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HOSOYA POLYNOMIAL AND DISTANCE-BASED TOPOLOGICAL INDICES OF BANANA TREE GRAPHS AND THEIR DERIVED GRAPHS

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Abstract The Hosoya polynomial of a graph is a powerful invariant that encodes distance-based structural information and serves as a generating function for several key topological indices. This paper presents a comprehensive study of the Hosoya polynomial for the class of Banana tree graphs. We provide a complete analysis of the distance distribution in Banana tree graphs, enumerating the number of vertex pairs at each possible distance. This enumeration is then used to derive an explicit closed-form formula for the Hosoya polynomial. Furthermore, we leverage this polynomial to compute exact expressions for four significant distance-based topological indices: the Wiener index, the Hyper-Wiener index, the Tratch-Stankevitch-Zefirov index, and the Harary index. In addition, we extend our analysis to five derived graphs of the Banana tree graph—namely, the line, subdivision, first and second semi–total, and total graphs—by deriving closed-form expressions for their Hosoya polynomials based on Gutman's relations.

1 Introduction

In mathematical chemistry and graph theory, topological indices are numerical descriptors derived from the structure of a molecular graph. These indices are instrumental in quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR) studies, enabling the prediction of the physicochemical and biological properties of chemical compounds without resorting to complex experiments [1, 2].

Among these descriptors, distance-based topological indices hold a prominent place. The most renowned is the Wiener index, introduced by Harry Wiener [14], which was initially used to correlate the molecular structure of alkanes with their boiling points. The Wiener index is defined as the sum of the distances between all unordered pairs of vertices in the graph. Generalizations of the Wiener index, such as the Hyper-Wiener index [11] and the Tratch-Stankevitch-Zefirov index [14], as well as the reciprocal-distance-based Harary index [8], have since been developed to capture more nuanced structural information. A recent related study is [9].

A significant breakthrough was Hosoya's introduction of the Hosoya polynomial (also known as the Wiener polynomial) in 1988 [4]. This polynomial elegantly consolidates distance information by enumerating the number of vertex pairs at each distance. A remarkable feature of this polynomial is that many distance-based indices can be derived directly from it through differentiation or integration, making it a master invariant for studying molecular graphs [13, 15–17].

This work focuses on a specific and interesting class of graphs: Banana tree graphs. A Banana tree graph $B_{n,k}$ is formed by connecting n copies of a star graph S_k (each with k leaves) to a single additional root vertex. The resulting graph has $nk + 1$ vertices and nk edges. These graphs are not only theoretically intriguing due to their tree-like and star-like hybrid structure but can also serve as simplified models for certain dendritic molecules or hierarchical network structures. Figure 1 illustrates the structure of $B_{3,5}$ (a description of the colored vertices will be given later).

Beyond their intrinsic combinatorial interest, banana-type graphs also arise naturally in applications from chemical graph theory and network modelling. In mathematical chemistry, molecular structures are represented as graphs, and many topological indices derived from these graphs are known to correlate with physical and chemical properties of compounds. In particular, highly branched organic macromolecules known as *dendrimers*—consisting of a central core and successive generations of branching units—exhibit tree-like architectures that are well captured by star- and banana-type graph models [5, 7, 10]. Banana tree graphs have therefore been employed as simplified structural motifs in the analysis and design of dendrimer systems.

