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The Evolution of the Concept of the Derivative: A Journey Through Time

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Abstract.Calculus is considered one of the fundamental topics in mathematics, playing a crucial role in the development of various sciences and influencing many fields. It relies primarily on basic concepts, including the derivative, which is used to describe and analyze changes in quantities. In this research, we provide a qualitative overview of the evolution of the concept of the derivative through the ages, focusing on key figures and significant concepts that have shaped this science. We trace its development from the geometric perspectives of the Greeks to the analytical approaches of Newton and Leibniz, who are regarded as the founders of modern calculus.

1 Introduction

The concept of calculus was introduced in mathematics through infinitesimal values that express precise changes in a state, used by Leibniz, Newton, Euler, Lagrange, Bernoulli, and many others in calculus. However, they encountered anomalous cases in algebraic procedures when involving infinitesimal quantities, which could not be explained by an accepted comprehensive model [13]. Here, the concept of the derivative was developed to represent a fundamental tool in the analysis of mathematical functions, enabling the study of the behavior of functions and their interaction with variables of time and space, making it a cornerstone of analytical mathematics and its applications [11, 15]. It is also considered a vital tool in physical analysis, allowing for the calculation of speeds, accelerations, and changes in forces and various physical fields [38]. What is known today as derivatives was initially used as a specialized means to solve specific problems. The general concept used to address these problems was then established within the discovery of calculus. Subsequently, the properties of the derivative were discovered and developed in various applications of mathematics and physics. Finally, a rigorous definition of the concept of the derivative was established within a constant law [15]. Historically, there were four steps in the development of today's concept of the derivative, which are listed here in chronological order: the derivative was first used; then discovered and developed; and finally defined. This was the result of the successive efforts of Western scholars after studying the works of Greek and Indian mathematicians and the discoveries of Arab Muslims in algebra.

Here, we will review some important pathways that scientists and mathematicians took in adopting different positions that ultimately led to the formation and description of the concept of the derivative, represented by three diverse approaches, which we list as follows:

1. The role of the Greeks in the emergence of the concept of the derivative.
2. The role of Muslim Arabs in constructing the concept of the derivative.
3. The role of the West in building and developing the concept of the derivative.

The concept of the derivative continued to evolve throughout history, as the idea of measuring instantaneous change in functions expanded thanks to many scientists such as Lagrange, Cauchy,

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and Augustin, among others, who contributed to the development of modern mathematical theories. Thus, scientists continued to build the foundations of calculus, aiding in the development of numerous scientific fields [8].

2 Historical Background

The origin of the discovery of the derivative in classical geometry lies in its use to solve problems. Ancient Greek mathematicians were also philosophers and emphasized the abstract and ideal nature of things. For them, relationships in the physical world were subject to change and did not represent absolute truth, while relationships in the ideal world were constant and absolute. Consequently, they were more interested in shapes than in rates of change, and the concept of calculus existed only in a few isolated cases [3]. Generally, understanding the derivative as an instantaneous rate of change or the slope of a tangent line became easier only after the innovations of algebra and analytic geometry.

The Role of the Greeks in the Emergence of the Concept of the Derivative

The study of tangent lines did not become widely explored until around 1635 AD, focusing on circles, conic sections, and more complex curves using synthetic geometry. Euclid (circa 300 BC), in his book *Elements*, defines the tangent line to a circle as a straight line produced so that it touches the circle at a single point without cutting it. His definition relies on the term "cut", meaning the line intersects the circle more than once, which limits this definition to circles only. This property does not hold for other curves, such as the Archimedean spiral, where the tangent can intersect the curve multiple times, as noted by Euclid [25].

After that, Euclid established a new result concerning tangents, now known as the *Alternate Segment Theorem*. This theorem states that if we have points A and B on a circle and T is a point on the tangent TP, then the measure of the tangent angle $\angle ATP$ is equal to the measure of the angle $\angle ABT$ in the alternate segment, as illustrated in Figure 1.

