_n)|\,dx + \frac{1}{h}\int_\Omega c(x)|u_n|^{\gamma(x)}\,|T_{2h}(u_n) - T_h(u_n)|\,dx \\ &\qquad + \frac{1}{h}\int_{\partial\Omega} |g_n|\,|T_{2h}(u_n) - T_h(u_n)|\,d\sigma \\ &\leq \frac{1}{h}\int_\Omega (|f_0(x)| + |c(x)|^{\frac{p(x)-1}{p(x)-1-\gamma(x)}})\,|T_{2h}(u_n) - T_h(u_n)|\,dx \\ &\qquad + \frac{1}{h}\int_{\partial\Omega} |g(x)|\,|T_{2h}(u_n) - T_h(u_n)|\,d\sigma \\ &\qquad + \frac{1}{2h}\int_\Omega |u_n|^{p(x)-1}\,|T_{2h}(u_n) - T_h(u_n)|\,dx. \end{aligned} \tag{3.48}$$

It follows that

$$\begin{aligned} &\frac{1}{h}\int_{\{h<|u_n|\leq 2h\}} \frac{a(x,|\nabla T_{2h}(u_n)|)|\nabla T_{2h}(u_n)|^2}{(1+|T_{2h}(u_n)|)^{\lambda(x)}}\,dx + \frac{1}{2}\int_{\{2h<|u_n|\}} |u_n|^{p(x)-1}\,dx \\ &\leq \int_{\{h<|u_n|\}} \left(|f_0(x)| + |c(x)|^{\frac{p(x)-1}{p(x)-1-\gamma(x)}}\right)dx + \int_{\partial\Omega\cap\{h<|u_n|\}} |g|\,d\sigma \longrightarrow 0 \qquad \text{as} \quad h\to\infty. \end{aligned} \tag{3.49}$$

Thus, we conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{h<|u_n|\leq 2h\}} \frac{a(x,|\nabla T_{2h}(u_n)|)|\nabla T_{2h}(u_n)|^2}{(1+|T_{2h}(u_n)|)^{\lambda(x)}}\,dx = 0 \tag{3.50}$$

and

$$\lim_{h\to\infty}\int_{\{2h<|u_n|\}} |u_n|^{p(x)-1}\,dx = 0. \tag{3.51}$$

Firstly, we will show that the sequence $(|u_n|^{p(x)-1})_n$ is uniformly equi-integrable.
Let $h > 0$, for any measurable subset $E \subset \Omega$, we have

$$\int_E |u_n|^{p(x)-1}\,dx \leq \int_{\{h<|u_n|\}} |u_n|^{p(x)-1}\,dx + \int_E |T_h(u_n)|^{p(x)-1}\,dx. \tag{3.52}$$

Thanks to (3.51), for all $\varepsilon > 0$, there exists $h_0(\varepsilon) > 0$ such that

$$\int_{\{h<|u_n|\}} |u_n|^{p(x)-1}\,dx \leq \frac{\varepsilon}{2} \qquad \text{for any} \quad h \geq h_0(\varepsilon). \tag{3.53}$$

On the other hand, there exists $\delta(\varepsilon,h) > 0$ small enough such that:

$$\int_E |T_h(u_n)|^{p(x)-1}\,dx \leq \frac{\varepsilon}{2} \qquad \text{for any} \quad E\subset\Omega \quad \text{with} \quad \text{meas}(E) \leq \delta(\varepsilon,h). \tag{3.54}$$

By combining (3.52) and (3.53) $-$ (3.54), we conclude that

$$\int_E |u_n|^{p(x)-1}\,dx \leq \varepsilon \qquad \text{for any} \quad E\subset\Omega \quad \text{with} \quad \text{meas}(E) \leq \delta(\varepsilon). \tag{3.55}$$

