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ZERO-DIVISOR GRAPHS OF BI-AMALGAMATION

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Abstract. Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and J and J' be two non-zero proper ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. The bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) is the subring of $B \times C$ given by:

$$A \bowtie^{f,g} (J, J') = \{(f(a) + j, g(a) + j')/a \in A, (j, j') \in J \times J'\}$$

This paper investigates the zero-divisor graphs of bi-amalgamations. Our objective is to characterize when the graph is complete, and compute its diameter and girth for various contexts of bi-amalgamations. The new results recover well-known results on amalgamations, and provide some original examples arising as bi-amalgamations.

1 Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. In 1988, Beck introduced the notion of zero-divisor graphs for his study of the coloring of a commutative ring (cf.[4]). He considered all elements of the ring where vertices of the graphs. D.D.Anderson and Nasser used in 1993 this same notion in [2]. Let R be a ring and let $Z(R)$ denote the set of zero-divisors of R and $Z(R)^* = Z(R) \setminus \{0\}$. The zero-divisor graphs of R, denoted by $\Gamma(R)$, is the graph whose vertices are the elements of $Z(R)^*$ and for distincts $a, b \in Z(R)^*$, there is an edge connecting a and b if and only if $ab = 0$. For two distinct vertices x and y in $\Gamma(R)$, the distance between x and y denoted by $d(x, y)$, is the length of the shortest path connecting x and y, if there is a path, otherwise $d(x, y) = \infty$. The diameter of $\Gamma(R)$ defined as follows: $diam(\Gamma(R)) := \sup\{d(x, y) / x$ and y are distinct vertices of $\Gamma(R)\}$. The girth of $\Gamma(R)$ denoted by $gr(\Gamma(R))$, is the length of a shortest cycle in $\Gamma(R)$, if there is a cycle in $\Gamma(R)$, otherwise $gr(\Gamma(R)) = \infty$. A graph is connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter less than or equal to one. A singleton graph is connected and of diameter zero. T. G. Lucas in [12], characterized when the diameter of zero-divisor graphs is equal to 0 or 1. On the other hand, it is clear that $\Gamma(R)$ is empty if and only if R is a domain. In [1], D.F.Anderson and P. S. Livingston considered only nonzero zero-divisors as vertices of the graph and proved that the zero-divisor graphs of a (commutative) ring is always connected with diameter less than or equal three. In [14], Mulay, and in [8], DeMeyer and Schneider examined the diameter and the girth of the zero-divisor graph of a ring. In [12], Lucas characterized the diameter of $\Gamma(R), \Gamma(R[X])$ and $\Gamma(R[[X]])$ with some properties of R. For reduced rings, he gave a complete characterization for all three graphs, and for non-reduced rings, he studied the diameter of $\Gamma(R)$ and $\Gamma(R[X])$. He gave an example of a reduced ring R, for which $diam(\Gamma(R))$ =$diam(\Gamma(R[X])) = 2$ and diam $(\Gamma(R[[X]])) = 3$. He also gave other examples of a reduced and non-reduced ring R, where diam $(\Gamma(R)) = 2$ and diam $(\Gamma(R[X]))$ = diam $(\Gamma(R[[X]])) = 3$. For more details on zero-divisor graphs, see [1, 2, 4, 8, 12, 14].
Let A be a ring and E an A-module, we called the trivial ring extension of A by E, the ring $A \ltimes E$ where the underlying group is $A \ltimes E$ and we defined a multiplication in $A \ltimes E$ by:

$(a,e)(b,f) = (ab, af+be)$; $\forall a,b \in A$, and $e,f \in E$. This notion is introduced by Nagata in [15] (often called the Nagata's idealization of E over A and denoted $A(+)E$).
In [3], Axtell and Stickles gave a complete characterization of the preservation of the diameter and girth under the Nagata's idealization. They characterized the girth of the zero-divisor graphs of the Nagata's idealization and when it is complete and they gave conditions for the zero-divisor graphs to have diameter 2.
Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two non-zero proper ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. In this setting, we can consider the following subring of $B \times C$:

$$A \bowtie^{(f,g)} (J, J') := \{(f(a)+j, g(a)+j') | a \in A, (j,j') \in J \times J'\}$$

called *the bi-amalgamated algebra of* A *with* (B,C) *along* (J,J') *with respect to* (f,g). This construction was introduced and studied in [11] as a natural generalization of duplications and amalgamated algebra of rings along an ideal. Given a ring homomorphism $f : A \to B$ and an ideal J of B, the bi-amalgamation $A \bowtie^{f,i} (J, f^{-1}(J))$ coincides with the amalgamated algebra of A with B along J with respect to f introduced and studied by D'Anna, Finocchiaro and Fontana in [5, 6, 9] as the following subring of $A \times B$:

$$A \bowtie^{f} J := \{(a, f(a)+j) \mid a \in A, j \in J\}$$

When $A = B$, $J = I$ an ideal of A and $f = id_A$, then $A \bowtie^{id_A} I$ is called the amalgamated duplication of A along I and denoted $A \bowtie I$ introduced and studied by D'Anna and Fontana in [7]. This construction can be seen as a bi-amalgamation as follows:

$$A \bowtie I = A \bowtie^{id_A, id_A} (I, I)$$

In [10], Kabbaj and Mimouni studied the notion of zero-divisor graphs of amalgamations, they characterized when the zero-divisor graphs of amalgamations is complete, and they computed the diameter and the girth of amalgamations. In [13], Maimani and Yassemi studied the diameter and girth of the graph of $A \bowtie I$.
In this paper, we investigate the zero-divisor graph of bi-amalgamations. Our aim is to characterize when the graph is complete and compute its diameter and girth for different contexts of bi-amalgamations. Precisely, sections 2 and 3 study, respectively, the cases when $diam(\Gamma(A \bowtie^{(f,g)} (J,J'))) = 2$, $diam(\Gamma(A \bowtie^{(f,g)} (J,J'))) = 3$ and $\Gamma(A \bowtie^{(f,g)} (J,J'))$ is complete. Section 4 will study the girth of $\Gamma(A \bowtie^{(f,g)} (J,J'))$. The new results are compared to previous works carried out in various settings of duplications and amalgamations [10, 13]. All results are backed with new and illustrative examples arising as bi-amalgamations.

2 When is diam($\Gamma(A \bowtie^{f,g} (J, J'))$)=2 or 3

Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two proper ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. We consider: $R := A \bowtie^{(f,g)} (J,J') = \{(f(a)+j, g(a)+j')/a \in A, (j,j') \in J \times J'\}$ called the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g). The aim of this section is to investigate the diameter of $\Gamma(R)$, precisely when $diam(\Gamma(R)) = 2$ or 3. Throughout, let $Z(J)$ denote the set of zero-divisors on J as an ideal of $f(A)+J$; that is, the elements of $f(A)+J$ that annihilate some non-zero element of J and $Z(J')$ denote the set of zero-divisors on J' as an ideal of $g(A)+J'$; that is, the elements of $g(A)+J'$ that annihilate some non-zero element of J'.

Consider the following conditions:
(C_1): For all $x \in Z(J)$, and all $j \in J : jx \neq 0$, then $ij = ix = 0$, for some $0 \neq i \in J$.
(C_2): For all $x \in Z(J')$ and all $j' \in J' : j'x \neq 0$, then $i'j' = i'x = 0$, for some $0 \neq i' \in J'$.
(C_3): For all $x, y \in Z(J); x \neq y$, then $ix = iy = 0$, for some $0 \neq i \in J$.
(C_4): For all $x, y \in Z(J'); x \neq y$, then $i'x = i'y = 0$, for some $0 \neq i' \in J'$.

