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A NEW PERSPECTIVE ON EIGENVALUES AND EIGENVECTORS OF BICOMPLEX LINEAR OPERATORS

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Abstract. This paper studies the spectral structure of bicomplex linear operators of the form $T = e_1T_1 + e_2T_2$ acting on $\mathbb{C}_2^n$. Owing to the presence of zero divisors in the bicomplex algebra, classical spectral theory does not fully capture partial degeneracy phenomena. Motivated by this structural feature, we introduce and systematically analyze the notions of modified eigenvalues and modified eigenspaces. We prove that a bicomplex number $\kappa_1e_1 + \kappa_2e_2$ is a modified eigenvalue of T if and only if at least one of its idempotent components belongs to the classical spectrum of the corresponding complex operator. This yields a complete structural characterization of the modified spectrum as $\Upsilon = (\Upsilon_1 \times_e \mathbb{C}_1) \cup (\mathbb{C}_1 \times_e \Upsilon_2)$. Furthermore, we provide a precise classification of modified eigenspaces according to the spectral configuration of T_1 and T_2. These results clarify the role of one-sided singularity and reveal the intrinsic anisotropic nature of bicomplex spectral theory.

1 Introduction

The theory of bicomplex numbers has developed into an active and rapidly growing area within complex and hypercomplex analysis. Over the past decades, substantial progress has been made in understanding their algebraic, analytic, and geometric structures. Foundational studies have established the topology of bicomplex spaces, differentiability and analyticity of bicomplex functions, matrix theory over bicomplex algebras, and extensions of classical complex analytic constructions to the bicomplex setting (see, for example, [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13]). These developments provide a coherent structural framework for investigating both linear and nonlinear phenomena in multicomplex spaces.

Section 1 introduces bicomplex numbers and presents the fundamental definitions required throughout the paper. In addition, we describe bicomplex matrices and linear transformations, together with the role of the idempotent decomposition that plays a central role in bicomplex analysis.

Section 2 investigates eigenvalues and eigenvectors in bicomplex spaces. In particular, we study eigenvalues and eigenvectors of a class of $\mathbb{C}_1$-linear maps of the form $T = e_1T_1 + e_2T_2$, and analyze several properties and structural results associated with such operators.

Section 3 focuses on eigenspaces and modified eigenspaces of bicomplex $\mathbb{C}_1$-linear maps of the form $T = e_1T_1 + e_2T_2$. We also establish the relationship between the set of classical eigenvalues and the set of modified eigenvalues for such bicomplex $\mathbb{C}_1$-linear operators.

The spectral structure of bicomplex linear operators differs from the classical complex case due to the presence of zero divisors. Motivated by this feature, we introduce the notions of modified eigenvalues and modified eigenspaces, which capture situations where an operator becomes singular with respect to one idempotent component while remaining regular with respect to the other.

Bicomplex numbers: A bicomplex number is an element that can be expressed in the following form:

$$\xi = u_1 + i_1u_2 + i_2u_3 + i_1i_2u_4, \;\; u_k \in \mathbb{R}, \;\; 1 \leqslant k \leqslant 4, \;\; \text{with} \;\; i_1i_2 \;=\; i_2i_1, \;\; i_1^2 = i_2^2 = -1,$$

where u_k are real numbers with $1 \leqslant k \leqslant 4$, and the unit vectors i_1 and i_2 satisfy the conditions $i_1i_2 = i_2i_1$ and $i_1^2 = i_2^2 = -1$. The set of all bicomplex numbers is denoted as $\mathbb{C}_2$, referred to as the bicomplex space. The symbols $\mathbb{C}_1,\ \mathbb{C}_0$, for convenience, denote the set of all complex numbers and the set of all real numbers respectively. The bicomplex space $\mathbb{C}_2$ can be described in the following two ways:
By considering the real components form:

$$\mathbb{C}_2 \;\; := \;\; \{u_1 + i_1u_2 + i_2u_3 + i_1i_2u_4 \;:\; u_1, u_2, u_3, u_4 \in \mathbb{C}_0\}$$

and the complex components form:

$$\mathbb{C}_2 \;\; := \;\; \{z_1 + i_2z_2 \;:\; z_1, z_2 \in \mathbb{C}_1\}. \tag{1.1}$$

It is important to note that $\mathbb{C}_2$ contains zero-divisors, which means it is not a field but rather an algebra over $\mathbb{C}_1$. In $\mathbb{C}_2$, there are exactly four idempotent elements; $0, 1, e_1$, and e_2. The nontrivial idempotent elements e_1 and e_2 are defined as follows:

$$e_1 := \frac{(1 + i_1i_2)}{2} \quad \text{and} \quad e_2 := \frac{(1 - i_1i_2)}{2}.$$

Notice that $e_1 + e_2 = 1\ ,\ e_1e_2 = e_2e_1 = 0$ and $e_1^2 = e_1,\ e_2^2 = e_2$.

Definition 1.1. [11] (Principal Ideal) The proper principal ideals generated by e_1 and e_2 are denoted as I_1 and I_2 and defined as:

$$I_1 =: \{\xi e_1 : \xi \in \mathbb{C}_2\} = \{\xi^- e_1 : \xi^- \in \mathbb{C}_1\},$$
$$I_2 =: \{\xi e_2 : \xi \in \mathbb{C}_2\} = \{\xi^+ e_2 : \xi^+ \in \mathbb{C}_1\}.$$

Singular and non-singular elements: An element $\xi \in \mathbb{C}_2$ is said to be non-singular if there exists $\eta \in \mathbb{C}_2$ such that $\xi.\eta = 1$. In other words, ξ possesses a multiplicative inverse within $\mathbb{C}_2$. Also, an element in $\mathbb{C}_2$ that lacks such a multiplicative inverse is called a singular element. Moreover, there is an infinite number of elements that lack a multiplicative inverse.
Notably, $\xi = z_1 + i_2z_2$ is classified as singular if and only if the absolute value of $z_1 + i_2z_2$ equals zero, written as $z_1^2 + z_2^2 = 0$. The set of all singular elements in $\mathbb{C}_2$ is denoted as $O_2 = I_1 \cup I_2$.