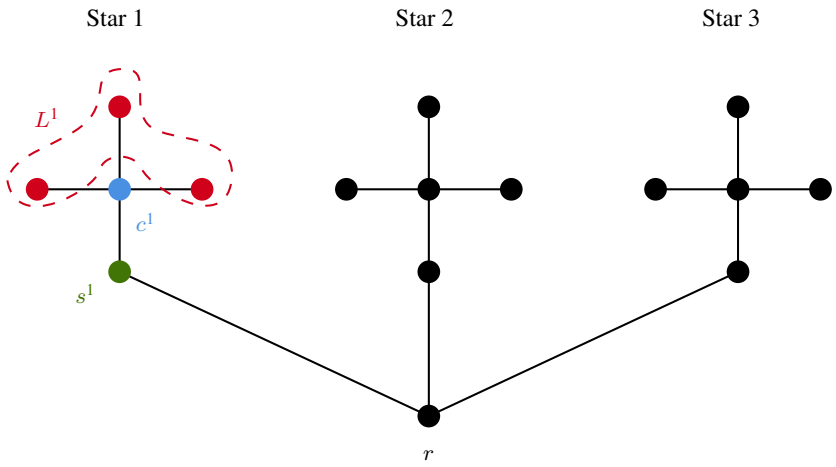


Figure 1. The Banana tree graph $B_{3,5}$.

In this paper, we completely characterize the distance distribution in $B_{n,k}$ and present an explicit formula for its Hosoya polynomial. We then use this polynomial to derive exact analytical expressions for the Wiener, Hyper-Wiener, Tratch–Stankevitch–Zefirov, and Harary indices. Furthermore, we extend our analysis to several derived graphs of the Banana tree graph—namely, the line graph, the subdivision graph, the first and second semi–total graphs, and the total graph—by establishing closed-form expressions for their Hosoya polynomials based on Gutman's transformation relations.

2 Preliminaries and Definitions

Let $G = (V(G), E(G))$ be a simple, connected, and undirected graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set. The *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is defined as the length of the shortest path connecting them. A *molecular graph* is a graph representing the carbon-atom skeleton of an organic molecule, where vertices correspond to atoms and edges correspond to chemical bonds.

Definition 2.1 (Hosoya Polynomial). The Hosoya polynomial of a graph G is defined as:

$$H(G, x) = \sum_{i \geq 1} d(G, i) x^i, \tag{2.1}$$

where $d(G, i)$ denotes the number of vertex pairs $\{u, v\} \subseteq V(G)$ such that $d(u, v) = i$.

Definition 2.2 (Banana Tree Graph). For integers $n \geq 1$ and $k \geq 3$, the *Banana tree graph* $B_{n,k}$ is constructed as follows:

- Take n disjoint copies of a star graph S_k, each with a center vertex and k leaves.
- Add one new vertex, called the *root* vertex r.
- For each copy i of the star S_k, connect the root vertex r to exactly one distinct leaf vertex of that star.

The graph $B_{n,k}$ has $nk+1$ vertices and nk edges.

Definition 2.3 (Topological Indices). Let G be a graph.

- *The Wiener index* is

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v) = \sum_{i\geq 1} i \cdot d(G,i). \tag{2.2}$$

- *The Hyper-Wiener index* is

$$WW(G) = \frac{1}{2} \sum_{\{u,v\}\subseteq V(G)} (d(u,v) + d(u,v)^2) = \frac{1}{2}(W(G) + \sum_{i\geq 1} i^2 \cdot d(G,i)). \tag{2.3}$$

- *The Tratch-Stankevitch-Zefirov index* is

$$TSZ(G) = \frac{1}{6} \sum_{\{u,v\}\subseteq V(G)} (d(u,v) + d(u,v)^2 + d(u,v)^3). \tag{2.4}$$

- *The Harary index* is

$$Ha(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d(u,v)} = \sum_{i\geq 1} \frac{d(G,i)}{i}. \tag{2.5}$$

These indices can be obtained from the Hosoya polynomial:

$$W(G) = \frac{d}{dx} H(G,x)\big|_{x=1}, \tag{2.6}$$

$$WW(G) = \frac{1}{2}\frac{d}{dx}\left(x\frac{d}{dx}H(G,x)\right)\big|_{x=1}, \tag{2.7}$$

$$TSZ(G) = \frac{1}{6}\frac{d}{dx}\left(x^2\frac{d}{dx}\left(x\frac{d}{dx}H(G,x)\right)\right)\big|_{x=1}, \tag{2.8}$$

$$Ha(G) = \int_0^1 \frac{H(G,x)}{x} dx. \tag{2.9}$$

3 Distance Distribution in Banana Tree Graphs

The structure of $B_{n,k}$ restricts the possible distances between any two vertices. The following proposition bounds these distances.