We start by giving necessary and sufficient conditions for the diameter of the zero-divisor graphs of the bi-amalgamation to be equal to 2 when $f(A)+J$ or $g(A)+J'$ is a domain.

Theorem 2.1. *Under the above notation, assume that J and J' are nonzero proper ideals of B and C, respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$. Then:*
(1) If $f(A) + J$ is a domain, then $diam(\Gamma(R)) = 2$ if and only if (C_2) and (C_4) hold.
(2) If $g(A) + J'$ is a domain, then $diam(\Gamma(R)) = 2$ if and only if (C_1) and (C_3) hold.

Before proving Theorem 2.1, we establishe the following lemma.

Lemma 2.2. *Under the above notation, suppose that J and J' are nonzero proper ideals of B and C, respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$. Let*
$V_1 := \{(0, j')/0 \neq j' \in J'\}$.
$V_2 := \{(j, 0)/0 \neq j \in J\}$.
$V_3 := \{(f(a) + j, g(a) + j')/0 \neq g(a) + j' \in Z(J')\}$.
$V_4 := \{(f(a) + j, g(a) + j')/0 \neq f(a) + j \in Z(J)\}$. *Then :*

(i) If $f(A) + J$ is a domain, then $Z(R)^* = V_1 \cup V_2 \cup V_4$.

(ii) If $g(A) + J'$ is a domain, then $Z(R)^* = V_1 \cup V_2 \cup V_3$.

Proof. (1) Assume that $f(A) + J$ is a domain. Let $x := (f(a) + j, g(a) + j') \in Z(R)^*$ and $0 \neq y := (f(b) + i, g(b) + i') \in R$ such that $xy = 0$, for all $(0, 0) \neq (i, i') \in J \times J'$. Then $(f(a) + j)(f(b) + i) = (g(a) + j')(g(b) + i') = 0$. If $f(a) + j = 0$, then $0 \neq g(a) + j' \in J'$ and so $x \in V_1$. If $f(a) + j \neq 0$, then $f(b) + i = 0$, therefore $0 \neq g(b) + i' \in J'$ and $(g(a)+j')(g(b)+i') = 0$, then $g(a)+j' = 0$ or $g(a)+j' \in Z(J')$, then $x \in V_2$ or $x \in V_4$. Hence $Z(R)^* \subseteq V_1 \cup V_2 \cup V_4$. In the otherwise, we have $V_1 \cup V_2 \subseteq Z(R)^*$. Let $x = (f(a)+i, g(a)+i') \in V_4$, then $0 \neq g(a) + i' \in Z(J')$, then there is $0 \neq k' \in J'$ such that $k'(g(a) + i') = 0$, then $(f(a) + i, g(a) + i')(0, k') = 0$ therefore $x \in Z(R)^*$. Hence $Z(R)^* = V_1 \cup V_2 \cup V_4$.
(2) For the same reasoning, it is easy to prove that if $g(A) + J'$ is a domain, then $Z(R)^* = V_1 \cup V_2 \cup V_3$. □

Proof of Theorem 2.1
(1) Assume that C_2 and C_4 hold. Notice that every vertex in V_1 is adjacent to every vertex in V_2. Next, let us handle five other possible distinct cases.

- Let $x_1 := (0, i') \neq (0, j') =: y_1 \in V_1$ and let $0 \neq j \in J$, then $x_1 - (j, 0) - y_1$ is a path in $\Gamma(R)$, then $d(x_1, y_1) \leq 2$.
- Let $x_2 := (i, 0) \neq (j, 0) =: y_2 \in V_2$ and let $0 \neq j' \in J'$, then $x_2 - (0, j') - y_2$ is a path in $\Gamma(R)$, then $d(x_2, y_2) \leq 2$.
- Let $x_1 := (0, i') \in V_1$ and $x_4 := (f(a) + j, g(a) + j') \in V_4$. If $i'(g(a) + j') = 0$, then $x_1x_4 = 0$, then $d(x_1, x_4) = 1$ (x_1 and x_4 are adjacents). If $i'(g(a) + j') \neq 0$, then by (C_2) there is $0 \neq k' \in J'$ such that $k'i' = k'(g(a) + j') = 0$. Thus $x_1 - (0, k') - x_4$ is a path in $\Gamma(R)$ and hence $d(x_1, x_4) = 2$.
- Let $x_2 := (j, 0) \in V_2$ and let $x_4 := (f(a) + j, g(a) + j') \in V_4$. The fact $x_4 \in V_4$, implies that there is $0 \neq k' \in J'$ such that $k'(g(a) + j') = 0$. Then $x_2 - (0, k') - x_4$ is a path in $\Gamma(R)$, so $d(x_2, x_4) = 2$.
- Let $x_4 := (f(a)+i, g(a)+i') \neq (f(b)+j, g(b)+j') =: y_4 \in V_4$. If $g(a)+i' = g(b)+j'$, then there exists $0 \neq k' \in J'$ such that $k'(g(a)+i') = k'(g(b)+j') = 0$, then $x_4 - (0, k') - y_4$ is a path in $\Gamma(R)$, then $d(x_4, y_4) = 2$. If $g(a) + i' \neq g(b) + j'$ by C_4, there exists $0 \neq k' \in J'$ such that $k'(g(a) + i') = k'(g(b) + j') = 0$, then $x_4 - (0, k') - y_4$ is a path in $\Gamma(R)$, so $d(x_4, y_4) = 2$.

Consequently $diam(\Gamma(R)) = 2$. Conversely, assume that $diam(\Gamma(R)) = 2$. Let $0 \neq x := g(a)+j' \in Z(J')$ and $i' \in J'$ with $i'(g(a)+j') \neq 0$. Let $0 \neq j \in J$ such that $f(a)+j \neq 0$, then $(f(a) + j, x)$ and $(0, i')$ are two non adjacent vertices in $\Gamma(R)$ or $diam(\Gamma(R)) = 2$, then there is a path in $\Gamma(R)$: $(f(a) + j, x) - (f(b) + k, g(b) + k') - (0, i')$, for some $(k, k') \in J \times J', b \in A$. Since $f(A) + J$ is a domain and $f(a) + j \neq 0$, then $f(b) + k = 0$, then $0 \neq g(b) + k' \in J'$ and $(g(b) + k')x = (g(b) + k')i' = 0$, so (C_2) holds. Let $x := g(a) + j' \neq g(b) + i' =: y \in Z(J')$. Let $0 \neq i, j \in J$ such that $0 \neq f(a)+j$, then $(f(a)+j, x)$ and $(f(b)+i, y)$ are two non adjacent vertices in $\Gamma(R)$ or $diam(\Gamma(R)) = 2$, then there is a path in $\Gamma(R)$: $(f(a) + j, x) - (f(c) +$

$r, g(c) + r') - (f(b) + i, y)$, for some $c \in A, (r, r') \in J \times J'$. Since $f(A) + J$ is a domain and $f(a) + j \neq 0$, then $(f(c) + r) = 0$, then $0 \neq g(c) + r' \in J'$ and $(g(c) + r')x = (g(c) + r')y = 0$, so (C_4) holds.
(2) Assume that (C_1) and (C_3) hold. In view of the proof of (1), we envisage only three cases:

- Let $x_1 := (0, i') \in V_1$ and $x_3 := (f(a) + j, g(a) + j') \in V_3$, then there is $0 \neq k \in J$ such that $k(f(a) + j) = 0$, then $x_1 - (k, 0) - x_3$ is a path in $\Gamma(R)$, then $d(x_1, x_3) = 2$
- Let $x_2 := (i, 0) \in V_2$ and $x_3 := (f(a) + j, g(a) + j') \in V_3$. If $i(f(a) + j) = 0$, then $x_2x_3 = 0$, then $d(x_2, x_3) = 1$. If $i(f(a) + j) \neq 0$, then by (C_1), there is $0 \neq k \in J$ such that $k(f(a) + j) = ki = 0$, then $x_2 - (k, 0) - x_3$ is a path in $\Gamma(R)$, then $d(x_2, x_3) = 2$.
- Let $x_3 := (f(a) + i, g(a) + i') \neq (f(b) + j, g(b) + j') =: y_3 \in V_3$. If $f(a) + i = f(b) + j$, then there is $0 \neq k \in J$ such that $k(f(a) + i) = k(f(b) + j) = 0$, then $x_3 - (k, 0) - y_3$ is a path in $\Gamma(R)$, then $d(x_3, y_3) = 2$. If $f(a) + i \neq f(b) + j$, then by (C_3), there is $0 \neq k \in J$ such that $k(f(a) + i) = k(f(b) + j) = 0$, then $x_3 - (k, 0) - y_3$ is a path in $\Gamma(R)$, then $d(x_3, y_3) = 2$.

Consequently $diam(\Gamma(R)) = 2$. Conversely, Assume that $diam(\Gamma(R)) = 2$. Let $0 \neq x := f(a) + i \in Z(J)$ and $j \in J$ with $j(f(a) + i) \neq 0$. Let $0 \neq i' \in J'$ such that $g(a) + i' \neq 0$. We have $(f(a) + i, g(a) + i')$ and $(j, 0)$ are two non adjacent vertices in $\Gamma(R)$, or $diam(\Gamma(R)) = 2$, then there is a path $(x, g(a) + i') - (f(b) + k, g(b) + k') - (j, 0)$. Since $g(A) + J'$ is a domain and $g(a) + i' \neq 0$, then $g(b) + k' = 0$, then $0 \neq f(b) + k \in J$ and $(f(b) + k)x = (f(b) + k)j = 0$, so (C_1) holds. Let $x := f(a) + i \neq f(b) + j =: y \in Z(J)$. Let $0 \neq i', j' \in J'$ such that $g(a) + i' \neq 0$. We have $(x, g(a) + i')$ and $(y, g(b) + j')$ are two non adjacent vertices in $\Gamma(R)$, or $diam(\Gamma(R)) = 2$, then there is a path: $(x, g(a) + i') - (f(c) + r, g(c) + r') - (y, g(b) + j')$. Since $g(A) + J'$ is a domain and $g(a) + i' \neq 0$, then $g(c) + r' = 0$, so $0 \neq f(c) + r \in J$ and $(f(c) + r)x = (f(c) + r)y = 0$. Thus (C_3) holds.

The special case where both $f(A)+J$ and $g(A)+J'$ are domains is given in the next corollary.

Corollary 2.3. *Let $f : A \to B, g : A \to C$ be two ring homomorphisms, J, J' be two nonzero proper ideals of B and C, respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$ and R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). Assume that f(A)+J and g(A)+J' are domains. Then $diam(\Gamma(R)) = 2$.*

Proof. If $f(A)+J$ and $g(A)+J'$ are domains, then C_1, C_2, C_3 and C_4 always hold since $Z(J) = Z(J') = 0$. Then by Theorem 2.1, $diam(\Gamma(R)) = 2$. □

Corollary 2.4. *[10, Corollary 3.2]: Let $f : A \to B$ be a ring homomorphism, J a non-zero proper ideal of B such that $0 \neq f^{-1}(J)$. If A and $f(A) + J$ are domains, then $diam(\Gamma(A \bowtie^f (J))) = 2$*

Next, consider the following conditions:
(C_5): For all $a \in Z(f^{-1}(J))$ and all $b \in f^{-1}(J) : ab \neq 0$, then $ac = bc = 0$ for some $0 \neq c \in f^{-1}(J)$.
(C_6): For all $a, b \in Z(f^{-1}(J)) : a \neq b$, then $ac = bc = 0$ for some $0 \neq c \in f^{-1}(J)$. The next result is a direct consequence of Theorem 2.1.

Corollary 2.5. *[10, Theorem 3.1]: Let $f : A \to B$ be a ring homomorphism, J a non-zero proper ideal of B, assume $0 \neq f^{-1}(J)$.*

(i) If A is a domain, then $diam(\Gamma(A \bowtie^f (J))) = 2$ if and only if (C_1) and (C_3) hold.

(ii) If $f(A) + J$ is a domain, then $diam(\Gamma(A \bowtie^f (J))) = 2$ if and only if (C_5) and (C_6) hold.

Next, we investigate conditions under which $diam(\Gamma(A \bowtie^{f,g} (J, J')) = 3$. For this purpose, let us negate the aforementioned conditions $(C1), (C2), (C3), (C4), (C_5), (C_6)$, to get the following:

$(\overline{C_1})$: There exist $x \in Z(J)$ and $j \in J$ with $jx \neq 0$ and for all $i \in J$, $ij \neq 0$ or $ix \neq 0$.

$(\overline{C_2})$: There exist $x \in Z(J')$ and $j' \in J'$ with $j'x \neq 0$ and for all $i' \in J'$, $i'j' \neq 0$ or $i'x \neq 0$.
$(\overline{C_3})$: There exist $x, y \in Z(J)$ with $x \neq y$ and for all $i \in J$, $ix \neq 0$ or $iy \neq 0$.
$(\overline{C_4})$: There exist $x, y \in Z(J')$ with $x \neq y$ and for all $i' \in J'$, $i'x \neq 0$ or $i'y \neq 0$.
$(\overline{C_5})$: There exist $a \in Z(f^{-1}(J))$ and $b \in f^{-1}(J)$ with $ab \neq 0$ and for all $c \in f^{-1}(J), ac \neq 0$ or $bc \neq 0$.
$(\overline{C_6})$: There exist $a, b \in Z(f^{-1}(J))$ with $a \neq b$ and for all $c \in f^{-1}(J), ac \neq 0$ or $bc \neq 0$.
As a straightforward application of Theorem 2.1, we get the next "dual" result.

Proposition 2.6. *Under the above notation, suppose that J and J' are nonzero proper ideals of B and C, respectively, with $0 \neq f^{-1}(J) = g^{-1}(J')$.*

(i) If $f(A) + J$ is a domain, then $diam(\Gamma(R)) = 3$ if and only if $(\overline{C_2})$ or $(\overline{C_4})$ hold.

(ii) If $g(A) + J'$ is a domain, then $diam(\Gamma(R)) = 3$ if and only if $(\overline{C_1})$ or $(\overline{C_3})$ hold.

Corollary 2.7. *[10, Proposition 3.8]: Let $f : A \to B$ be a ring homomorphism, J a non-zero proper ideal of B such that $0 \neq f^{-1}(J)$.*

(i) If A is a domain, then $diam(\Gamma(A \bowtie^f (J))) = 3$ if and only if $(\overline{C_1})$ or $(\overline{C_3})$ hold.

(ii) If $f(A) + J$ is a domain, then $diam(\Gamma(A \bowtie^f (J))) = 3$ if and only if $(\overline{C_5})$ or $(\overline{C_6})$ hold.