The following result mentioned in [11] can be verified easily.

Proposition 1.2. *Let $\xi \in \mathbb{C}_2$ be a bicomplex number. Then we have*

ξ is singular if and only if $\quad \xi \in I_1 \cup I_2$

in other words, ξ is non-singular if and only if $\quad \xi \notin I_1 \cup I_2$.

Idempotent representation of bicomplex numbers: [11] Every bicomplex number $\xi = z_1 + i_2z_2$ can be uniquely represented as the complex combination of elements e_1 and e_2 in the following form:

$$\xi = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2,$$

where complex numbers $(z_1 - i_1z_2)$ and $(z_1 + i_1z_2)$ are called idempotent components of ξ and will be denoted by ξ^- and ξ^+ respectively. Therefore the number $\xi = z_1 + i_2z_2$ can also be written as $\xi = \xi^-e_1 + \xi^+e_2$, where $\xi^- = z_1 - i_1z_2$ and $\xi^+ = z_1 + i_1z_2$. Moreover the idempotent representation of the product of elements $\xi, \eta \in \mathbb{C}_2$ can be seen easily to be as:

$$\xi \cdot \eta = \left(\xi^-\eta^-\right) e_1 + \left(\xi^+\eta^+\right) e_2.$$

Cartesian product: [8] The n-times Cartesian product of $\mathbb{C}_2$ denoted by $\mathbb{C}_2^n$ and consists of all n-tuples of bicomplex numbers of the form $(\xi_1, \xi_2, \ldots, \xi_n)$. Each component ξ_i belongs to $\mathbb{C}_2$ for $i = 1, 2, \ldots, n$. Formally, this is expressed as:

$$\mathbb{C}_2^n =: \{(\xi_1, \xi_2, \ldots, \xi_n) :\ \xi_i \in \mathbb{C}_2;\ i = 1, 2, \ldots, n\}.$$

Furthermore, using idempotent representation, each element of $\mathbb{C}_2^n$ can be expressed uniquely as follows:

$$(\xi_1, \xi_2, \ldots, \xi_n) = (\xi_1^-, \xi_2^-, \ldots, \xi_n^-)e_1 + (\xi_1^+, \xi_2^+, \ldots, \xi_n^+)e_2,$$

such that $(\xi_1^-, \xi_2^-, \ldots, \xi_n^-)$ and $(\xi_1^+, \xi_2^+, \ldots, \xi_n^+)$ are n-tuples of complex numbers from the space $\mathbb{C}_1^n$.

Cartesian idempotent product: [8] The Cartesian product in bicomplex spaces is defined as follows: for any subsets $S_1, S_2 \subseteq \mathbb{C}_1^n$, their idempotent product, denoted $S_1 \times_e S_2$, is defined as a subset of $\mathbb{C}_2^n$ given by

$$S_1 \times_e S_2 =: \{e_1 x + e_2 y : x \in S_1, y \in S_2\}. \tag{1.2}$$

Analogous to the concept of cartesian product of bicomplex number, for any subsets $H_1, H_2 \subseteq \mathbb{C}_2^{m\times n}$, their idempotent product, denoted as $H_1 \times_e H_2$ is defined to be a subset of $\mathbb{C}_2^{m\times n}$ given as:

$$H_1 \times_e H_2 =: \{e_1 A + e_2 B : A \in H_1, B \in H_2\}. \tag{1.3}$$

Similarly for linear maps, for any non-empty subsets A_1, A_2 of the space of $\mathbb{C}_1$-linear maps $\mathrm{Hom}(\mathbb{C}_1^n, \mathbb{C}_1^m)$, their idempotent product, written as $A_1 \times_e A_2$, is defined to be a subset of $\mathrm{Hom}(\mathbb{C}_2^n, \mathbb{C}_2^m)$ given as

$$A_1 \times_e A_2 =: \{e_1 T_1 + e_2 T_2 : \ T_1 \in A_1, T_2 \in A_2\}. \tag{1.4}$$

Remark 1.3. For the convenience, we denote the set of all $\mathbb{C}_1$-linear maps from $\mathbb{C}_1^n$ to $\mathbb{C}_1^m$ by L_1^{nm}, instead of $\mathrm{Hom}(\mathbb{C}_1^n, \mathbb{C}_1^m)$ and set of all $\mathbb{C}_1$-linear maps from $\mathbb{C}_2^n$ to $\mathbb{C}_2^m$ by L_2^{nm}. Clearly both L_1^{nm} and L_2^{nm} are vector spaces over $\mathbb{C}_1$. Hence, we can see that

$$\dim(L_1^{nm}) = mn \ \text{ and } \ \dim(L_2^{nm}) = 4mn. \tag{1.5}$$

Definition 1.4. (**Bicomplex matrix**): A bicomplex matrix of order $m \times n$ is denoted by $A = [\xi_{ij}]_{m\times n}, \xi_{ij} \in \mathbb{C}_2$. We let the set of all bicomplex matrices of order $m \times n$ be denoted by $\mathbb{C}_2^{m\times n}$, i.e., we have

$$\mathbb{C}_2^{m\times n} =: \Big\{[\xi_{ij}] : \ \xi_{ij} \in \mathbb{C}_2; \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n\Big\}. \tag{1.6}$$

The usual matrix addition and scalar multiplication make the space $\mathbb{C}_2^{m\times n}$ to be a vector space over the field $\mathbb{C}_1$. This immediately indicates that the dimension of $\mathbb{C}_2^{m\times n}$ over $\mathbb{C}_1$ is 2mn, i.e. we have

$$\dim(\mathbb{C}_2^{m\times n}(\mathbb{C}_1)) = 2mn. \tag{1.7}$$

Analogous to the idempotent representation of bicomplex numbers, any bicomplex matrix $A = [\xi_{ij}]_{m\times n} \in \mathbb{C}_2^{m\times n}$ can be decomposed uniquely as

$$A = e_1 \, A^- \ + \ e_2 \, A^+, \tag{1.8}$$

where $A^- = \left[\xi_{ij}^-\right]_{m\times n}$, $A^+ = \left[\xi_{ij}^+\right]_{m\times n}$ are complex matrices.