Proposition 3.1. *For all integers $n \geq 2$, $k \geq 3$, and for any two distinct vertices u, v in $B_{n,k}$, the distance between them is bounded by $1 \leq d(u,v) \leq 6$.*

Proof. Since $B_{n,k}$ is connected, $d(u,v) \geq 1$. The diameter of the graph is realized by the path between two non-root, non-center leaves in two different stars: one must traverse from a leaf to its center (length 1), from this center to the unique leaf connected to the root (length 1), from this leaf to the root (length 1), from the root to a leaf in another star (length 1), from this leaf to its center (length 1), and finally from this center to another leaf (length 1). This path has a total length of 6. Hence, $d(u,v) \leq 6$ for all $u, v \in V(B_{n,k})$. □

To enumerate the pairs $d^i_{n,k} = |\{\{u,v\} : d(u,v) = i\}|$, we define the following vertex types for $B_{n,k}$:

- r: the root vertex;
- for each star $i \in [n] = \{1, 2, ..., n\}$:

- c^i: the center vertex of the i-th star,
- s^i: the leaf vertex in the i-th star connected to the root r,
- L^i: the set of the remaining $k-2$ leaves in the i-th star.

See Star 1 in Figure 1 for an example of the vertex type definition described above.

The distance between any two vertices depends on their type and location. The following lemma provides a complete case analysis.

Lemma 3.2 (Distance cases in $B_{n,k}$). *Let $B_{n,k}$ be as in Definition 2.2, with root r, centers c_i for $i \in [n]$, special leaves s_i, and vertex sets L_i. For any distinct vertices u, v in $B_{n,k}$, the distance $d(u,v)$ is determined as follows:*

- *Within the same star i:*
  - $d(c_i, x) = 1$ *for all* $x \in L_i \cup \{s_i\}$.
  - $d(x, y) = 2$ *for all distinct* $x, y \in L_i \cup \{s_i\}$.
- *Root to vertices of star i:*
  - $d(r, s_i) = 1, \quad d(r, c_i) = 2, \quad$ *and* $d(r, x) = 3$ *for all* $x \in L_i$.
- *Between different stars $i \neq j$:*
  - $d(s_i, s_j) = 2$.
  - $d(c_i, s_j) = 3$.
  - $d(c_i, c_j) = 4$.
  - $d(c_i, x) = 5$ *for all* $x \in L_j$.
  - $d(s_i, x) = 4$ *for all* $x \in L_j$.
  - $d(x, y) = 6$ *for all* $x \in L_i$ *and* $y \in L_j$.

Proof. Fix $n \geq 1$ and $k \geq 3$. Recall that $B_{n,k}$ consists of n disjoint stars, and a root vertex r adjacent to exactly one distinguished leaf s_i in each star i. For each $i \in [n]$, let c_i be the center of the i-th star and let L_i be the set of its remaining $k-2$ leaves.

- Distances within the same star i: Inside the i-th star, the center c_i is adjacent to every leaf in $L_i \cup \{s_i\}$, hence $d(c_i, x) = 1$ for all $x \in L_i \cup \{s_i\}$. If $x \neq y$ are two leaves in $L_i \cup \{s_i\}$, then the unique path between them in the star is $x - c_i - y$, of length 2. Therefore $d(x, y) = 2$ for all distinct $x, y \in L_i \cup \{s_i\}$.
- Distances from the root to vertices of star i: By construction, r is adjacent to s_i, so $d(r, s_i) = 1$. Moreover, the unique path from r to c_i is $r - s_i - c_i$, hence $d(r, c_i) = 2$. Finally, for any $x \in L_i$, the unique path from r to x is $r - s_i - c_i - x$, so $d(r, x) = 3$.
- Distances between two different stars $i \neq j$: Observe that the only edges connecting different stars go through the root r, and r is adjacent precisely to the vertices $s_1, \ldots, s_n$. Consequently, for vertices lying in distinct stars, any path must pass through r, and the shortest path is obtained by concatenating shortest paths to and from r.