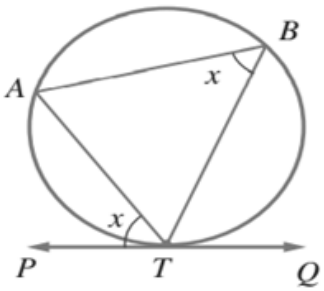


Figure 1. The Alternate Segment Theorem for Conic Sections

Archimedes (287–212 BC) developed a method for determining the tangent to the curve generated by a moving wheel. He defined the shape of the motion as follows: "For every straight line drawn in a plane, it rotates at a uniform constant rate around one end and returns to the position from which it started. If a point moves simultaneously with the rotation of the line at a uniform rate along the straight line starting from the fixed end, the point will the point traces a spiral in the plane in the plane" [24].

Archimedes divided the motion of the point into two components:a uniform radial motion away from the origin and a circular motion around the origin. By combining these two motions and analyzing the resulting polygon of velocities, he determined the direction of motion and thus identified the tangent to the curve, as illustrated in Figure 2.

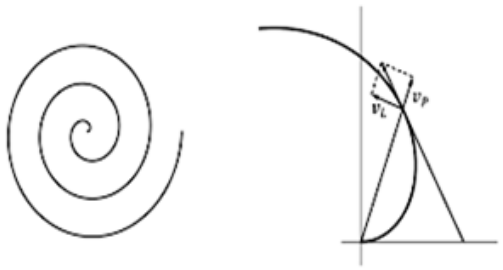


Figure 2. The Spiral of Archimedes with Three Turns, Interpreting the Tangent Line as a Result of Combined Motion

Archimedes' theorem (the tangent to the wheel) states that: If P is any point on the first turn of the spiral and OT is drawn perpendicular to OP, then OT will intersect the tangent at P to the spiral at some point T, and if the circle drawn with center O and radius OP intersects the initial line at K, then OT is equal to the arc of this circle between K and P measured in the "forward" direction of the spiral [20].

This result is illustrated in the attached figure, which shows a spiral created by the initial line during the KO alternately counterclockwise. Essentially, OT is drawn vertically to OP and is equal to the length of the arc KP. The tangent line to the spiral at point P is found by connecting P and T.

Indeed, constructing a straight line whose length equals the arc of a circle is generally impossible using a classical protractor and straightedge method, as the length may involve a multiple of π, which is transcendental and therefore cannot be constructed. Since every constructible number is a root of some polynomial equations with rational coefficients, it is assumed that the tangent line can be constructed [18].

Archimedes proves his theorem through a proof by contradiction, and we include a detailed outline below of his method of proof, where he used *neusis* (a construction devised by Archimedes to describe the plane for solving problems in the plane). A straight line must be drawn through a specific point that intersects two given curves at a segment of a certain length [9].

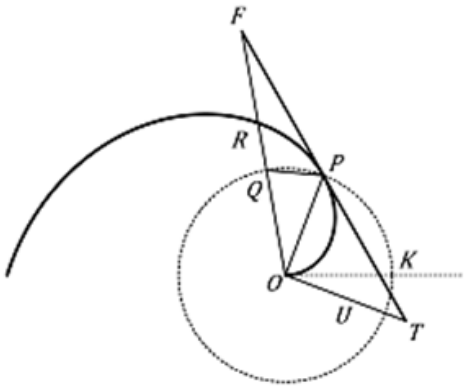


Figure 3. Archimedes' Neusis Construction

What we hope to achieve here is to prove that OT is equal to the length of the arc KP, and this is done as follows:

First: Assume that OT is greater than the arc KP. We draw OU such that $KP < OU < OT$ (see Figure 3). Using the *neusis* construction, it is possible to draw OQF such that:

$$\frac{OP}{OU} = \frac{FQ}{PQ}$$

$$\frac{OF}{OQ} = \frac{\text{arc } KQ}{\text{arc } KP} < \frac{OR}{OP}$$

$$\frac{FQ}{OQ} < \frac{PQ}{OU} < \frac{\text{arc } PQ}{\text{arc } KP}$$

But $OQ = OP$, so $OF < OR$, which is impossible. Therefore, OT is not greater than the arc KP.