Definition 1.5. (see definition [8, 2.3]) For any given $T_1, T_2 \in L_1^{nm}$, we can define a map $T\colon \mathbb{C}_2^n \to \mathbb{C}_2^m$ by the following rule:

$$T(\xi_1, \xi_2, \ldots, \xi_n) =: e_1 \cdot T_1(\xi_1^-, \ldots, \xi_n^-) + e_2 \cdot T_2(\xi_1^+, \ldots, \xi_n^+).$$

Clearly, T is a $\mathbb{C}_1$-linear map and can be written as $e_1 T_1 + e_2 T_2$. Thus the set of all such linear maps is the idempotent product $L_1^{nm} \times_e L_1^{nm}$, i.e., we have

$$L_1^{nm} \times_e L_1^{nm} \ \ =: \ \ \{ \, e_1 T_1 + e_2 T_2 \in L_2^{nm} : \ \ T_1, T_2 \in L_1^{nm} \} \, . \tag{1.9}$$

For convenience, the set of all such type of $T = e_1T_1 + e_2T_2 \colon \mathbb{C}_2^n \to \mathbb{C}_2^n$ linear operators is denoted by $L_1^n \times_e L_1^n$. The idempotent product $L_1^{nm} \times_e L_1^{nm}$ is a subspace of L_2^{nm} over the field $\mathbb{C}_1$. Consequently, $L_1^{nm} \times_e L_1^{nm}$ has dimension $2mn$. That is

$$\dim(L_1^{nm} \times_e L_1^{nm}(\mathbb{C}_1)) = 2mn. \tag{1.10}$$

The vector spaces $\mathbb{C}_2^{m\times n}$ and $L_1^{nm} \times_e L_1^{nm}$ are isomorphic because they have the same dimension over the field $\mathbb{C}_1$. So the matrix representation of $T = e_1T_1 + e_2T_2$ with respect to the ordered bases $\mathcal{B}_1, \mathcal{B}_2$ for $\mathbb{C}_1^n$ and $\mathbb{C}_1^m$ respectively is defined as

$$[T]_{\mathcal{B}_2}^{\mathcal{B}_1} =: e_1[T_1]_{\mathcal{B}_2}^{\mathcal{B}_1} + e_2[T_2]_{\mathcal{B}_2}^{\mathcal{B}_1}. \tag{1.11}$$

Where $[T_1]_{\mathcal{B}_2}^{\mathcal{B}_1}$ and $[T_2]_{\mathcal{B}_2}^{\mathcal{B}_1}$ are called the matrix of T_1 and T_2 relative to the pair of ordered bases $\mathcal{B}_1$ and $\mathcal{B}_2$. In the special case when $\mathbb{C}_1^n = \mathbb{C}_1^m$, the matrix representation of the operator $T = e_1T_1 + e_2T_2$ with respect to basis $\mathcal{B}$ for $\mathbb{C}_1^n$ is simplified to $[T]_{\mathcal{B}}$ from $[T]_{\mathcal{B}}^{\mathcal{B}}$. Thus, we have

$$[T]_{\mathcal{B}} = e_1[T_1]_{\mathcal{B}} + e_2[T_2]_{\mathcal{B}}. \tag{1.12}$$

Proposition 1.6. *(See [8]) Let $T, S \in L_1^{nm} \times_e L_1^{nm}$ be any elements such that $T = e_1T_1 + e_2T_2$ and $S = e_1S_1 + e_2S_2$. Then, we have*

(i) $T + S = e_1(T_1 + S_1) + e_2(T_2 + S_2)$.

(ii) $\alpha T = e_1(\alpha T_1) + e_2(\alpha T_2); \quad \forall \alpha \in \mathbb{C}_1$.

Theorem 1.7. *(See, [8]) Let $T \in L_1^{nm} \times_e L_1^{nm}$ so that $T = e_1T_1 + e_2T_2$ for some $T_1, T_2 \in L_1^{nm}$. Then, we have*

(i) $\ker(e_1T_1 + e_2T_2) = \ker T_1 \times_e \ker T_2$.

(ii) $Im\,(e_1T_1 + e_2T_2) = Im\,T_1 \times_e Im\,T_2$.

2 Eigenvalues and eigenvectors associated with $T = e_1T_1 + e_2T_2$

Earlier, [8] introduced an idempotent method for the matrix representation of linear maps of the form $T = e_1T_1 + e_2T_2 : \mathbb{C}_2^n \to \mathbb{C}_2^m$. This establishes a one-to-one correspondence between bicomplex matrices $A = [\xi_{ij}]_{n\times n}$ and such operators on $\mathbb{C}_2^n$. Through this correspondence, spectral properties of bicomplex matrices may be studied via their associated linear operators. In this section, we investigate eigenvalues and eigenvectors of operators of the above form.

Definition 2.1 (Singular and Non-Singular Vectors). A vector $(\xi_1, \ldots, \xi_n) \in \mathbb{C}_2^n$ is called *non-singular* if at least one component ξ_i is non-singular (i.e., invertible) in $\mathbb{C}_2$. Otherwise, it is called *singular*.

If a vector is non-singular, then $(\xi_1^-, \ldots, \xi_n^-) \neq 0$ and $(\xi_1^+, \ldots, \xi_n^+) \neq 0$. The converse need not hold.