Thus, the sequence $(|u_n|^{p(x)-1})_n$ is uniformly equi-integrable, and since $|u_n|^{p(x)-1} \longrightarrow |u|^{p(x)-1}$ a.e. in $\Omega$. In view of Vitali's Theorem, we conclude that

$$|u_n|^{p(x)-1} \longrightarrow |u|^{p(x)-1} \qquad \text{strongly in} \quad L^1(\Omega). \tag{3.56}$$

Moreover, we have for any $\varepsilon > 0$ there exists $\beta(\varepsilon) > 0$ such that

$$\begin{aligned} \int_E |f_n(x,u_n)|\ dx &\leq \int_E |f_0(x)|\ dx + \int_E c(x)|u_n|^{\gamma(x)}\ dx \\ &\leq \int_E (|f_0(x)| + |c(x)|^{\frac{p(x)-1}{p(x)-1-\gamma(x)}})\ dx + \int_E |u_n|^{p(x)-1} \\ &\leq \varepsilon, \end{aligned} \tag{3.57}$$

for any $E \subset \Omega$ with meas$(E) < \beta(\varepsilon)$.
Thus, the sequence $(f_n(x,u_n))_n$ is uniformly equi-integrable, and since $f_n(x,u_n) \to f(x,u)$ almost everywhere in $\Omega$. Using the Vitali's theorem, we deduce that

$$f_n(x,u_n) \longrightarrow f(x,u) \quad \text{ strongly in } L^1(\Omega). \tag{3.58}$$

**Step 4: Convergence almost everywhere of gradients**

In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \ldots$, various real-valued functions of real variables that converges to 0 as $n$ tends to infinity. Similarly, we define $\varepsilon_i(h)$, and $\varepsilon_i(n,h)$.
Let $h > k > 0$, and we consider the function $S_h(\cdot) : \mathbb{R} \longmapsto \mathbb{R}$ defined by:

$$S_h(r) = 1 - \frac{|T_{2h}(r) - T_h(r)|}{h}.$$

By choosing $S_h(u_n)(T_k(u_n) - T_k(u))$ as a test function for the approximate problem (3.31), and since $0 \leq S_h(r) \leq 1$ for any $r \in \mathbb{R}$ we get

$$\begin{aligned} &\int_\Omega \frac{a(x,|\nabla u_n|)\nabla u_n}{(1+|T_n(u_n)|)^{\lambda(x)}} \cdot (\nabla T_k(u_n) - \nabla T_k(u)) S_h(u_n)\ dx \\ &\quad - \frac{1}{h} \int_{\{h \leq |u_n| \leq 2h\}} \frac{a(x,|\nabla u_n|)|\nabla u_n|^2}{(1+|T_n(u_n)|)^{\lambda(x)}} |T_k(u_n) - T_k(u)|\ dx \\ &\leq \int_\Omega |u_n|^{p(x)-1} S_h(u_n) |T_k(u_n) - T_k(u)|\ dx + \int_\Omega |f_n(x,u_n)| S_h(u_n) |T_k(u_n) - T_k(u)|\ dx \\ &\quad + \int_{\partial\Omega} |g_n| S_h(u_n) |T_k(u_n) - T_k(u)|\ d\sigma \\ &\leq \int_\Omega |u_n|^{p(x)-1} |T_k(u_n) - T_k(u)|\ dx + \int_\Omega |f_n(x,u_n)| |T_k(u_n) - T_k(u)|\ dx \\ &\quad + \int_{\partial\Omega} |g_n| |T_k(u_n) - T_k(u)|\ d\sigma. \end{aligned} \tag{3.59}$$

For the first term on the right-hand side of (3.59), thanks to (3.56) we have $|u_n|^{p(x)-1} \longrightarrow |u|^{p(x)-1}$ strongly in $L^1(\Omega)$, and since $T_k(u_n) - T_k(u) \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$, then

$$\varepsilon_1(n) = \int_\Omega |u_n|^{p(x)-1} |T_k(u_n) - T_k(u)|\ dx \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.60}$$