Next, assume that OT is less than the arc KP, so we draw OU'. We now have $OU' > OT$ but $OU' <$ arc KP (see Figure 4). Using the *neusis* construction, it is possible to draw OKF' in this way:

$$\frac{FQ'}{PG} = \frac{OP}{OU'}$$

$$\frac{F'Q'}{OQ'} = \frac{PG}{OU'} > \frac{\text{arc } PQ'}{\text{arc } KP} \quad \text{and} \quad \frac{OF'}{OQ'} < \frac{\text{arc } KQ'}{\text{arc } KP} < \frac{OR'}{OP}$$

But $OQ' = OP$, so $OF' < OR'$, which is impossible. Therefore, OT is not less than the arc KP.

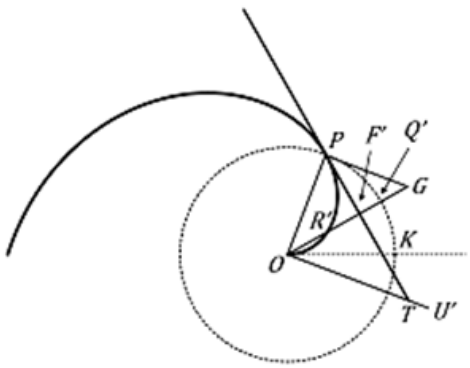


Figure 4. Archimedes' Neusis Construction

And since OT is neither greater nor less than the arc KP, it follows that it is equal to KP, which is what Archimedes aimed to prove.

In general, Archimedes' treatment of tangent lines is far removed from the ideas of calculus. His perspective was fixed and did not include any notion of rates of change, which is not surprising given that the concepts of tangents and slopes as rates of change rely on the notions of functions and limits. There is no recognition in Greek mathematics of the need to use limits, whether to determine areas or tangents. Although he presented these ideas vaguely to address the topic of tangents specifically, we cannot attribute the discovery of calculus to Archimedes. He did not observe the relationship between integration and differentiation, which is fundamental to calculus as a complementary branch of mathematics.

Apollonius (262-190 BC) described several methods for determining the tangent to conic sections (parabolas, ellipses, and hyperbolas), all of which were somewhat different. However, these were merely geometric problems studied for their very limited interests, and the ancient Greeks did not recognize any common factor or further value in these theories.

The Role of Muslim Arabs in Building the Concept of Derivative

Muslim mathematicians made significant contributions to mathematics during the Islamic conquests, which expanded across Persia, the Middle East, Central Asia, North Africa, Libya, and parts of post-eighth-century India. They built upon the mathematical knowledge inherited from Greece and India, refining and expanding it in various fields.

For instance, one of the most notable contributions came from the Indian mathematician Brahmagupta in the seventh century, who presented the concept of derivatives in a simple and primitive form in his book *Brahmasphutasiddhanta*. Similarly, Aryabhata, around the same period, applied the idea of integration to calculate the area of a circle and other simple figures in his work *Aryabhatiya* [35].

These contributions played a key role in preserving, transmitting, and developing mathematical knowledge, laying the foundation for later advances in algebra, geometry, and calculus across the Islamic world and Europe.

It is likely that early Arabic algebra was influenced by Indian mathematics to a much greater extent than is acknowledged by most contemporary historians, many of whom assume a Babylonian influence transmitted in unknown ways. Additionally, Greek influence was prevalent in Arabic mathematics, indicating that it served as a living continuation of Greek mathematics [36].

The Quran encouraged the accumulation of knowledge, and there was rapid expansion of Islam from the time of Muhammad's return to Mecca in 630 CE from his exile in Medina, to the Muslim conquests of lands stretching from Spain to the borders of China by 715 CE. The Golden Age of Islamic sciences and mathematics flourished throughout the medieval period, from the ninth to the fifteenth centuries. Efforts were immediately undertaken to translate key Greek and Indian mathematical and astronomical works into Arabic [27].