Example 2.2. The vectors $(1, 0, \ldots, 0)$ and $(0, 1, \ldots, 0)$ are non-zero vectors of $\mathbb{C}_1^n$. However,

$$e_1(1, 0, \ldots, 0) + e_2(0, 1, \ldots, 0) = (e_1, e_2, 0, \ldots, 0)$$

is a singular vector in $\mathbb{C}_2^n$, since both e_1 and e_2 are zero divisors in $\mathbb{C}_2$.

Remark 2.3. Every vector $(\xi_1, \ldots, \xi_n) \in \mathbb{C}_2^n$ admits the idempotent decomposition

$$(\xi_1, \ldots, \xi_n) = e_1(\xi_1^-, \ldots, \xi_n^-) + e_2(\xi_1^+, \ldots, \xi_n^+).$$

Moreover, the vector is non-zero if and only if either $(\xi_1^-, \ldots, \xi_n^-) \neq 0$ or $(\xi_1^+, \ldots, \xi_n^+) \neq 0$.

Definition 2.4 (Classical and Modified Eigenvalues). Let $T = e_1T_1 + e_2T_2 \in L_1^n \times_e L_1^n$ be a $\mathbb{C}_1$-linear operator on $\mathbb{C}_2^n$.

A scalar $\lambda \in \mathbb{C}_1$ is called a **classical eigenvalue** of T if there exists a non-zero vector $(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}_2^n$ such that

$$T(\xi_1, \xi_2, \ldots, \xi_n) = \lambda(\xi_1, \xi_2, \ldots, \xi_n).$$

Such a vector is called a **classical eigenvector** associated with λ.

A scalar $\kappa \in \mathbb{C}_2$ is called a **modified eigenvalue** of T if there exists a non-zero vector $(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}_2^n$ such that

$$T(\xi_1, \xi_2, \ldots, \xi_n) = \kappa(\xi_1, \xi_2, \ldots, \xi_n).$$

Such a vector is called a **modified eigenvector** associated with κ.

Remark 2.5. Every classical eigenvalue of T is automatically a modified eigenvalue. However, the converse need not hold, since the bicomplex algebra $\mathbb{C}_2$ contains zero divisors and admits decomposition along idempotent directions.

Observe that classical eigenvalues are precisely those modified eigenvalues that lie in the embedded subfield $\mathbb{C}_1 \subset \mathbb{C}_2$. Thus, the notion of modified eigenvalue properly extends the classical spectral concept while retaining it as a special case.

The following example shows that not every modified eigenvalue is a classical eigenvalue.

Example 2.6. Let $T_1, T_2 : \mathbb{C}_1^2 \to \mathbb{C}_1^2$ be defined by

$$T_1(x, y) = (x, 0), \qquad T_2(x, y) = (x, y).$$

Define

$$T = e_1 T_1 + e_2 T_2 : \mathbb{C}_2^2 \to \mathbb{C}_2^2.$$

A straightforward computation shows that $(1, e_2)$ is a classical eigenvector (and hence a modified eigenvector) associated with the eigenvalue 1.

Moreover, $(e_1, 0)$ is a modified eigenvector of T associated with the modified eigenvalue $e_1 + re_2$, where $r \in \mathbb{C}_1$. For $r \neq 1$, the scalar $e_1 + re_2 \notin \mathbb{C}_1$, and hence it is not a classical eigenvalue.

Definition 2.7. Let $\eta = \eta^- e_1 + \eta^+ e_2 \in \mathbb{C}_2$ and $T = e_1 T_1 + e_2 T_2 \in L_1^{nm} \times_e L_1^{nm}$. We define the map ηT by

$$\eta T := e_1(\eta^- T_1) + e_2(\eta^+ T_2).$$

Then $\eta T \in L_1^{nm} \times_e L_1^{nm}$.

Theorem 2.8. *Let $T = e_1 T_1 + e_2 T_2 \in L_1^n \times_e L_1^n$ and $\kappa = \kappa_1 e_1 + \kappa_2 e_2 \in \mathbb{C}_2$. Then:*

(i) T is singular if and only if either T_1 is singular or T_2 is singular.

(ii) $(T - \kappa I)$ is singular if and only if either $(T_1 - \kappa_1 I)$ is singular or $(T_2 - \kappa_2 I)$ is singular.

Proof. (1) By Theorem 1.7, we have

$$\ker(T) = \ker(e_1 T_1 + e_2 T_2) = \ker(T_1) \times_e \ker(T_2).$$

Hence

$$T \text{ is singular} \iff \ker(T_1) \times_e \ker(T_2) \neq \{0\}.$$

By Remark 2.3, a vector of the form

$$e_1(\xi_1^-, \ldots, \xi_n^-) + e_2(\xi_1^+, \ldots, \xi_n^+)$$

is non-zero if and only if at least one of $(\xi_1^-, \ldots, \xi_n^-)$ or $(\xi_1^+, \ldots, \xi_n^+)$ is non-zero. Therefore,

$$\ker(T_1) \times_e \ker(T_2) \neq \{0\}$$

if and only if

$$\ker(T_1) \neq \{0\} \quad \text{or} \quad \ker(T_2) \neq \{0\}.$$

This is equivalent to the singularity of T_1 or T_2.

(2) Using Proposition 1.6, we write

$$T - \kappa I = e_1(T_1 - \kappa_1 I) + e_2(T_2 - \kappa_2 I).$$

By part (1) of this theorem, we obtain

$$(T - \kappa I) \text{ is singular} \iff (T_1 - \kappa_1 I) \text{ is singular or } (T_2 - \kappa_2 I) \text{ is singular.}$$

□

As a consequence, we obtain the following corollary for $\lambda \in \mathbb{C}_1$.

Corollary 2.9. *Let $T = e_1T_1 + e_2T_2 \in L_1^n \times_e L_1^n$, $I = e_1I + e_2I \in L_1^n \times_e L_1^n$ be the identity linear operator and $\lambda \in \mathbb{C}_1$, then $(T - \lambda I)$ is singular if and only if either $(T_1 - \lambda I)$ is singular or $(T_2 - \lambda I)$ is singular.*

Remark 2.10. The above result shows that singularity in the bicomplex setting is completely determined by the idempotent components. Thus the zero divisor structure of $\mathbb{C}_2$ induces a componentwise spectral behavior.