  First, $s_i - r - s_j$ has length 2, so $d(s_i, s_j) = 2$. Next, the path $c_i - s_i - r - s_j$ has length 3, so $d(c_i, s_j) = 3$. Similarly, $c_i - s_i - r - s_j - c_j$ has length 4, so $d(c_i, c_j) = 4$. If $x \in L_j$, then $s_i - r - s_j - c_j - x$ has length 4, hence $d(s_i, x) = 4$. Also $c_i - s_i - r - s_j - c_j - x$ has length 5, hence $d(c_i, x) = 5$. Finally, for $x \in L_i$ and $y \in L_j$, the path

  $$x - c_i - s_i - r - s_j - c_j - y$$

  has length 6, so $d(x, y) = 6$.

All cases listed in the statement follow, completing the proof. □

Remark 3.3. We have $\mathrm{diam}(B_{n,k}) = 6$, if and only if $n \geq 2$ and $k \geq 3$.

Vertex u	Vertex v	Distance $d(u,v)$	Condition
c^i	$x \in L^i \cup \{s^i\}$	1	Same star i
$x \in L^i$	$y \in L^i$	2	$x \neq y$, same star i
r	s^i	1	For any star i
r	c^i	2	For any star i
r	$x \in L^i$	3	For any star i
s^i	s^j	2	$i \neq j$
c^i	s^j	3	$i \neq j$
c^i	c^j	4	$i \neq j$
c^i	$x \in L^j$	5	$i \neq j$
s^i	$x \in L^j$	4	$i \neq j$
$x \in L^i$	$y \in L^j$	6	$i \neq j$

Table 1. Distance between vertices u and v in $B_{n,k}$ based on their types.

Table 1 summarize the results in Lemma 3.2. Using the distance cases in Lemma 3.2, we can now count the number of vertex pairs at each distance.

Lemma 3.4. *For all integers $n \geq 1$, $k \geq 3$, the number of vertex pairs $d^i_{n,k}$ at distance i in $B_{n,k}$ is given by:*

$$d^i_{n,k} = \begin{cases} nk, & \text{if } i = 1; \\ n\binom{k-1}{2} + \binom{n}{2} + n, & \text{if } i = 2; \\ 2\binom{n}{2} + n(k-2), & \text{if } i = 3; \\ \binom{n}{2} + 2\binom{n}{2}(k-2), & \text{if } i = 4; \\ 2\binom{n}{2}(k-2), & \text{if } i = 5; \\ \binom{n}{2}(k-2)^2, & \text{if } i = 6. \end{cases} \tag{3.1}$$

Proof. The counts are derived by summing the contributions from each case in Table 1.

- d^1: Pairs at distance 1 come from edges. Each of the n stars has k edges, giving nk pairs.
- d^2: (i) Non-center pairs within a star: $n \times \binom{k-1}{2}$. (ii) Pairs of s-vertices from different stars: $\binom{n}{2}$. (iii) Pairs (r, c^i): n. Summing these gives the result.
- d^3: (i) Pairs (c^i, s^j) and (c^j, s^i) for $i \neq j$: $2\binom{n}{2}$. (ii) Pairs $(r, x \in L^i)$: $n(k-2)$.
- d^4: (i) Pairs of centers (c^i, c^j): $\binom{n}{2}$. (ii) Pairs $(s^i, x \in L^j)$ and $(s^j, x \in L^i)$ for $i \neq j$: $2\binom{n}{2}(k-2)$.
- d^5: Pairs $(c^i, x \in L^j)$ and $(c^j, x \in L^i)$ for $i \neq j$: $2\binom{n}{2}(k-2)$.
- d^6: Pairs $(x \in L^i, y \in L^j)$ for $i \neq j$: $\binom{n}{2}(k-2)^2$.