As a result, new fields in mathematics were developed, such as algebra and trigonometry, and a great deal of creative work was accomplished. For instance, analytic geometry and calculus emerged, with many Muslim scholars playing a significant role in the discovery of calculus concerning derivatives, areas, volumes, and partial differential equations. They tackled numerous problems, including the issue of infinity, which posed a significant challenge to mathematics that extended into the 19th century. Many works were produced that included algorithms for

extracting square roots, cube roots, and approximation processes. Notable examples include Al-Hassar's book *Kitab al-Bayan wa al-Tadhkarah*, Ibn al-Yasamin's *Talqih al-Afkar fi al-A'mal bi Rusum al-Ghubar*, Ibn al-Mun'im's *Fiqh al-Hisab*, and two works by Ibn al-Banna titled *Takhlees A'mal al-Hisab* and *Raf al-Hijab* [19].

Formulas were developed to determine precise or approximate values for numbers based on the nature of the integers A and B. Mathematicians in the Islamic world extensively discussed these methods, including figures such as Thabit ibn Qurra, Ibn al-Banna, Al-Tusi, Ibn Haydor, Ibrahim ibn Sinan, Al-Quhi, Banu Musa, and Ibn al-Haytham [32].

Thabit ibn Qurra (826–901 CE) was a pioneer in the development of early integral calculus concepts. He laid the groundwork for this field by determining the volume of a paraboloid around its axis in 870 CE. Through his methods, scholars were able to solve some complex problems and perform intricate calculations [1].

Thabit ibn Qurra's approach was closer to the modern concept of definite integration, as he divided the axis of integration into unequal sections corresponding to successive odd numbers, unlike Archimedes. The same principle applied to his method for finding the volume of the solid generated by rotating a parabola around its axis, where he demonstrated that this volume equals half the volume of the cylinder circumscribing the solid. In his first work, he employed the method of summing integrals, a technique previously used by Greek scholars for other geometric shapes besides the parabola [27].

Thabit begins by dividing the segment around the axis, let's say the X-axis, using points

whose coordinates correspond to successive squares of the integers. $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

The corresponding y-coordinates in the figure would be $X\ n$:

$$\begin{aligned} x_n &= n^2, \\ y_n &= \sqrt[n]{P}, \\ (x_1, y_1) &= (1^2, \sqrt[1]{P}) = (1, P), \\ (x_2, y_2) &= (2^2, \sqrt[2]{P}) = (4, \sqrt{P}), \\ (x_3, y_3) &= (3^2, \sqrt[3]{P}), \\ &\vdots \\ (x_n, y_n) &= (n^2, \sqrt[n]{P}) \end{aligned}$$

Ibn al-Qurra bases his calculations on fifteen preliminary arithmetic problems, including the computation of the sum of the first n odd numbers and the sum of the corresponding line segments. Moreover, he determines the area of the parabola through a method known as *squaring the parabola*. Let OAB denote the parabolic segment whose axis is OM (see Figure 5). Then we divide the line segment OM into n proportional parts corresponding to the odd numbers

$$1, 3, 5, \ldots, (2n-1).$$

The length of the chord (i.e., twice the y-coordinate) passing through the j-th division point is $2j-1$, where $j = 1, 2, \ldots, n$. This is proportional to the even number $2j$, and hence

$$LK = 2(2j-1)\sqrt{P}.$$

The equation of the parabola is

$$Y^2 = Px.$$

At this stage, we connect the vertex of the parabola to the endpoints of the successive chords, as illustrated in Figure 5..