3 Structural Description of the Modified Spectrum and Eigenspaces

In this section, we develop a structural description of the modified spectrum and the associated eigenspaces of $T = e_1T_1 + e_2T_2$. By introducing spectral sets associated with T and its idempotent components, we establish precise relationships between classical eigenvalues, modified eigenvalues, and their corresponding eigenspaces. This framework reveals how the bicomplex spectral structure is governed by the interaction of two underlying complex spectral theories.

Theorem 3.1. *Let $T = e_1T_1 + e_2T_2 \in L_1^n \times_e L_1^n$ and let $I = e_1I + e_2I$ denote the identity operator on $\mathbb{C}_2^n$. Then a bicomplex number $\kappa = \kappa_1e_1 + \kappa_2e_2 \in \mathbb{C}_2$ is a modified eigenvalue of T if and only if the operator $(T - \kappa I)$ is singular.*

Proof. Suppose that $\kappa \in \mathbb{C}_2$ is a modified eigenvalue of T. Then there exists a non-zero vector $x = (\xi_1, \ldots, \xi_n) \in \mathbb{C}_2^n$ such that

$$Tx = \kappa x.$$

This implies

$$(T - \kappa I)x = 0, \quad x \neq 0.$$

Hence $\ker(T - \kappa I) \neq \{0\}$, and therefore $(T - \kappa I)$ is singular.

Conversely, suppose that $(T - \kappa I)$ is singular. Then there exists a non-zero vector $x = (\xi_1, \ldots, \xi_n) \in \mathbb{C}_2^n$ such that

$$(T - \kappa I)x = 0.$$

Thus

$$Tx = \kappa x.$$

Hence κ is a modified eigenvalue of T.

Therefore, κ is a modified eigenvalue of T if and only if $(T - \kappa I)$ is singular. □

As an immediate consequence of Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let $T = e_1T_1 + e_2T_2 \in L_1^n \times_e L_1^n$ and let $I = e_1I + e_2I$ denote the identity operator. A complex number $\lambda \in \mathbb{C}_1$ is an eigenvalue of T if and only if $(T - \lambda I)$ is singular.*

Definition 3.3. Let $T = e_1T_1 + e_2T_2$ be a bicomplex linear operator on $\mathbb{C}_2^n$. We denote by

$$\Upsilon := \sigma_{\text{mod}}(T), \qquad \Upsilon_1 := \sigma(T_1), \qquad \Upsilon_2 := \sigma(T_2),$$

the modified spectrum of T and the classical spectra of T_1 and T_2, respectively. The idempotent product of classical spectra of Υ_1 and Υ_2 is defined by

$$\Upsilon_1 \times_e \Upsilon_2 = \{\kappa_1e_1 + \kappa_2e_2 : \kappa_1 \in \Upsilon_1, \kappa_2 \in \Upsilon_2\}.$$

The introduction of these spectral sets allows a precise comparison between classical and modified spectral structures. The following theorem establishes the fundamental relationship between Υ and the idempotent product $\Upsilon_1 \times_e \Upsilon_2$.

Theorem 3.4. *If $\kappa_1 \in \Upsilon_1$ and $\kappa_2 \in \Upsilon_2$, then $\kappa_1e_1 + \kappa_2e_2$ is a modified eigenvalue of T. Equivalently,*

$$\Upsilon_1 \times_e \Upsilon_2 \subset \Upsilon.$$

Proof. Let us suppose that $\kappa = \kappa_1 e_1 + \kappa_2 e_2 \in \Upsilon_1 \times_e \Upsilon_2$, then

$$\begin{aligned}
&\kappa_1 \text{ is an eigenvalue of } T_1 \text{ and } \kappa_2 \text{ is an eigenvalue of } T_2\\
\Rightarrow\ &\exists \quad 0 \neq (v_1, v_2, \ldots, v_n) \in \mathbb{C}_1^n \quad \text{and} \quad 0 \neq (w_1, w_2, \ldots, w_n) \in \mathbb{C}_1^n \quad \text{such that}\\
&T_1(v_1, v_2, \ldots, v_n) = \kappa_1(v_1, v_2, \ldots, v_n) \quad \text{and} \quad T_2(w_1, w_2, \ldots, w_n) = \kappa_2(w_1, w_2, \ldots, w_n)\\
\Rightarrow\ &e_1T_1(v_1, v_2, \ldots, v_n) + e_2T_2(w_1, w_2, \ldots, w_n) = e_1\kappa_1(v_1, v_2, \ldots, v_n) + e_2\kappa_2(w_1, w_2, \ldots, w_n)\\
\Rightarrow\ &(e_1T_1 + e_2T_2)[e_1(v_1, v_2, \ldots, v_n) + e_2(w_1, w_2, \ldots, w_n)]\\
&\qquad = \kappa[e_1(v_1, v_2, \ldots, v_n) + e_2(w_1, w_2, \ldots, w_n)]\\
&\text{In view of 2.3, } e_1(v_1, v_2, \ldots, v_n) + e_2(w_1, w_2, \ldots, w_n) \neq 0\\
\Rightarrow\ &\kappa = \kappa_1 e_1 + \kappa_2 e_2 \quad \text{is a modified eigenvalue of} \quad T = e_1T_1 + e_2T_2.\\
\Rightarrow\ &\kappa = \kappa_1 e_1 + \kappa_2 e_2 \in \Upsilon.
\end{aligned}$$

Therefore, $\Upsilon_1 \times_e \Upsilon_2 \subset \Upsilon$.
This completes the proof. □

Remark 3.5. In view of 2.6, the set $\Upsilon_1 = \{0, 1\}$ and $\Upsilon_2 = \{1\}$ but $1e_1 + 2e_2$ is a modified eigenvalue of $T = e_1T_1 + e_2T_2$. It yields that the set $\Upsilon_1 \times_e \Upsilon_2$ is properly contained in the set Υ. That is, in case of 2.6,

$$\Upsilon_1 \times_e \Upsilon_2 \subsetneq \Upsilon. \tag{3.1}$$

We will prove further (3.1) for arbitrary Υ, Υ_1 and Υ_2. The following corollary characterizes the classical eigenvalues of T in terms of the spectra of its idempotent components.