Similarly, in view of (3.58) we have $f_n(x,u_n) \longrightarrow f(x,u)$ strongly in $L^1(\Omega)$, then

$$\varepsilon_2(n) = \int_\Omega |f_n(x,u_n)| |T_k(u_n) - T_k(u)|\ dx \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.61}$$

Concerning the last term of the right-hand side of (3.59), we have $g_n(x) \to g(x)$ strongly in $L^1(\partial\Omega)$ and since $T_k(u_n) - T_k(u) \rightharpoonup 0$ weak$-*$ in $L^\infty(\partial\Omega)$, then

$$\varepsilon_3(n) = \int_{\partial\Omega} |g_n| |T_k(u_n) - T_k(u)|\ d\sigma \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.62}$$

By combining (3.59) and (3.60) – (3.62) we conclude that

$$\begin{aligned}
&\int_{\{|u_n|\le k\}} \frac{a(x,|\nabla u_n|)\nabla u_n}{(1+|T_n(u_n)|)^{\lambda(x)}}\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\\
&\quad+\int_{\{k<|u_n|\le 2h\}} \frac{a(x,|\nabla u_n|)\nabla u_n}{(1+|T_n(u_n)|)^{\lambda(x)}}\cdot(\nabla T_k(u_n)-\nabla T_k(u))S_h(u_n)\,dx\\
&\le \frac{1}{h}\int_{\{h<|u_n|\le 2h\}} \frac{a(x,|\nabla u_n|)|\nabla u_n|^2}{(1+|T_n(u_n)|)^{\lambda(x)}}|T_k(u_n)-T_k(u)|\,dx+\varepsilon_4(n).
\end{aligned}\tag{3.63}$$

It follows that

$$\begin{aligned}
&\int_\Omega\Big(\frac{a(x,|\nabla T_k(u_n)|)\nabla T_k(u_n)}{(1+|T_k(u_n)|)^{\lambda(x)}}-\frac{a(x,|\nabla T_k(u)|)\nabla T_k(u)}{(1+|T_k(u_n)|)^{\lambda(x)}}\Big)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\\
&\quad+\int_\Omega \frac{a(x,|\nabla T_k(u)|)\nabla T_k(u)}{(1+|T_k(u_n)|)^{\lambda(x)}}\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\\
&\quad-\int_{\{k<|u_n|\le 2h\}} \frac{a(x,|\nabla T_{2h}(u_n)|)\nabla T_{2h}(u_n)}{(1+|T_{2h}(u_n)|)^{\lambda(x)}}\cdot\nabla T_k(u)S_h(u_n)\,dx\\
&\le \frac{1}{h}\int_{\{h<|u_n|\le 2h\}} \frac{a(x,|\nabla u_n|)|\nabla u_n|^2}{(1+|T_n(u_n)|)^{\lambda(x)}}|T_k(u_n)-T_k(u)|\,dx+\varepsilon_4(n).
\end{aligned}\tag{3.64}$$

For the second term on the left-hand side of (3.64), In view of Lebesgue's dominated convergence theorem we have $\frac{a(x,|\nabla T_k(u)|)\nabla T_k(u)}{(1+|T_k(u_n)|)^{\lambda(x)}} \longrightarrow \frac{a(x,|\nabla T_k(u)|)\nabla T_k(u)}{(1+|T_k(u)|)^{\lambda(x)}}$ strongly in $(L^{p'(\cdot)}(\Omega))^N$, and since $\nabla T_k(u_n)\rightharpoonup \nabla T_k(u)$ weakly in $(L^{p(\cdot)}(\Omega))^N$, then

$$\varepsilon_6(n)=\int_\Omega \frac{a(x,|\nabla T_k(u)|)\nabla T_k(u)}{(1+|T_k(u_n)|)^{\lambda(x)}}\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\longrightarrow 0\quad\text{as}\quad n\to\infty.\tag{3.65}$$