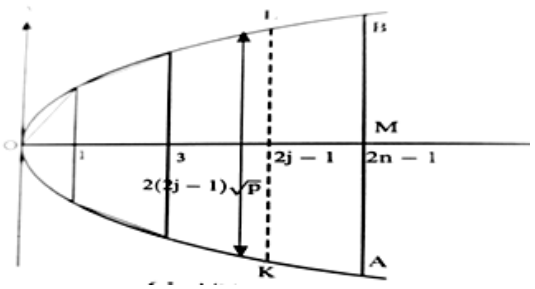


Figure 5. The Conic Section

Then this Arab scholar calculated the squaring of the parabola through long and detailed computations, leading to integral expressions of the form

$$\int_a^b x^n\, dx$$

where n is a fractional value. He used this approach to divide the interval of integration into unequal parts, thereby advancing beyond the work of the mathematician Fermat in the mid-seventeenth century [34].

If Ibn al-Qurra pioneered early ideas related to calculus by determining the volume of the solid generated by the rotation of the parabola around its axis, then al-Kuhi (940–1000 AD) further developed this field by explaining how to construct a spherical segment equivalent to another known spherical segment, such that the lateral surface area of the former equals that of a given fixed spherical segment [21].

Then came Abu Ali al-Hasan Ibn al-Haytham (965–1040 AD), one of the greatest Arab mathematicians. He was born in Basra (northern Iraq) and later moved to Egypt. He wrote more than ninety books and made significant contributions to astronomy, optics, algebra, geometry, and number theory. He was among the scholars who investigated cubic equations using conic sections, and he derived the volume of the solid generated by the rotation of a parabola about its axis (i.e., the x- or y-axis) [37].

Building on the works of Ibn al-Qurra and al-Kuhi, Ibn al-Haytham made substantial contributions to the early development of infinitesimal ideas. In his well-known treatise *The Area of the Paraboloid*, he presented a method for calculating the volume generated by the rotation of a parabola about a given axis, providing a solution that had not been achieved by his predecessors. He also established four different formulas giving the sums of the natural numbers for the first four powers, namely,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Ibn al-Haytham devised a general method for calculating sums of the form

$$\sum_{k=1}^{n} k^m,$$

for a natural number m, provided that the sums

$$\sum_{k=1}^{n} k^x, \qquad x = 1, 2, 3, \ldots, m-1,$$

are already known. This idea corresponds to what is now referred to as *mathematical induction*.

It is worth noting that Ghiyath al-Din al-Kashi discussed related results in his book *The Key to Arithmetic* (1400 AD), where he established inequalities of the form

$$\frac{8}{15}(n-1)n^4 < \sum_{k=1}^{n}(n^2-k^2)^2 < \frac{8}{15}\, n \cdot n^4,$$

$$\sum_{k=0}^{n-1}\left(n^2-k^2\right)^2 > \frac{8}{15}\, n \cdot n^4$$

This allowed for the derivation of the following results:

(i) The volume of the solid of revolution generated by rotating triangle $\triangle OAB$ around OA is equal to half the volume of the cylinder whose height is OA and whose base radius is OB,as shown in Figure 6.

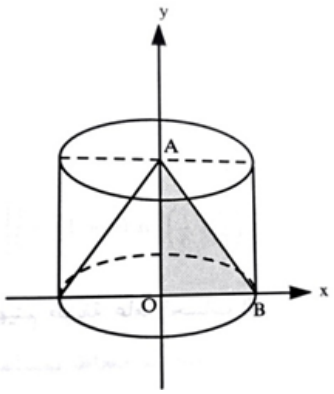


Figure 6. The Solid Generated by the Rotation of OAB about OA

(ii) The volume of the solid of revolution formed by rotating the parabola OAB around the axis OA is equal to $\frac{8}{15}$ of the volume of the cylinder whose height is OA and whose base radius is OB.