Corollary 3.6. *Let $T = e_1T_1 + e_2T_2 \in L_1^n \times_e L_1^n$. A complex number $\lambda \in \mathbb{C}_1$ is an eigenvalue of T if and only if $\lambda \in \Upsilon_1 \cup \Upsilon_2$.*

As an immediate consequence of Theorems 2.8 and 3.1, we obtain the following characterization.

Corollary 3.7. *Let $T = e_1T_1 + e_2T_2 \in L_1^n \times_e L_1^n$, then $\kappa = \kappa_1 e_1 + \kappa_2 e_2 \in \mathbb{C}_2$ is a modified eigenvalue of T if and only if either $\kappa_1 \in \Upsilon_1$ or $\kappa_2 \in \Upsilon_2$.*

Theorem 3.8. *Let $T = e_1T_1 + e_2T_2 \in L_1^n \times_e L_1^n$. Then T admits a modified eigenvalue if and only if T admits a (classical) eigenvalue.*

Proof. Suppose that a modified eigenvalue of T exists. We use 3.6 and 3.7.

$$\begin{aligned}
&\Rightarrow \quad \text{there exists } \kappa = \kappa_1 e_1 + \kappa_2 e_2 \in \Upsilon \text{ such that either} \quad \kappa_1 \in \Upsilon_1 \quad \text{or} \quad \kappa_2 \in \Upsilon_2.\\
&\Rightarrow \quad \kappa_1 \in \Upsilon_1 \cup \Upsilon_2 \text{ or } \kappa_2 \in \Upsilon_1 \cup \Upsilon_2\\
&\Rightarrow \quad \text{either} \quad \kappa_1 \quad \text{is an eigenvalue of} \quad T \quad \text{or} \quad \kappa_2 \quad \text{is an eigenvalue of} \quad T\\
&\Rightarrow \quad T \text{ admits an eigenvalue.}
\end{aligned}$$

Conversely, if T has a complex eigenvalue, then by definition it is also a modified eigenvalue. Hence T has a modified eigenvalue. □

The following result exhibits the abundance of modified eigenvalues arising from a single classical eigenvalue of an idempotent component.

Theorem 3.9. *Let $T = e_1T_1 + e_2T_2 \in L_1^n \times_e L_1^n$. If $\kappa_1 \in \Upsilon_1$, then $\kappa_1 e_1 + we_2 \in \Upsilon$ for every $w \in \mathbb{C}_1$.*

Proof. Suppose that $\kappa_1 \in \Upsilon_1$ is a classical eigenvalue of T_1. Then there exists a non-zero vector $(v_1, \ldots, v_n) \in \mathbb{C}_1^n$ such that

$$T_1(v_1, \ldots, v_n) = \kappa_1(v_1, \ldots, v_n).$$

Consider the vector $x = e_1(v_1, \ldots, v_n) \in \mathbb{C}_2^n$. By Remark 2.3, $x \neq 0$.
Now, $T(x) = (e_1T_1 + e_2T_2)\big(e_1(v_1, \ldots, v_n)\big) = e_1T_1(v_1, \ldots, v_n) = e_1\kappa_1(v_1, \ldots, v_n) = (\kappa_1 e_1)x$.
Hence, $T(x) = (\kappa_1 e_1 + we_2)x \quad$ for every $w \in \mathbb{C}_1$, since $e_2x = 0$. Therefore, $\kappa_1 e_1 + we_2$ is a modified eigenvalue of T for every $w \in \mathbb{C}_1$.
This completes the proof. □

The following corollary provides the analogous strip phenomenon corresponding to the second idempotent component.

Corollary 3.10. *If $\kappa_2 \in \Upsilon_2$, then $ze_1 + \kappa_2 e_2 \in \Upsilon \quad$ for every $z \in \mathbb{C}_1$.*

We now state the main structural theorem describing the modified spectrum.

Theorem 3.11. *Let $T = e_1T_1 + e_2T_2 \in L_1^n \times_e L_1^n$. Then the modified spectrum of T satisfies*

$$\Upsilon = (\Upsilon_1 \times_e \mathbb{C}_1) \cup (\mathbb{C}_1 \times_e \Upsilon_2),$$

that is,

$$\Upsilon = \{\kappa_1 e_1 + \kappa_2 e_2 : \kappa_1 \in \Upsilon_1 \text{ or } \kappa_2 \in \Upsilon_2\}.$$

Proof. Let $\kappa = \kappa_1 e_1 + \kappa_2 e_2 \in \mathbb{C}_2$. By Theorem 3.1, a bicomplex number κ is a modified eigenvalue of T if and only if $(T - \kappa I)$ is singular. Using the idempotent decomposition,

$$T - \kappa I = e_1(T_1 - \kappa_1 I) + e_2(T_2 - \kappa_2 I).$$

By Theorem 2.8 (Part 2), the operator $(T - \kappa I)$ is singular if and only if either $(T_1 - \kappa_1 I)$ is singular or $(T_2 - \kappa_2 I)$ is singular. By definition of the classical spectra Υ_1 and Υ_2, this is equivalent to

$$\kappa_1 \in \Upsilon_1 \quad \text{or} \quad \kappa_2 \in \Upsilon_2.$$

Therefore,

$$\kappa \in \Upsilon \quad \Longleftrightarrow \quad \kappa_1 \in \Upsilon_1 \text{ or } \kappa_2 \in \Upsilon_2.$$