This completes the proof. □

Proposition 3.5. *The total number of vertex pairs in $B_{n,k}$ is consistent with the binomial coefficient:*

$$\sum_{i=1}^{6} d^i_{n,k} = \binom{nk+1}{2}. \tag{3.2}$$

Proposition 3.5 serves as a validation check for the counts in Lemma 3.4.

4 Hosoya Polynomial and Distance-Based Indices of Banana Tree Graphs

Using the distance distribution from Lemma 3.4, we can directly state the Hosoya polynomial.

Theorem 4.1. *The Hosoya polynomial of the Banana tree graph $B_{n,k}$ for $n \geq 1$, $k \geq 3$ is:*

$$\begin{aligned} H(B_{n,k},x) = \; & nkx + \frac{n}{2}\left(n+k^2-3k+2\right)x^2 + n\left(n+k-3\right)x^3 \\ & + \binom{n}{2}(2k-3)\,x^4 + 2\binom{n}{2}(k-2)x^5 + \binom{n}{2}(k-2)^2x^6. \end{aligned} \tag{4.1}$$

This expression can be simplified algebraically for specific applications. We now derive the topological indices.

Proposition 4.2. *For all integers $n \geq 1$, and $k \geq 3$, we have the following.*

- *Wiener Index of $B_{n,k}$:*

$$W(B_{n,k}) = 3n^2k^2 + 3n^2k - 2nk^2 - 2nk - 16n. \tag{4.2}$$

- *Hyper-Wiener Index of $B_{n,k}$:*

$$WW(B_{n,k}) = -\frac{5}{4}nk + \frac{7}{4}n^2k^2 + 2n^2k - \frac{5}{4}nk^2 - 10n. \tag{4.3}$$

- *Tratch-Stankevitch-Zefirov Index of $B_{n,k}$:*

$$TSZ(B_{n,k}) = -\frac{1}{2}nk + \frac{2}{3}n^2k^2 + \frac{5}{6}n^2k - \frac{1}{3}nk^2 - 4n. \tag{4.4}$$

- *Harary Index of $B_{n,k}$:*

$$Ha(B_{n,k}) = \frac{2}{15}nk - 2nk^2 + \frac{104}{15}n^2 + 3n^2k^2 - \frac{9}{5}n^2k - \frac{88}{5}n. \tag{4.5}$$

Table 2 shows some values of the studied topological indices for small values of n and k. Figure 2 illustrates a plot of the Wiener index $W(B_{n,k})$ for $n, k \in [2, 10]$, demonstrating its quadratic growth in both parameters, as predicted by Equation 4.2.

Graph	Wiener Index	Hyper-Wiener Index	TSZ Index	Harary Index
$B_{2,3}$	56	104.25	142.67	30.93
$B_{3,3}$	174	387.75	577.00	85.20
$B_{2,4}$	120	257.00	378.67	52.53
$B_{3,4}$	354	897.00	1410.00	136.27

Table 2. Computed topological indices for small Banana tree graphs.