Concerning the third term of the right-hand side of (3.64), the sequence

$$\Big(\frac{a(x,|\nabla T_{2h}(u_n)|)|\nabla T_{2h}(u_n)|}{(1+|T_{2h}(u_n)|)^{\lambda(x)}}\Big)_n$$

is uniformly bounded in $(L^{p'(\cdot)}(\Omega))^N$, then there exists a measurable function $\varphi_{2h}\in(L^{p'(\cdot)}(\Omega))^N$ such that $\frac{a(x,|\nabla T_{2h}(u_n)|)|\nabla T_{2h}(u_n)|}{(1+|T_{2h}(u_n)|)^{\lambda(x)}}\rightharpoonup\varphi_{2h}$ weakly in $(L^{p'(\cdot)}(\Omega))^N$, and since $\nabla T_k(u)S_h(u_n)\to\nabla T_k(u)S_h(u)$ strongly in $(L^{p'(\cdot)}(\Omega))^N$, it follows that

$$\begin{aligned}
\varepsilon_7(n)&=\int_{\{k<|u_n|\le 2h\}} \frac{a(x,|\nabla T_{2h}(u_n)|)\nabla T_{2h}(u_n)}{(1+|T_{2h}(u_n)|)^{\lambda(x)}}\cdot\nabla T_k(u)S_h(u_n)\,dx\\
&\longrightarrow\int_{\{k<|u|\le 2h\}}\varphi_{2h}\cdot\nabla T_k(u)S_h(u)\,dx=0\qquad\text{as}\quad n\to\infty.
\end{aligned}\tag{3.66}$$

For the first term of the right-hand side of (3.64), thanks to (3.50) we have

$$\begin{aligned}
\varepsilon_8(h)&=\frac{1}{h}\int_{\{h<|u_n|\le 2h\}} \frac{a(x,|\nabla u_n|)|\nabla u_n|^2}{(1+|T_n(u_n)|)^{\lambda(x)}}|T_k(u_n)-T_k(u)|\,dx\\
&\le\frac{2k}{h}\int_{\{h<|u_n|\le 2h\}} \frac{a(x,|\nabla u_n|)|\nabla u_n|^2}{(1+|T_n(u_n)|)^{\lambda(x)}}\,dx\longrightarrow 0\quad\text{as}\quad n,h\to\infty.
\end{aligned}\tag{3.67}$$

By combining (3.64) and (3.65) – (3.67), we conclude that

$$\begin{aligned}
&\frac{1}{(1+k)^{\lambda_+}}\int_\Omega\Big(a(x,|\nabla T_k(u_n)|)\nabla T_k(u_n)-a(x,|\nabla T_k(u)|)\nabla T_k(u)\Big)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\\
&\le\int_\Omega\Big(\frac{a(x,|\nabla T_k(u_n)|)\nabla T_k(u_n)}{(1+|T_k(u_n)|)^{\lambda(x)}}-\frac{a(x,|\nabla T_k(u)|)\nabla T_k(u)}{(1+|T_k(u_n)|)^{\lambda(x)}}\Big)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx\\
&\le\varepsilon_9(n,h).
\end{aligned}\tag{3.68}$$

Having in mind that $T_k(u_n) \to T_k(u)$ strongly in $L^{p(\cdot)}(\Omega)$, we conclude that

$$\begin{aligned}&\int_\Omega \Big(a(x,|\nabla T_k(u_n)|)\nabla T_k(u_n) - a(x,|\nabla T_k(u)|)\nabla T_k(u)\Big)\cdot(\nabla T_k(u_n) - \nabla T_k(u))\,dx\\ &+\int_\Omega (|T_k(u_n)|^{p(x)-2}T_k(u_n) - |T_k(u)|^{p(x)-2}T_k(u))(T_k(u_n) - T_k(u))\,dx \longrightarrow 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{3.69}$$