Let us assume the X-axis along OB and the Y-axis along OA. Suppose that $OA = h$, representing the height of the segment, and $OB = r$, representing the radius of the cylinder. Then, the equation of the parabola shown in Figure 7 is given by::

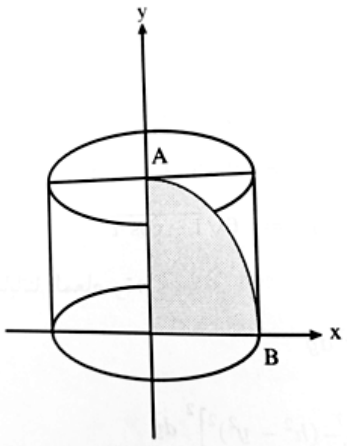


Figure 7. The Solid of Revolution Generated by Rotating OAB about the OA Axis

The equation of the parabola is given by:

$$y^2 = \frac{-h^2}{r}(x - r)$$

Note that in Figure (7), $h = 6$ and $r = 3$ were used. The intermediate equations were also employed to plot the ellipses and the parabola as follows:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$\frac{x^2}{9} - \frac{(y-6)^2}{4} = 1$$

$$y^2 = -12(x - 3)$$

The intermediate equations are:

$$x = f(t) = 3\cos t$$

$$y_1 = F(t) = 2\sin t$$

$$y_2 = G(t) = 2\sin(t+6)$$

$$y_3 = H(t) = 6\sqrt{1-\cos t}$$

We then have, based on our contemporary calculations:

$$\begin{aligned} V &= \int_0^h \pi x^2\, dy \\ &= \int_0^h \pi \left[\frac{r}{h^2}\left(h^2 - y^2\right)\right]^2 dy \\ &= \frac{\pi r^2}{h^4}\int_0^h \left(h^2 - y^2\right)^2 dy \\ &= \frac{\pi r^2}{h^4}\int_0^h \left(h^4 - 2h^2y^2 + y^4\right) dy \\ &= \frac{\pi r^2}{h^4}\left[h^4 y - \frac{2}{3}h^2y^3 + \frac{y^5}{5}\right]_0^h \\ &= \frac{\pi r^2}{h^4}\cdot h^5\left(1 - \frac{2}{3} + \frac{1}{5}\right) \\ &= \frac{8}{15}\pi r^2 h \end{aligned}$$

This result reached by Ibn al-Haytham was not known to the Greek scholars. Archimedes conducted calculations equivalent to integral calculations of the type:

$$\int_0^a x\, dx, \quad \int_0^a x^2\, dx$$

While Ibn al-Haytham conducted calculations equivalent to integrals, as we have seen earlier, of the type:

$$\int_0^a x\, dx, \quad \int_0^a x^2\, dx, \quad \int_0^a x^4\, dx$$

It is noteworthy that some of these results introduced by Ibn al-Haytham were not discovered in Europe until the first half of the seventeenth century, when general rules for the integrals of exponential functions were established [33].

In the eleventh century, the mathematician and Persian poet Omar Khayyam (1048–1131 AD) made significant developments in the theory of cubic equations, proving the possibility of having more than one solution for a cubic equation through what is known as geometric resolution (Khayyam's resolution) [5]. To solve the equation:

$$x^3 + m^2x = n, \quad n > 0$$

Khayyam constructed a parabola corresponding to his equation:

$$y = \frac{x^2}{m}.$$

He created a circle with a diameter defined by the point $\left(0, \frac{n^2}{m}\right)$ drawn on the x-axis. Then he drew the vertical line passing through the intersection of the circle and the parabola. The solution to the equation is the distance from the origin to the point where this vertical line intersects the x-axis, as shown in the accompanying Figure 8:

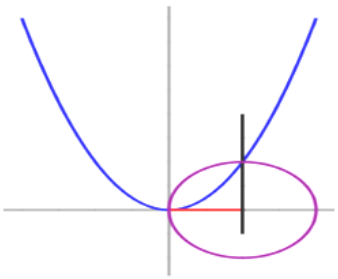


Figure 8. Geometric Resolution of the Cubic Equation by Omar Khayyam

With the development of algebra and methods for solving cubic equations pioneered by Omar Khayyam and Ibn al-Qurra, along with contributions from many other scholars, Sharaf al-Din al-Tusi later continued this work by analyzing cubic equations and arriving at a formula closer to the derivative.