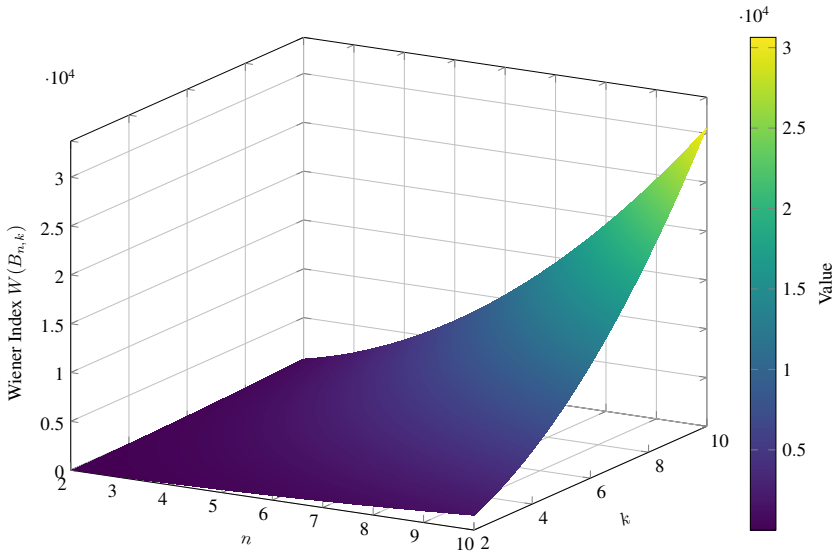


Figure 2. Plot of the Wiener index $W(B_{n,k})$ for $n, k \in [2, 10]$, demonstrating its quadratic growth in both parameters, as predicted by Equation 4.2.

5 Hosoya Polynomial of Derived Graphs of Banana Tree Graph

Let G be a connected graph on n vertices, and let $L(G)$, $S(G)$, $ST_1(G)$, $ST_2(G)$, and $T(G)$ denote the *line graph*, the *subdivision graph*, the *first semi–total graph*, the *second semi–total graph*, and the *total graph* of G, respectively. These graphs are defined as follows.

Definition 5.1 (The line graph). The line graph $L(G)$ of G has as vertices the edges y_1, y_2,...,y_m of G, $V(L(G)) = E(G)$. Two vertices of $L(G)$ are adjacent if, and only if, the respective edges of G are incident.

Definition 5.2 (The subdivision graph). The subdivision graph $S(G)$ of G is obtained by inserting a new vertex on each edge of G. The subdivision graph can be viewed as consisting of the vertices of both G and $L(G)$, $VS(G) = V(G) \cup V(L(G))$. In view of this we we denoted the vertices of $S(G)$ by x_1,x_2,...,x_n,y_1,y_2,...,y_m and say that the vertices x_1,x_2,...,x_n originate from G whereas the vertices y_1,y_2,...,y_m originate from $L(G)$.
The vertices of the below described semi-total graphs and of the total graph are labeled in the same manner as the vertices of the subdivision graph.

Definition 5.3 (The first semi-total graph). The first semi-total graph $ST_1(G)$ of G is obtained by adding new edges to $S(G)$, connecting all pairs of vertices originating from G and being adjacent in G.Thus $V(ST_1(G)) = V(S(G))$ and $E(ST_1(G)) = E(S(G)) \cup E(G)$.

Definition 5.4 (The second semi-total graph). The second semi-total graph $ST_2(G)$ of G is obtained by adding new edges to $S(G)$, connecting all pairs of vertices originating from $L(G)$ and being adjacent in $L(G)$. Thus $V(ST_2(G)) = V(S(G))$ and $E(ST_2(G)) = E(S(G)) \cup E(L(G))$.

Definition 5.5 (The total graph). The total graph $T(G)$ of G is obtained by adding new edges to $S(G)$, connecting all pairs of vertices originating from G and being adjacent in G and all pairs of vertices originating from $L(G)$ and being adjacent in $L(G)$. Thus $V(T(G)) = V(S(G))$ and $E(T(G)) = E(S(G)) \cup E(G) \cup E(L(G))$.

Figure 3 illustrates the five derived graphs of the Banana tree graph $B_{3,5}$ described above.

Gutman [3] established linear relations between the Hosoya polynomial of a tree G and those of its derived graphs defined above.