Thanks to Lemma 3.2, we obtain

$$\begin{cases} T_k(u_n) \to T_k(u) & \text{strongly in} \quad W^{1,p(\cdot)}(\Omega),\\ \nabla u_n \to \nabla u & \text{a.e. in} \quad \Omega. \end{cases} \tag{3.70}$$

Moreover, we have $\dfrac{a(x,|\nabla T_h(u_n)|)|\nabla T_h(u_n)|^2}{(1+|T_n(u_n)|)^{\lambda(x)}}$ tends to $\dfrac{a(x,|\nabla T_h(u)|)|\nabla T_h(u)|^2}{(1+|u|)^{\lambda(x)}}$ almost everywhere in $\Omega$, and in view of Fatou's lemma and (3.50) we conclude that

$$\begin{aligned}&\lim_{h\to\infty}\frac{1}{h}\int_\Omega \frac{a(x,|\nabla T_h(u)|)|\nabla T_h(u)|^2}{(1+|u|)^{\lambda(x)}}\,dx\\ &\le \lim_{h\to\infty}\liminf_{n\to\infty}\frac{1}{h}\int_\Omega \frac{a(x,|\nabla T_h(u_n)|)|\nabla T_h(u_n)|^2}{(1+|T_n(u_n)|)^{\lambda(x)}}\,dx\\ &\le \lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_\Omega \frac{a(x,|\nabla T_h(u_n)|)|\nabla T_h(u_n)|^2}{(1+|T_n(u_n)|)^{\lambda(x)}}\,dx = 0.\end{aligned} \tag{3.71}$$

Moreover, we conclude that

$$\frac{a(x,|\nabla T_k(u_n)|)\nabla T_k(u_n)}{(1+|T_k(u_n)|)^{\lambda(x)}} \rightharpoonup \frac{a(x,|\nabla T_k(u)|)\nabla T_k(u)}{(1+|T_k(u)|)^{\lambda(x)}} \quad \text{weakly in} \quad (L^{p'(\cdot)}(\Omega))^N. \tag{3.72}$$

**Step 4: Passage to the limit**

Let $k>0$ and $\varphi \in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, we set $M = \|\varphi\|_\infty + k$.
By taking $T_k(u_n-\varphi)$ as a test function in the approximate problem (3.31), we have

$$\begin{aligned}&\int_\Omega \frac{a(x,|\nabla u_n|)\nabla u_n}{(1+|T_n(u_n)|)^{\lambda(x)}}\nabla T_k(u_n-\varphi)\,dx + \int_\Omega |u_n|^{p(x)-2}u_n T_k(u_n-\varphi)\,dx\\ &= \int_\Omega f_n(x,u_n)T_k(u_n-\varphi)\,dx + \int_{\partial\Omega} g_n(x)T_k(u_n-\varphi)\,d\sigma.\end{aligned} \tag{3.73}$$

For the first term on the left-hand side of (3.73). We have $\{|u_n-\varphi|\le k\}\subset\{|u_n|\le M\}$, and in view of Fatou's Lemma, we obtain