Hence,

$$\Upsilon = (\Upsilon_1 \times_e \mathbb{C}_1) \cup (\mathbb{C}_1 \times_e \Upsilon_2).$$

□

Remark 3.12. Theorem 3.11 shows that the modified spectrum is not simply the idempotent product $\Upsilon_1 \times_e \Upsilon_2$ of the classical spectra. Rather, it consists of two spectral strips:

$$\Upsilon_1 \times_e \mathbb{C}_1 = \{\kappa_1 e_1 + we_2 : \kappa_1 \in \Upsilon_1,\ w \in \mathbb{C}_1\},$$

$$\mathbb{C}_1 \times_e \Upsilon_2 = \{ze_1 + \kappa_2 e_2 : z \in \mathbb{C}_1,\ \kappa_2 \in \Upsilon_2\}.$$

Thus modified eigenvalues arise from one-sided spectral degeneracy of the idempotent components. In particular, even in finite dimensions, the modified spectrum is typically infinite whenever at least one of the component operators T_1 or T_2 has a nonempty spectrum. This reflects the intrinsic strip-type geometry of bicomplex spectral theory and highlights the anisotropic nature of its spectral structure.

Remark 3.13. It is evident that $\Upsilon_1 \times_e \Upsilon_2 \subsetneq (\Upsilon_1 \times_e \mathbb{C}_1) \cup (\mathbb{C}_1 \times_e \Upsilon_2)$, that is for arbitrary Υ, Υ_1 and Υ_2, $\Upsilon_1 \times_e \Upsilon_2 \subsetneq \Upsilon$.

Definition 3.14 (Eigenspace and Modified Eigenspace). Let $T = e_1T_1 + e_2T_2 \in L_1^n \times_e L_1^n$ be a linear operator on $\mathbb{C}_2^n$. Let $\lambda \in \mathbb{C}_1$ be an eigenvalue of T and $\kappa = \kappa_1 e_1 + \kappa_2 e_2 \in \mathbb{C}_2$ be a modified eigenvalue of T. The eigenspace and modified eigenspace of T corresponding to λ and κ are defined by

$$\begin{aligned} (E)_\lambda(T) &= \{(\xi_1, \ldots, \xi_n) \in \mathbb{C}_2^n : T(\xi_1, \ldots, \xi_n) = \lambda(\xi_1, \ldots, \xi_n)\}. \\ (ME)_\kappa(T) &= \{(\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{C}_2^n : T(\eta_1, \eta_2, \ldots, \eta_n) = \kappa(\eta_1, \eta_2, \ldots, \eta_n)\}. \end{aligned}$$

It is straightforward to verify that the modified eigenspace $(ME)_\kappa(T)$ of T associated with the modified eigenvalue κ forms a subspace of $\mathbb{C}_2^n$. The eigenspace of T_1 and T_2 associated with the eigenvalue z and w are denoted by $E_z(T_1)$ and $E_w(T_2)$, respectively.

The following theorems 3.15, 3.17 and 3.19 are immediate consequence of 3.7 and 3.14.

Theorem 3.15. *Let $T = e_1T_1 + e_2T_2$ and let $\kappa = \kappa_1e_1 + \kappa_2e_2 \in \Upsilon$ such that $\kappa_1 \in \Upsilon_1$ and $\kappa_2 \notin \Upsilon_2$. Then*

$$(ME)_\kappa(T) = E_{\kappa_1}(T_1) \times_e \{0\}.$$

Proof. Suppose that $(\eta_1, \eta_2, \ldots, \eta_n) \in (ME)_\kappa(T)$.

$$\begin{aligned}
&\Leftrightarrow T(\eta_1, \eta_2, \ldots, \eta_n) &&= \kappa(\eta_1, \eta_2, \ldots, \eta_n)\\
&\Leftrightarrow (e_1T_1 + e_2T_2)(\eta_1, \eta_2, \ldots, \eta_n) &&= (\kappa_1e_1 + \kappa_2e_2)(\eta_1, \eta_2, \ldots, \eta_n)\\
&\Leftrightarrow e_1T_1(\eta_1^-, \eta_2^-, \ldots, \eta_n^-) + e_2T_2(\eta_1^+, \eta_2^+, \ldots, \eta_n^+) &&= e_1\kappa_1(\eta_1^-, \eta_2^-, \ldots, \eta_n^-)\\
&&&\quad + e_2\kappa_2(\eta_1^+, \eta_2^+, \ldots, \eta_n^+)\\
&\Leftrightarrow T_1(\eta_1^-, \eta_2^-, \ldots, \eta_n^-) = \kappa_1(\eta_1^-, \eta_2^-, \ldots, \eta_n^-) &&\text{ and } T_2(\eta_1^+, \eta_2^+, \ldots, \eta_n^+) = \kappa_2(\eta_1^+, \eta_2^+, \ldots, \eta_n^+)\\
&\Leftrightarrow (\eta_1^-, \eta_2^-, \ldots, \eta_n^-) \in E_{\kappa_1}(T_1) &&\text{ and } (\eta_1^+, \eta_2^+, \ldots, \eta_n^+) = 0, \text{ as } \kappa_2 \notin \Upsilon_2\\
&\Leftrightarrow (\eta_1, \eta_2, \ldots, \eta_n) &&\in E_{\kappa_1}(T_1) \times_e \{0\}.
\end{aligned}$$

Hence, $(ME)_\kappa(T) = E_{\kappa_1}(T_1) \times_e \{0\}$.
This completes the proof. □

The following Corollary 3.16 is immediate consequence of 3.15.