Theorem 5.6 (Gutman [3]). *Let G be a tree graph, then we have*

$$H(L(G),x) = \frac{1}{x}H(G,x) - (|V(G)|-1), \tag{5.1}$$

$$H(S(G),x) = \left(1+\frac{1}{x}\right)^2 H(G,x^2) - (|V(G)|-1), \tag{5.2}$$

$$H(ST_1(G),x) = 4H(G,x) - (|V(G)|-1)x, \tag{5.3}$$

$$H(ST_2(G),x) = x\left(1+\frac{1}{x}\right)^2 H(G,x) - (|V(G)|-1), \tag{5.4}$$

$$H(T(G),x) = \left(3+\frac{1}{x}\right) H(G,x) - (|V(G)|-1). \tag{5.5}$$

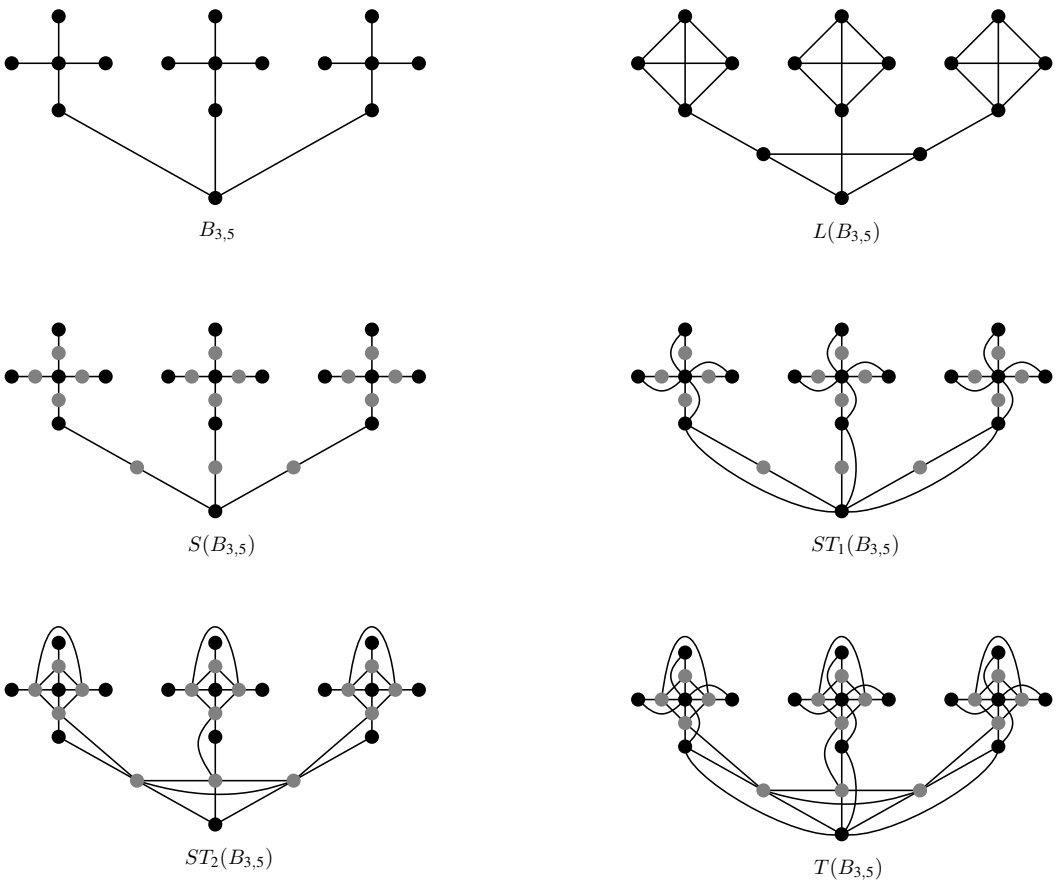


Figure 3. Banana graph $B_{3,5}$ and its five derived graphs $L(B_{3,5})$,$S(B_{3,5})$,$ST_1(B_{3,5})$,$ST_2(B_{3,5})$, and $T(B_{3,5})$.

Using Theorem 5.6 and the Hosoya polynomial of $B_{n,k}$ obtained in Theorem 4.1, we now derive explicit expressions for the Hosoya polynomials of the five derived graphs.