Proposition 2.8. *Let $f : A \to B, g : A \to C$ be two ring homomorphisms, J, J' be two nonzero proper ideals of B and C, respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). Then :*

(i) If J and J' are regulars, $I_0 = f^{-1}(J) = g^{-1}(J')$ is radical and $Z(f(A) + J)$ or $Z(g(A) + J')$ is not an ideal, then $diam(\Gamma(R)) = 3$.

(ii) If $Z(A)$ is not an ideal, f injective, $I_0 = f^{-1}(J) = g^{-1}(J')$ is radical, $f^{-1}(Z(B)) \subseteq Z(A)$ and $g^{-1}(Z(C)) \subseteq Z(A)$, then $diam(\Gamma(R)) = 3$.

Proof. (1) Claim 1: $Z(R)$ is not an ideal.
Let $(0, 0) \neq (j, j') \in J \times J'$, it's clear that $(0, j'), (j, 0) \in Z(R)$ but (j, j') is regular if J and J' are regulars, as desired.
Claim 2: If L_1 is a minimal prime ideal in $f(A)+J$, then $\overline{L_1} = (L_1 \times (g(A)+J')) \cap R$ is minimal prime ideal in R. Let L_1 be a proper ideal of $f(A) + J$, assume that $L_1 \in spec(f(A) + J)$ is minimal prime in $f(A) + J$. Let $P \in spec(R)$ such that $P \subseteq \overline{L_1}$. If $J \times J' \subseteq P$, then there exists $L \in spec(f(A) + J), J \subseteq L$ and $L' \in spec(g(A) + J'), J' \subseteq L'$ such that $P = \overline{L} = \overline{L'}$ by [11, Proposition 5.3]. Let $f(a)+j \in L \in spec(f(A)+J), a \in A, j \in J$, then $(f(a)+j, g(a)+j') \in R$ for all $j' \in J'$ and $(f(a) + j, g(a) + j') \in L \times (g(A) + J')$, then $(f(a) + j, g(a) + j') \in (L \times (g(A)+J')) \cap R = \overline{L} = P \subseteq \overline{L_1}$, then $(f(a)+j, g(a)+j') \in \overline{L_1} = (L_1 \times (g(A)+J')) \cap R$, then $(f(a)+j, g(a)+j') \in L_1 \times (g(A)+J')$, so $f(a)+j \in L_1 \Rightarrow L \subseteq L_1$. Then by minimality, $L = L_1$, then $\overline{L} = \overline{L_1} = P$. If $J \times J' \nsubseteq P$, then there exists $L \subseteq spec(f(A) + J)$, (or in $spec(g(A) + J')$) such that $J \nsubseteq L$, (or $J' \nsubseteq L$) and $P = \overline{L}$ by [11, Proposition 5.3]. Say $L \subset spec(f(A)+J), J \nsubseteq L$ and $P = \overline{L} = (L \times (g(A)+J')) \cap R \subseteq \overline{L_1}$. Let $0 \neq f(a)+j \in L; a \in A, j \in J$, then $(f(a)+j, g(a)+j') \in R$ for all $j' \in J'$ and $(f(a)+j, g(a)+j') \in L \times (g(A)+J'$, then $(f(a) + j, g(a) + j') \in (L \times (g(A) + J')) \cap R = \overline{L} \subseteq \overline{L_1}$, then $(f(a) + j, g(a) + j') \in L_1 \times (g(A) + J')$, then $f(a) + j \in L_1$, therefore $L \subseteq L_1$. Then by minimality $L = L_1$, then $\overline{L} = \overline{L_1}$. Hence $P = \overline{L_1}$. In both cases, we have $P = \overline{L_1}$, then $\overline{L_1}$ is minimal prime in R.
Claim 3: If L_2 is minimal prime in $g(A) + J'$, then $\overline{L_2} = (f(A) + J) \times L_2) \cap R$ is minimal prime in R. To prove this, use the same reasoning as in claim 2.
Now, by claim 1, $Z(R)$ is not an ideal. If R is a non-reduced ring, then $diam(\Gamma(R)) = 3$ by [12, Corollary 2.5]. Assume that R is reduced, since I_0 is radical, $f(A) + J$ is reduced, $Nil(C) \cap J' = 0$, $g(A) + J'$ is reduced, and $Nil(B) \cap J = 0$. Suppose that $f(A) + J$ is reduced, and $Nil(C) \cap J' = 0$. Assume that $diam(\Gamma(R)) \leq 2$, then by [12, Theorem 2.2], R has exactly two minimal prime ideals. If $diam(\Gamma(f(A) + J)) = 3$, then by [12, Theorem 2.6(4)] $f(A) + J$ has more than two minimal prime ideals, then claim 2 yields that R has more than two minimal prime ideals, a contradiction. Hence $diam(\Gamma(f(A) + J)) \leq 2$, $Z(f(A) + J)$ is not an ideal of $f(A) + J$ reduced, then $f(A) + J$ has exactly two distinct minimal primes L_1, L_2 and by claim 2 R has two distinct minimal primes $\overline{L_1}, \overline{L_2}$. On the other hand, R reduced equivalent that $\alpha \times_{A/I_0} \beta$

is reduced, with $\alpha : f(A) + J \to A/I_0$ and $\beta : g(A) + J' \to A/I_0$ be two ring homomorphisms and $I_0 = f^{-1}(J) = g^{-1}(J')$, $ker(f) = J$ and $ker(g) = J'$. The fact that $\alpha \times_{A/I} \beta$ is reduced, implies that $Nilp(f(A) + J) \cap J = Nilp(g(A) + J') \cap J' = 0$ by [5, Proposition 4.9]. Since $Nilp(g(A) + J') \cap J' = 0$, then there is one minimal prime ideal L in $g(A) + J'$, with $J' \nsubseteq L$, then by claim 3, $\overline{L} = ((f(A) + J) \times L) \cap R$ is a minimal prime ideal in R; hence R has three minimal prime ideals, a contradiction. It completes the proof of (1).
(2) We claim that if $Z(A)$ is not an ideal, f injective, $f^{-1}(Z(B)) \subseteq Z(A)$, and $g^{-1}(Z(C)) \subseteq Z(A)$, then $Z(R), Z(f(A) + J)$ and $Z(g(A) + J')$ are not ideals. Let $0 \neq a, b \in Z(A)$ such that $a - b \notin Z(A)$. The fact that $a, b \in Z(A)$, yields that there are $0 \neq s, t \in A$ such that $as = 0 = bt$, then $(f(a), g(a)), (f(b), g(b)) \in Z(R)$, $f(a), f(b) \in Z(f(A)+J)$ and $g(a), g(b) \in Z(g(A)+J')$ (since f is injective). If $f(a - b) = 0$, then $f(a - b) \in Z(B)$, then $a - b \in f^{-1}(Z(B)) \subseteq Z(A)$. If $g(a - b) = 0$, for the same reasoning we have $a - b \in Z(A)$. Hence we can suppose that $0 \neq f(a - b)$ and $0 \neq g(a - b)$. If $(f(a - b), g(a - b)) \in Z(R)$, then there exists $0 \neq (f(s)+k, g(s)+k') \in R$ such that $f(a-b)(f(s)+k) = 0 = g(a-b)(g(s)+k')$. If $f(s)+k = 0$, then $0 \neq g(s) + k' \in J'$ and $g(a - b)(g(s) + k') = 0$, then $g(a - b) \in Z(C)$, then $a - b \in Z(A)$. If $g(s) + k' = 0$ for tha same reasoning we have also $a - b \in Z(A)$. On the otherwise we have also $a - b \in Z(A)$. If $f(a - b) \in Z(f(A) + J)$, then there exists $0 \neq f(s) + k \in f(A) + J$ such that $f(a - b)(f(s) + k) = 0$, then $f(a - b) \in Z(B)$, therefore $a - b \in Z(A)$. For the same reasoning, if $g(a - b) \in Z(g(A) + J')$, then $a - b \in Z(A)$. Consequently $Z(R), Z(f(A) + J)$ and $Z(g(A) + J')$ are not ideals . After this, we use the same reasoning as (1) and prove that $diam(\Gamma(R)) = 3$. □