$$\begin{aligned}&\liminf_{n\to\infty}\int_\Omega \frac{a(x,|\nabla u_n|)\nabla u_n}{(1+|T_n(u_n)|)^{\lambda(x)}}\nabla T_k(u_n-\varphi)\,dx\\ &= \liminf_{n\to\infty}\int_{\{|u_n-\varphi|\le k\}} \frac{a(x,|\nabla T_M(u_n)|)\nabla T_M(u_n)}{(1+|T_M(u_n)|)^{\lambda(x)}}\cdot(\nabla T_M(u_n)-\nabla\varphi)\,dx\\ &= \liminf_{n\to\infty}\int_{\{|u_n-\varphi|\le k\}} \Big(\frac{a(x,|\nabla T_M(u_n)|)\nabla T_M(u_n)}{(1+|T_M(u_n)|)^{\lambda(x)}}\\ &\qquad -\frac{a(x,|\nabla\varphi|)\nabla\varphi}{(1+|T_M(u_n)|)^{\lambda(x)}}\Big)\cdot(\nabla T_M(u_n)-\nabla\varphi)\,dx\\ &\quad + \lim_{n\to\infty}\int_{\{|u_n-\varphi|\le k\}} \frac{a(x,|\nabla\varphi|)\nabla\varphi}{(1+|T_M(u_n)|)^{\lambda(x)}}\cdot(\nabla T_M(u_n)-\nabla\varphi)\,dx\\ &\ge \int_{\{|u-\varphi|\le k\}} \Big(\frac{a(x,|\nabla T_M(u)|)\nabla T_M(u)}{(1+|T_M(u)|)^{\lambda(x)}} - \frac{a(x,|\nabla\varphi|)\nabla\varphi}{(1+|T_M(u)|)^{\lambda(x)}}\Big)\cdot(\nabla T_M(u)-\nabla\varphi)\,dx\\ &\quad + \int_{\{|u-\varphi|\le k\}} \frac{a(x,|\nabla\varphi|)\nabla\varphi}{(1+|T_M(u)|)^{\lambda(x)}}\cdot(\nabla T_M(u)-\nabla\varphi)\,dx\\ &= \int_{\{|u-\varphi|\le k\}} \frac{a(x,|\nabla T_M(u)|)\nabla T_M(u)}{(1+|T_M(u)|)^{\lambda(x)}}\cdot(\nabla T_M(u)-\nabla\varphi)\,dx\\ &= \int_\Omega \frac{a(x,|\nabla u|)\nabla u}{(1+|u|)^{\lambda(x)}}\cdot\nabla T_k(u-\varphi)\,dx.\end{aligned} \tag{3.74}$$

Moreover, thanks to (3.56) we have $|u_n|^{p(x)-2}u_n \to |u|^{p(x)-2}u$ strongly in $L^1(\Omega)$, and since $T_k(u_n-\varphi)$ tends to $T_k(u-\varphi)$ weak $-*$ in $L^\infty(\Omega)$, then

$$\int_\Omega |u_n|^{p(x)-2}u_n T_k(u_n-\varphi)\,dx \longrightarrow \int_\Omega |u|^{p(x)-2}uT_k(u-\varphi)\,dx \qquad \text{as} \quad n\to\infty. \tag{3.75}$$

Similarly, using (3.58) we have $f_n(x,u_n)\to f(x,u)$ strongly in $L^1(\Omega)$, then

$$\int_\Omega f_n(x,u_n)T_k(u_n-\varphi)\,dx \longrightarrow \int_\Omega f(x,u)T_k(u-\varphi)\,dx. \tag{3.76}$$

For the second term on the right-hand side of (3.73), we have $g_n(x)\to g(x)$ strongly in $L^1(\partial\Omega)$, and since $T_k(u_n-\varphi)$ tends to $T_k(u-\varphi)$ weak $-*$ in $L^\infty(\partial\Omega)$, it follows that

$$\int_{\partial\Omega} g_n(x)T_k(u_n-\varphi)\,d\sigma \longrightarrow \int_{\partial\Omega} g(x)T_k(u-\varphi)\,d\sigma. \tag{3.77}$$

By combining (3.73) and (3.74) $-$ (3.77), we conclude that

$$\begin{aligned}&\int_\Omega \frac{a(x,|\nabla u|)\nabla u}{(1+|u|)^{\lambda(x)}}\nabla T_k(u-\varphi)\,dx + \int_\Omega |u|^{p(x)-2}uT_k(u-\varphi)\,dx\\ &\le \int_\Omega f(x,u)T_k(u-\varphi)\,dx + \int_{\partial\Omega} gT_k(u-\varphi)\,d\sigma,\end{aligned} \tag{3.78}$$

which conclude the proof of Theorem 3.4.

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