Proposition 5.7. *For all integers $n \geq 1$, and $k \geq 3$, we have the following.*

- *Hosoya polynomial of Line graph of $B_{n,k}$:*

$$\begin{aligned} H(L(B_{n,k}),x) = {} & \frac{n}{2}\left(n+k^2-3k+2\right)x + n\left(n+k-3\right)x^2 \\ & + \binom{n}{2}(2k-3)\,x^3 + 2\binom{n}{2}(k-2)x^4 + \binom{n}{2}(k-2)^2x^5. \end{aligned} \tag{5.6}$$

- *Hosoya polynomial of subdivision graph of $B_{n,k}$:*

$$\begin{aligned} H(S(B_{n,k}),x) &= 2nkx + \frac{n}{2}\left(k^2-k+4\right)x^2 + n\left(2n+k^2-2k\right)x^3 \\ &+ \frac{n}{2}\left(2nk+6n+k^2-k-10\right)x^4 + n\left(3nk-4n-2k+2\right)x^5 \\ &+ \binom{n}{2}(k^2+4k-7)x^6 + \binom{n}{2}(2k^2-2k+4)x^7 + \binom{n}{2}(k-2)^2x^8 \end{aligned} \tag{5.7}$$

- *Hosoya polynomial of first semi-total graph of $B_{n,k}$:*

$$\begin{aligned} H(ST_1(B_{n,k}),x) &= 3nkx + 2n\left(n+k^2-3k+2\right)x^2 + 4n\left(n+k-3\right)x^3 \\ &+ 4\binom{n}{2}\left(2k-3\right)x^4 + 8\binom{n}{2}(k-2)x^5 + 4\binom{n}{2}(k-2)^2x^6. \end{aligned} \tag{5.8}$$

- *Hosoya polynomial of second semi-total graph of $B_{n,k}$:*

$$\begin{aligned} H(ST_2(B_{n,k}),x) &= \frac{n}{2}\left(n+k^2+k+4\right)x + n\left(2n+k^2-k\right)x^2 \\ &+ n\left(nk+3n+k^2-2k-4\right)x^3 + n\left(3nk-4n-2k+2\right)x^4 \\ &+ \binom{n}{2}(k^2+4k-7)x^5 + \binom{n}{2}(2k^2-2k+4)x^6 \\ &+ \binom{n}{2}(k-2)^2x^7. \end{aligned} \tag{5.9}$$

- *Hosoya polynomial of total graph of $B_{n,k}$:*

$$\begin{aligned} H(T(B_{n,k}),x) &= \frac{n}{2}\left(n+2k^2+5\right)x + \frac{n}{2}\left(5n+3k^2-7k+3\right)x^2 \\ &+ \frac{n}{2}\left(2nk+3n+4k-15\right)x^3 + \binom{n}{2}\left(8k-13\right)x^4 \\ &+ \binom{n}{2}(k^2+4k-8)x^5 + 3\binom{n}{2}(k-2)^2x^6. \end{aligned} \tag{5.10}$$

6 Conclusion

In this paper, we have presented a complete analysis of the distance-based topological properties of Banana tree graphs $B_{n,k}$. We provided a detailed enumeration of vertex pairs at each distance, which led to an explicit formula for the Hosoya polynomial. From this generating function, we derived closed-form expressions for four significant topological indices: the Wiener, Hyper-Wiener, Tratch-Stankevitch-Zefirov, and Harary indices. Our results are rigorously proven and validated through computational checks. Moreover, we have generalized our findings to several derived graphs of $B_{n,k}$, including $L(B_{n,k})$, $S(B_{n,k})$, $ST_1(B_{n,k})$, $ST_2(B_{n,k})$, and $T(B_{n,k})$, for which we have established explicit formulas for their Hosoya polynomials. These results, obtained using Gutman's transformation relations, further demonstrate the robustness of the analytical approach introduced for the Banana tree family.

The methods employed here can be applied to other classes of tree-like graphs or graph operations. Future work could focus on studying the Hosoya polynomial of generalized tree structures, graphs with different branching factors, or the impact of edge/vertex deletion on these indices within this family of graphs.

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