Example 2.9. Let $A := Z_4$, $B := Z_2[X]$, $J := XB$, $C := Z_4$ and $J' := 2Z_4$. Consider two ring homomorphisms $f : A \to B$ defined by: $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$ and $g : A \to C$ defined by: $g(a) = a$ for all $a \in A$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). It's easy to show that $f(A) + J$ is a domain and $g(A) + J' := Z_4$ is a domain, $Z(J') = \{0, 2\}$. For all $x \in Z(J')$ and all $j' \in J'$, we have $xj' = 0$, then (C_2) holds. For all $x, y \in Z(J')$ with $x \neq y$, we have $2x = 2y = 0$ $(2 \in J')$, then (C_4) holds. Therefore, by Theorem 2.1(1), the diameter of the zero-divisor graphs of R is 2.

Example 2.10. Let $A := Z_4$, $B := Z_2[X]/(X^2)$, $J := \{0, x\}$, $C := Z_3$, $J' := 2Z_3$, where x denotes the class of X modulo X^2. Consider two ring homomorphisms $f : A \to B$ and $g : A \to C$ defined by: $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$ and $g(0) = g(2) = 0$ and $g(1) = g(3) = 1$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). It's easy to show that $g(A) + J' = Z_3$ is a domain and $Z(J) = \{0, 1, x, 1 + x\}$. We have $1 + x \in Z(J)$, $x \in J$ with $x(1 + x) = x \neq 0$ and for all $0 \neq j \in J, j(1 + x) \neq 0$, then $\overline{C_1}$ holds. We have also $1, 1 + x \in Z(J), 1 \neq 1 + x$ and for all $0 \neq j \in J, j1 \neq 0$ and $j(1 + x) \neq 0$, then $\overline{C_3}$ holds. Then by Proposition 2.6(2), the diameter of the zero-divisor graphs of R is 3.

3 When $\Gamma(A \bowtie^{f,g} (J, J'))$ is complete

Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms, J and J' two proper ideals of B and C respectively such that $f^{-1}(J) = g^{-1}(J')$. Let $R := A \bowtie^{(f,g)} (J, J') = \{(f(a)+j, g(a)+ j')/a \in A, (j, j') \in J \times J'\}$ be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). Our aim in this section is to establish necessary and sufficient conditions for $\Gamma(R)$ to be complete. Throughout, let:

$$Z(A) \bowtie^{(f,g)} (J, J') := \{(f(a) + j, g(a) + j')/a \in Z(A), (j, j') \in J \times J'\}$$

and for any $(j, j') \in J \times J'$, let:

$$(0 : j) = \{x \in f(A) + J/xj = 0\} \quad and \quad (0 : j') = \{x \in g(A) + J'/xj' = 0\}$$

In what follows, we give necessary and sufficient conditions for $\Gamma(R)$ to be complete.

Theorem 3.1. *Let $f : A \to B, g : A \to C$ be two ring homomorphisms, J, J' be two nonzero proper ideals of B and C respectively such that $0 \neq f^{-1}(J) = g^{-1}(J')$ and R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). If f is injective and $Z(f(A)+J) \subseteq J$ or $Z(g(A) + J') \subseteq J'$ then the following conditions are equivalent:*

(i) $\Gamma(R)$ *is complete;*

(ii) $Z(A)^2 = 0, J^2 = J'^2 = 0$, $Z(A) = f^{-1}(0 : j) = g^{-1}(0 : j')$ *for all* $(0,0) \neq (j,j') \in J \times J'$*;*

(iii) $Z(R)^2 = 0$.

Moreover, if any one of the three condition holds, then $Z(R) = Z(A) \bowtie^{((f,g))} (J, J')$ *and* $gr(\Gamma(R)) = 3$.

For the proof of Theorem 3.1, we need the lemma given below.

Lemma 3.2. *Under the above notation, suppose that* J *and* J' *are nonzero proper ideals of* B *and* C*, respectively, such that* $0 \neq f^{-1}(J) = g^{-1}(J')$*. Let:*
$V_1 := \{(0, j')/0 \neq j' \in J'\}$.
$V_2 := \{(j, 0)/0 \neq j \in J\}$.
$V_3 := \{(f(a) + j, g(a) + j')/0 \neq g(a) + j' \in Z(J')\}$.
$V_4 := \{(f(a) + j, g(a) + j')/0 \neq f(a) + j \in Z(J)\}$.
$V_5 := Z(f(A) + J)^* \times Z(g(A) + J')^*$.
Then : $Z(R) \subseteq V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$.

Proof. Let $x = (f(a) + j, g(a) + j') \in Z(R), a \in A, (j, j') \in J \times J'$, then there is $0 \neq y = (f(s)+i, g(s)+i') \in R$ such that $xy = 0, s \in A, (i, i') \in J \times J'$. Therefore $(f(a)+j)(f(s)+i) = 0$ and $(g(a)+j')(g(s)+i') = 0$. If $f(a)+j = 0$, then $f(a) \in J$, then $a \in f^{-1}(J) = g^{-1}(J')$ then $g(a) \in J'$, so $0 \neq g(a)+j' \in J'$. Hence $x \in V_1$. If $g(a)+j' = 0$, then $0 \neq f(a)+j \in J$. Hence $x \in V_2$. Assume that $0 \neq f(a) + j$ and $0 \neq g(a) + j'$. If $f(s) + i = 0$, then $0 \neq g(s) + i' \in J'$ and $(g(a) + j')(g(s) + i') = 0$, then $g(a) + j' \in Z(J')$. Thus $x \in V_3$. If $g(s) + i' = 0$, then $0 \neq f(s)+i \in J$ and $(f(a)+j)(f(s)+i) = 0$, then $f(a)+j \in Z(J)$. Thus $x \in V_4$. On the other hand $f(a) + j \in Z(f(A) + J)^*$ and $g(a) + j' \in Z(g(A) + J')^*$. Thus $x \in V_5$, as desired. □