Corollary 3.16. *If $\kappa = \kappa_1e_1 + \kappa_2e_2 \in \Upsilon$ with $\kappa_1 \in \Upsilon_1$ and $\kappa_2 \notin \Upsilon_2$, then every modified eigenvector of T associated with κ is singular. Moreover,*

$$(ME)_\kappa(T) = e_1\, E_{\kappa_1}(T_1).$$

Theorem 3.17. *Let $\kappa = \kappa_1e_1 + \kappa_2e_2 \in \Upsilon$ such that $\kappa_1 \notin \Upsilon_1$ and $\kappa_2 \in \Upsilon_2$. Then*

$$(ME)_\kappa(T) = \{0\} \times_e E_{\kappa_2}(T_2).$$

Proof. The proof follows from the same idempotent decomposition argument used in Theorem 3.15. Indeed, if $(\eta_1, \eta_2, \ldots, \eta_n) \in (ME)_\kappa(T)$, then writing

$$(\eta_1, \eta_2, \ldots, \eta_n) = e_1(\eta_1^-, \eta_2^-, \ldots, \eta_n^-) + e_2(\eta_1^+, \eta_2^+, \ldots, \eta_n^+)$$

and comparing idempotent components yields

$$T_1(\eta_1^-, \eta_2^-, \ldots, \eta_n^-) = \kappa_1(\eta_1^-, \eta_2^-, \ldots, \eta_n^-), \qquad T_2(\eta_1^+, \eta_2^+, \ldots, \eta_n^+) = \kappa_2(\eta_1^+, \eta_2^+, \ldots, \eta_n^+).$$

Since $\kappa_1 \notin \Upsilon_1$, we must have $(\eta_1^-, \eta_2^-, \ldots, \eta_n^-) = 0$, while $(\eta_1^+, \eta_2^+, \ldots, \eta_n^+) \in E_{\kappa_2}(T_2)$. Hence

$$(ME)_\kappa(T) = \{0\} \times_e E_{\kappa_2}(T_2).$$

This completes the proof. □

The following Corollary 3.18 is immediate consequence of 3.17.

Corollary 3.18. *If $\kappa_1 \notin \Upsilon_1$ and $\kappa_2 \in \Upsilon_2$, then every modified eigenvector of T associated with $\kappa = \kappa_1e_1 + \kappa_2e_2$ is singular. Moreover,*

$$(ME)_\kappa(T) = e_2\, E_{\kappa_2}(T_2).$$

Theorem 3.19. *If $\kappa_1 \in \Upsilon_1$ and $\kappa_2 \in \Upsilon_2$, then $(ME)_\kappa(T) = E_{\kappa_1}(T_1) \times_e E_{\kappa_2}(T_2)$.*

Proof. Suppose that $(\eta_1, \eta_2, \ldots, \eta_n) \in (ME)_\kappa(T)$

$$\begin{aligned}
&\Leftrightarrow T(\eta_1, \eta_2, \ldots, \eta_n) &&= \kappa(\eta_1, \eta_2, \ldots, \eta_n)\\
&\Leftrightarrow (e_1T_1 + e_2T_2)(\eta_1, \eta_2, \ldots, \eta_n) &&= (\kappa_1e_1 + \kappa_2e_2)(\eta_1, \eta_2, \ldots, \eta_n)\\
&\Leftrightarrow e_1T_1(\eta_1^-, \eta_2^-, \ldots, \eta_n^-) + e_2T_2(\eta_1^+, \eta_2^+, \ldots, \eta_n^+) &&= e_1\kappa_1(\eta_1^-, \eta_2^-, \ldots, \eta_n^-)\\
&&&\quad + e_2\kappa_2(\eta_1^+, \eta_2^+, \ldots, \eta_n^+)\\
&\Leftrightarrow T_1(\eta_1^-, \eta_2^-, \ldots, \eta_n^-) = \kappa_1(\eta_1^-, \eta_2^-, \ldots, \eta_n^-) &&\text{ and } T_2(\eta_1^+, \eta_2^+, \ldots, \eta_n^+) = \kappa_2(\eta_1^+, \eta_2^+, \ldots, \eta_n^+)\\
&\Leftrightarrow (\eta_1^-, \eta_2^-, \ldots, \eta_n^-) \in E_{\kappa_1}(T_1) &&\text{ and } (\eta_1^+, \eta_2^+, \ldots, \eta_n^+) \in E_{\kappa_2}(T_2)\\
&\Leftrightarrow (\eta_1, \eta_2, \ldots, \eta_n) &&\in E_{\kappa_1}(T_1) \times_e E_{\kappa_2}(T_2).
\end{aligned}$$

Hence, $(ME)_\kappa(T) = E_{\kappa_1}(T_1) \times_e E_{\kappa_2}(T_2)$.
This completes the proof. □

4 Open Problem

In Section 3 we established that each modified eigenspace of $T = e_1T_1 + e_2T_2$ admits the structural decomposition $(ME)_\kappa(T) = E_{\kappa_1}(T_1) \times_e E_{\kappa_2}(T_2)$.

This naturally raises the question of how different modified eigenspaces interact with one another.

In classical complex linear algebra, eigenspaces corresponding to distinct eigenvalues intersect trivially and their sum is direct. However, in the bicomplex setting, the presence of zero divisors and partial spectral degeneracy may alter this behavior.

Problem. Let $\kappa = \kappa_1 e_1 + \kappa_2 e_2, \kappa' = \kappa_1' e_1 + \kappa_2' e_2$ be two distinct modified eigenvalues of T. Under what conditions does the following equality hold?

$$(ME)_\kappa(T) + (ME)'_\kappa(T) = (ME)_\kappa(T) \oplus (ME)'_\kappa(T).$$

Equivalently, when do distinct modified eigenspaces intersect trivially?

This problem amounts to determining how the direct-sum decomposition of classical eigenspaces of T_1 and T_2 interacts with the idempotent product structure of $\mathbb{C}_2^n$. A complete answer would provide a refined understanding of spectral decompositions in bicomplex linear algebra. A complete resolution of this problem would determine whether bicomplex spectral theory admits a decomposition theorem analogous to the classical direct-sum decomposition of eigenspaces, or whether zero divisors fundamentally obstruct such a structure.

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