Proof of Theorem 3.1
(1) ⇒ (2): Assume that $\Gamma(R)$ is complete. We claim that $Z(A)^2 = 0$. Assume that there exists $0 \neq a, b \in Z(A)$ such that $ab \neq 0$. Since f is injective, then $f(ab) \neq 0$. The fact that $a \in Z(A)$ implies that there exists $0 \neq s \in A : as = 0$, then $f(a)f(s) = g(a)g(s) = 0$. But f is injective, then $0 \neq f(s)$, then $(f(s), g(s)) \in R^*$ and $(f(a), g(a))(f(s), g(s)) = 0$, therefore $(f(a), g(a)) \in Z(R)$. The fact that $b \in Z(A)$ yields for the same reasoning that $(f(b), g(b)) \in Z(R)$. Since $\Gamma(R)$ complete, then $(f(a), g(a))(f(b), g(b)) = 0$ equivalent that $(f(ab), g(ab)) = 0$ equivalent that $f(ab) = g(ab) = 0$, a contradiction since f is injective and $0 \neq ab$. Hence $Z(A)^2 = 0$. Next We claim that $J^2 = J'^2 = 0$. Let $(0,0) \neq (j, j') \in J \times J'$, it's clear that $(j, 0), (0, j') \in Z(R)$. If $\mid J \mid \geq 3$ and $\mid J' \mid \geq 3$, then for all $0 \neq i \in J, i \neq j$ and all $0 \neq i' \in J', i' \neq j'$, we have $(i, 0)(j, 0) = 0$ and $(0, i')(0, j') = 0$, ($\Gamma(R)$ is complete), then $ij = i'j' = 0$. Hence $(j, j')(i, 0) = (j, j')(0, i') = 0$, then $(j, j') \in Z(R)$. Since $\Gamma(R)$ complete, then $(j, j')(j, 0) = (j, j')(0, j') = 0$, then $j^2 = j'^2 = 0$. If $\mid J \mid = \mid J' \mid = 2$. Let $0 \neq i \in J$ and $0 \neq i' \in J'$. If $i^2 = i$ and $i'^2 = i'$, then $i(1 - i) = i'(1 - i') = 0$, then $(1 - i, 1 - i')(i, i') = 0$. Since J and J' are proper ideals of B and C, respectively, then $0 \neq 1 - i$ and $0 \neq 1 - i'$. Then $(i, i') \in Z(R)$, then $(i, i')(i, 0) = (i, i')(0, i') = 0$ since $\Gamma(R)$ is complete. Then $i^2 = i'^2 = 0$, equivalent that $i = i' = 0$, a contradiction. In both cases we have $J^2 = J'^2 = 0$ if the zero-divisor graphs is complete. Next, We claim that $Z(A) = f^{-1}(0 : j) = g^{-1}(0 : j')$, for all $0 \neq j \in J$, $0 \neq j' \in J'$. Let $0 \neq a \in Z(A)$, then there exists $0 \neq s \in A : as = 0$, then $f(a)f(s) = g(a)g(s) = 0$. Since f is injective, then $0 \neq (f(s), g(s)) \in R$, then $(f(a), g(a)) \in Z(R)$ and we have $(j, j')(0, j') = 0$, then $(j, j') \in Z(R)^*$. Then $(f(a), g(a))(j, j') = 0$, since $\Gamma(R)$ is complete. Then $jf(a) = j'g(a) = 0$ for any $(0,0) \neq (j, j') \in J \times J'$, then $f(a) \in (0 : j)$ and $g(a) \in (0 : j')$. Then $a \in f^{-1}(0 : j)$ and $a \in g^{-1}(0 : j')$. Hence $Z(A) \subset f^{-1}(0 : j)$ and $Z(A) \subset g^{-1}(0 : j')$. Suppose that for any $(0,0) \neq (j, j') \in J \times J'$, we have $jf(a) = j'g(a) = 0$, for $0 \neq a \in A$, then $(f(a), g(a))(j, 0) = (f(a), g(a))(0, j') = 0$. Let $0 \neq c \in f^{-1}(J) = g^{-1}(J')$ such that $c \notin ker(f) \cup ker(g)$, then $(f(a), g(a))(f(c), 0) = (f(a), g(a))(0, g(c)) = 0$, then $f(ac) = g(ac) = 0$. Since f is injective, then $ac = 0$, then $a \in Z(A)$. Hence $Z(A) = f^{-1}(0 : j) = g^{-1}(0 : j')$, for any $(0,0) \neq (j, j') \in J \times J'$.
(2) ⇒ (3): Assume that $Z(A)^2 = 0, J^2 = J'^2 = 0$ and $Z(A) = f^{-1}(0 : j) = g^{-1}(0 : j')$, for all

$(j,j') \in J \times J'$. Let $x = (f(a)+j, g(a)+j') \in Z(R)^*$, $a \in A, (j,j') \in J \times J'$, then by lemma 3.2, $x \in V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$. If $x \in V_1 \cup V_2$, it's easy to show that $a \in Z(A)$. If $x \in V_3$, then $0 \neq g(a)+j' \in Z(J')$, then there exists $0 \neq k' \in J'$ such that $k'(g(a)+j') = 0$, then $kg(a) = 0$, so $a \in g^{-1}(0 : k') = Z(A)$. If $x \in V_4$, then $0 \neq f(a)+j \in Z(J)$ there exists $0 \neq k \in J$ such that $k(f(a)+j) = 0$, then $kf(a) = 0$, then $a \in f^{-1}(0 : k) = Z(A)$. If $x \in V_5$, then $f(a)+j \in Z(f(A)+J)^*$ and $g(a)+j' \in Z(g(A)+J')^*$. Suppose that $Z(f(A)+J) \subseteq J$, since $J^2 = 0$, then for all $0 \neq i \in J, i(f(a)+j) = 0$, then $a \in f^{-1}(0 : i) = Z(A)$. Hence $Z(R) \subseteq Z(A) \bowtie^{(f,g)} (J, J')$. On the other hand, let $(f(a)+j, g(a)+j') \in Z(A) \bowtie^{(f,g)} (J, J')$, then $a \in Z(A), (j,j') \in J \times J'$. Then there is $0 \neq b \in Z(A)$ such that $ab = 0$ since $Z(A)^2 = 0$, then $f(ab) = g(ab) = 0$. Then $(f(a)+j, g(a)+j')(f(b), g(b)) = (f(ab)+jf(b), g(ab)+j'g(b)) = (jf(b), j'g(b))$. The fact that $b \in Z(A)$, implies that for all $(0,0) \neq (j,j') \in J \times J'; jf(b) = j'g(b) = 0$. Then $(f(a)+j, g(a)+j')(f(b), g(b)) = 0$, then $(f(a)+j, g(a)+j') \in Z(R)$. Hence $Z(R) = Z(A) \bowtie^{(f,g)} (J, J')$. Let $x \neq y \in Z(A) \bowtie^{(f,g)} (J, J')$, it's easy to prove that $xy = 0$, then $Z(R)^2 = 0$.
(3) $\Rightarrow$ (1): It's trivial by the Anderson, Livingston definition. The "moreover" is proved by (2) $\Rightarrow$ (3) and the fact that if $0 \neq a \in f^{-1}(J) = g^{-1}(J')$, then $(f(a),0) - (f(a),g(a)) - (0,g(a)) - (f(a),0))$ is a cycle in $\Gamma(R)$ of length 3, then $gr(\Gamma(R)) = 3$.

Example 3.3. Let $A := Z_4$, $B := Z_4 \times Z_4$, $J := 2Z_4 \times 2Z_4$ be an ideal of B, $f : A \to B$ be the ring homomorphism defined by: $f(a) = (a,a)$, for all $a \in A$, then f is injective, $f^{-1}(J) = \{0,2\}$. Let $C = Z_2 \times Z_2[X]/(X^2)$, $J' := ((0,\overline{X}))$ be an ideal of C, $g : A \to C$ be the ring homomorphism defined by: $g(0) = g(2) = 0$ and $g(1) = g(3) = 1$. It' easy to show that $g^{-1}(J') = \{0,2\} = f^{-1}(J)$. Let R be the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g) . It's easy to show that $Z(A) = \{0,2\} = f^{-1}(0 : (0 : \overline{X})) = g^{-1}(0 : j)$, $0 \neq j \in J$, $J^2 = 0 = J'^2$, $Z(A)^2 = 0$ and $Z(f(A)+J) = \{(0,0),(0,2),(2,0),(2,2)\} = J$. Thus by Theorem 3.1, $\Gamma(R)$ is complete.

4 On the girth of the zero-divisors graph of bi-amalgamations

This section deals with the girth of the zero-divisor graphs of bi-amalgamations $R := A \bowtie^{(f,g)} (J, J')$ for some various settings of $f(A)+J$ and $g(A)+J'$.

Consider the following conditions:
w_1: For any distinct elements $0 \neq i, j \in J, ij \neq 0$.
w_2: For any distinct elements $0 \neq i', j' \in J'; i'j' \neq 0$.
w_3: For any elements $0 \neq a, b \in f^{-1}(J), a \neq b; ab \neq 0$.
$\overline{w_1}$: There are two distinct elements $0 \neq i, j \in J$ with $ij = 0$.
$\overline{w_2}$: There are two distinct elements $0 \neq i', j' \in J'$ with $i'j' = 0$.
$\overline{w_3}$: There are two distinct elements $0 \neq a, b \in f^{-1}(J)$ with $ab = 0$.

Theorem 4.1. *Under the above notation, suppose that J and J' are nonzero proper ideals of B and C, respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$.*

(i) If $\overline{w_1}$ or $\overline{w_2}$ hold, then $gr(\Gamma(R)) = 3$.

(ii) If $f(A)+J$ is a domain and $\mid J' \mid = 2$ or $g(A)+J'$ is a domain and $\mid J \mid = 2$, then $gr(\Gamma(R)) = \infty$.

(iii) If $f(A)+J$ is a domain, $\mid J' \mid \geq 3$ and w_2 holds or $g(A)+J'$ is a domain, $\mid J \mid \geq 3$ and w_1 holds, then $gr(\Gamma(R)) = 4$.

Proof. (1): Let $0 \neq i, j \in J, i \neq j$ with $ij = 0$, let $0 \neq j' \in J'$, clearly, $(i,0) - (0,j') - (j,0) - (i,0)$ is a cycle in $\Gamma(R)$ of length 3, then $gr(\Gamma(R)) = 3$. Let $0 \neq i', j' \in J', i' \neq j'$ with $i'j' = 0$, let $0 \neq j \in J$. Then $(0,j') - (j,0) - (0,i') - (0,j')$ is a cycle in $\Gamma(R)$ of length 3, then $gr(\Gamma(R)) = 3$.
(2): Assume $f(A)+J$ is a domain and $\mid J' \mid = 2$. Suppose that $\Gamma(R)$ contain a cycle of length n, say: $x_1 - x_2 - x_3 - ... - x_n - x_1$ with $x_k := (f(a_k)+i_k, g(a_k)+i'_k)$, for $k = 1, ..., n$. If $f(a_1)+i_1 \neq 0$, then $f(a_2)+i_2 = f(a_n)+i_n = 0$, so $0 \neq g(a_2)+i'_2 \in J'$ and $0 \neq g(a_n)+i'_n \in J'$ with $g(a_2)+i'_2 \neq g(a_n)+i'_n$, it's a contradiction with the hypothesis. If $f(a_1)+i_1 = 0$,

then $0 \neq g(a_1) + i_1 \in J'$. Since $(f(a_2) + i_2)(f(a_3) + i_3) = 0$, then $f(a_2) + i_2 = 0$ or $f(a_3) + i_3 = 0$, then $0 \neq g(a_2) + i'_2 \in J'$ or $0 \neq g(a_3) + i'_3 \in J'$ with $g(a_1) + i'_1 \neq g(a_2) + i'_2$ or $g(a_1) + i'_1 \neq g(a_3) + i'_3$; it's contradiction. With the same reasoning, in both cases we have a contradiction with our hypothesis, then $gr(\Gamma(R)) = \infty$. Assume that $g(A) + J'$ is a domain and $| J | = 2$. Suppose that $\Gamma(R)$ contains a cycle of length n, say: $x_1 - x_2 - ... - x_n - x_1$ with $x_k := (f(a_k)+i_k, g(a_k)+i'_k)$, for $k = 1, ..., n$. If $g(a_1)+i'_1 \neq 0$, then $g(a_2)+i'_2 = g(a_n)+i'_n = 0$, then $0 \neq f(a_2)+i_2 \in J$ and $0 \neq f(a_n)+i_n \in J$ with $f(a_2)+i_2 \neq f(a_n)+i_n$, it's a contradiction. If $g(a_1) + i'_1 = 0$, then $0 \neq f(a_1) + i'_1 \in J$. Since $(g(a_2) + i'_2)(g(a_3) + i'_3) = 0$ and $g(A) + J'$ is a domain, then $g(a_2) + i'_2 = 0$ or $g(a_3) + i'_3 = 0$, then $f(a_1) + i_1 \neq f(a_2) + i_2 \in J$ or $f(a_1) + i_1 \neq f(a_3) + i_3 \in J$, a contradiction. Then $gr(\Gamma(R)) = \infty$.
(3): Assume that $f(A) + J$ is a domain, $| J' | \geq 3$ and (C_2) holds. Let $0 \neq i', j' \in J', i' \neq j'$, let $0 \neq i \in J$. clearly, $(0, j') - (i, 0) - (0, i') - (i^2, 0) - (0, j')$ is a cycle in $\Gamma(R)$ of length 4, then $gr(\Gamma(R)) \leq 4$.
Suppose that $gr(\Gamma(R)) = 3$, and let $x_1 - x_2 - x_3 - x_1$ a cycle in $\Gamma(R)$ of length 3 with $x_k := (f(a_k) + i_k, g(a_k) + i'_k)$, for $k = 1, 2, 3$. If $f(a_1) + i_1 \neq 0$, then $f(a_2) + i_2 = f(a_3) + i_3 = 0$, then $0 \neq g(a_2) + i'_2 \in J'$ and $0 \neq g(a_3) + i'_3 \in J'$ with $g(a_2) + i'_2 \neq g(a_3) + i'_3$ and $(g(a_2) + i'_2)(g(a_3) + i'_3) = 0$, a contradiction. If $f(a_1) + i_1 = 0$, then $0 \neq g(a_1) + i'_1 \in J'$. Since $(f(a_2) + i_2)(f(a_3) + i_3) = 0$ then $f(a_2) + i_2 = 0$ or $f(a_3) + i_3 = 0$, then $0 \neq g(a_2) + i'_2 \in J'$ or $0 \neq g(a_3) + i'_3 \in J'$ with $g(a_1) + i'_1 \neq g(a_2) + i'_2$ and $(g(a_1) + i'_1)(g(a_2) + i'_2) = 0$ or $g(a_1) + i'_1 \neq g(a_3) + i'_3$ and $(g(a_1) + i'_1)(g(a_3) + i'_3) = 0$, a contradiction. Then $gr(\Gamma(R)) = 4$.
Assume that $g(A) + J'$ is a domain, $| J | \geq 3$ and (C_1) holds. Let $0 \neq i, j \in J, i \neq j$, let $0 \neq j' \in J'$, then $(i, 0) - (0, j') - (j, 0) - (0, j'^2) - (i, 0)$ is a cycle in $\Gamma(R)$ of length 4, then $gr(\Gamma(R)) \leq 4$. Suppose that $gr(\Gamma(R)) = 3$ and let $x_1 - x_2 - x_3 - x_1$ a cycle in $\Gamma(R)$ of length 3 with $x_k := (f(a_k) + i_k, g(a_k) + i'_k)$, for $k = 1, 2, 3$. If $g(a_1) + i'_1 \neq 0$, then $g(a_2)+i'_2 = g(a_3)+i'_3 = 0$, then $0 \neq (f(a_2)+i_2), (f(a_3)+i_3) \in J$ with $f(a_2)+i_2 \neq f(a_3)+i_3$ and $(f(a_2) + i_2)(f(a_3) + i_3) = 0$, a contradiction. If $g(a_1) + i'_1 = 0$, then $0 \neq f(a_1) + i_1 \in J$. Since $(g(a_2) + i'_2)(g(a_3) + i'_3) = 0$ and $g(A) + J'$ is a domain, then $g(a_2) + i'_2 = 0$ or $g(a_3) + i'_3 = 0$, then $0 \neq f(a_2) + i_2 \in J$ or $0 \neq f(a_3) + i_3 \in J$ with $f(a_1) + i_1 \neq f(a_2) + i_2$ and $(f(a_1) + i_1)(f(a_2) + i_2) = 0$ or $(f(a_1) + i_1)(f(a_3) + i_3) = 0$, a contradiction. Hence in both cases we have $gr(\Gamma(R)) = 4$. □

Corollary 4.2. *[10, Theorem 4.1] Let $f : A \to B$ be a ring homomorphism, J a non-zero proper ideal of B such that $0 \neq f^{-1}(J)$.*

(i) If $\overline{w_1}$ or $\overline{w_3}$ hold, then $gr(\Gamma(A \bowtie^f (J))) = 3$.

(ii) If A is a domain and $| J | = 2$ or $| f^{-1}(J) | = 2$ and $f(A) + J$ is a domain, then $gr(\Gamma(A \bowtie^f (J))) = \infty$.

(iii) If A is a domain, $| J | \geq 3$ and (w_1) holds or $f(A) + J$ is a domain, $| f^{-1}(J) | \geq 3$ and (w_3) holds, then $gr(\Gamma(A \bowtie^f (J))) = 4$.

Next we give a result when $f(A) + J$ and $g(A) + J'$ are domains.

Corollary 4.3. *Under the above notation, suppose that J and J' are nonzero proper ideals of B and C, respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$. Assume that $f(A) + J$ and $g(A) + J'$ are domains. Then $gr(\Gamma(R)) = 4$.*

Corollary 4.4. *[10, Corollary 4.2] Let $f : A \to B$ a ring homomorphism, J a nonzero proper ideal of B. Assume that both A and $f(A) + J$ are domains with $0 \neq f^{-1}(J)$. Then $gr(\Gamma(A \bowtie^f (J))) = 4$.*

The special case, where neither $f(A) + J$ nor $g(A) + J'$ is domain, is given below.

Proposition 4.5. *Let $f : A \to B, g : A \to C$ be two ring homomorphisms, J, J' be two nonzero proper ideals of B and C, respectively, such that $0 \neq f^{-1}(J) = g^{-1}(J')$. If $Z(A) \neq 0$ and f and g are injectives. Then $gr(\Gamma(A \bowtie^{(f,g)} (J, J'))) = 3$.*

Proof. Assume that $Z(A) \neq 0$, f and g are injectives. Let $0 \neq a, b \in A$ such that $ab = 0$ ($a = b$ is a possibility), let $0 \neq c \in f^{-1}(J) = g^{-1}(J')$. If $ca = 0$ or $cb = 0$, then $c \in f^{-1}(J) \cap Z(A)$. If $ca \neq 0$ and $cb \neq 0$, then $ca \in f^{-1}(J) \cap Z(A)$. Then without loss generality, we assume that

$f^{-1}(J) \cap Z(A) \neq \{0\}$. Let $0 \neq x \in f^{-1}(J) \cap Z(A)$ and let $0 \neq y \in A$ such that $xy = 0$, then $f(x)f(y) = g(x)g(y) = 0$. Since f and g injectives, then $0 \neq f(x), f(y)$ and $0 \neq g(x), g(y)$. Hence , $(f(y), g(y)) - (f(x), 0) - (0, g(x)) - (f(y), g(y))$ is a cycle in $\Gamma(A \bowtie^{(f,g)} (J, J'))$ of length 3, then $gr(\Gamma(A \bowtie^{(f,g)} (J, J'))) = 3$. □

Corollary 4.6. *[10, Proposition 4.4] Let $f : A \to B$ be a ring homomorphism, J a non-zero proper ideal of B. Assume that $Z(A) \neq 0$ and f injective with $0 \neq f^{-1}(J)$. Therefore $gr(\Gamma(A \bowtie^f (J))) = 3$.*

Example 4.7. Let $A := Z_6$, $B := Z_6 \times Z_6$, $C := Z_8 \times Z_8$, $J := 2Z_6 \times Z_6$ and $J' := 2Z_8 \times Z_8$. Consider the ring homomorphisms : $f : A \to B$ defined by: $f(a) = (a, a)$ for all $a \in A$ and $g : A \to C$ defined by: $g(a) = (a, a)$ for all $a \in A$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). It's easy to show that $\overline{w_1}$ holds, then by Theorem 4.1(1), $gr(\Gamma(R)) = 3$.

Example 4.8. Let $A := Z_4$, $B := Z_2[X]$, $J := XB$, $C := Z_4$, $J' := 2Z_4$. Consider the ring homomorphisms $f : A \to B$ and $g : A \to C$ defined by:$f(0) = f(2) = 0$, $f(1) = f(3) = 1$ and $g(a) = a$ for all $a \in A$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). It's easy to show that $f(A) + J$ is a domain and $\mid J' \mid = 2$, then by Theorem 4.1(2), $gr(\Gamma(R)) = \infty$.

Example 4.9. Let $A := Z_4$, $B := Z_2[X]/(X^2)$, $J := \{0, x\}(\mid J \mid = 2)$, where x denotes the class of X modulo (X^2) and $f : A \to B$ defined by: $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$. Let $C := Z_3$, $J' := 2Z_3$ and $g : A \to C$ defined by: $g(0) = g(2) = 0$ and $g(1) = g(3) = 1$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). It's easy to show that $g(A) + J' = Z_3$ is a domain and $\mid J \mid = 2$, then by Theorem 4.1(2), $gr(\Gamma(R)) = \infty$.

Example 4.10. Let $A := Z_4$, $B := Z_2[X]$, $J := XB$, $C := Z_6$ and $J' := 2Z_6$. Consider the ring homomorphisms $f : A \to B$ defined by : $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$ and $g : A \to C$ defined by : $g(a) = a$ for all $a \in A$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). It's easy to show that $\mid J' \mid \geq 3$, $f(A) + J$ is a domain and (w_2) holds, then by Theorem 4.1(3), $gr(\Gamma(R)) = 4$.

Example 4.11. Consider the bi-amalgamation R defined as in Example 4.7. Clearly $Z(A) \neq 0$ and f and g are injectives, then by Proposition 4.5, $gr(\Gamma(R)) = 3$.

Example 4.12. Let $A := Z_4$, $B := Z_2[X]$, $J := XB$, $C := Z_5$, $J' := 2Z_5$. Consider the ring homomorphisms $f : A \to B$ defined by: $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$ and $g : A \to C$ defined by: $g(a) = a$. Let R be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). It's easy to show that $f(A) + J$ and $g(A) + J'$ are domains, then by Corollary 4.3, $gr(\Gamma(R)) = 4$.

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