Sharaf al-Din al-Tusi, or Sharaf al-Din al-Muẓaffar ibn Muhammad al-Tusi (1213–1135 AD), was a mathematician, astronomer, and mechanical engineer from Tus.

He studied various mathematical topics, including number theory, astronomical tables, and astrology, in Aleppo and Mosul. Al-Tusi gained significant fame in teaching mathematics, mentoring scholars such as Kamal al-Din ibn Yunus and Muhammad ibn Abd al-Karim al-Harithy. Students of knowledge came to him from all directions [30].

Al-Tusi played an important and fundamental role in the field of algebra. The algebraic term (derivative) was used in his discussions of various equations in a systematic way. He conceptualized functions without using them in the conventional sense. His method for solving numerical equations is considered closest to Newton's and Raphson's methods, and even more aligned with the Ruffini–Horner method [29]. By observing the numerical problems that al-Tusi solved, it is evident that he consistently used expressions that are algebraically identical to the first-degree derivative. In this context, despite the lack of a clear reference to functions—especially when identifying the correct positive root of a numerical equation using the method of successive approximations—al-Tusi's method also allows for obtaining the negative root of an equation, such as:

$$x^3 + a_1 x^2 + a_2 x = N$$

It is sufficient to replace the variable x with the variable $-x$. The algebraic function (derivative) was utilized in discussing the existence of roots of algebraic equations. The equations addressed by al-Tusi were, for him, examples of algebraic equations for which roots had previously been proven to exist. His aim was to formulate the theory of equations of degree less than or equal to three in a verbal manner [31].

The main steps relied upon by al-Tusi:

- Classifying equations into two categories: the first category includes equations that have roots, while the second includes equations that cannot be solved unless certain conditions are met. Then, he proceeds with the analysis.
- By a method called affine transformation, which is $x \to a - x$ or $x \to x - a$, he transforms the equations that need to be solved into other equations whose solutions are known.

- He does not study the maximum value, nor area, nor volume, but he studies limits.
- If one does not find a root of the cubic equation, he studies a quadratic equation, which is equivalent to the result of dividing the cubic equation by $(x - r)$, where r is the root that was found.
- When he encounters a previously unknown quadratic equation, such as

$$x^2 - bx = c,$$

he uses affine transformation to convert it into a known type of equation.

- In other words, al-Tusi was able to use the concept of the derivative to calculate the maximum value by his method of handling equations that have no positive real solution, which are the following equations:

$$x^3 + ax^2 + c = bx \quad (1)$$
$$x^3 + c = bx \quad (2)$$
$$x^3 + c = ax^2 \quad (3)$$
$$x^3 + c = ax^2 + bx \quad (4)$$
$$x^3 + bx + c = ax^2 \quad (5)$$

Each of these equations has two positive solutions, or one double root and a negative root, which al-Tusi does not acknowledge and considers the equation impossible to solve. He then solves it using the following method:

Al-Tusi writes each type of these equations in the following form:

$$f(x) = c = E.$$

Since $f(0) = 0$ in equation (1), it is written in the following form:

$$f(x) = bx - ax^2 - x^3 = c.$$

To determine the domain of the function $f(x)$, he gives conditions on the variable x so that $f(x) > 0$. By taking the derivative, he denotes x_0 for a quadratic equation equivalent to $f'(x) = 0$. Then, $f(x)$ reaches its maximum value $f(x_0)$ when $x = x_0$, where $f'(x)$ denotes the derivative of the function $f(x)$. It can then be shown that $f(x_0)$ is indeed the maximum of $f(x)$ by observing:

$$\begin{cases} x < x_0 \Rightarrow f(x) < f(x_0) \\ x > x_0 \Rightarrow f(x) < f(x_0) \end{cases}$$

From this proof, we conclude:

- If $f(x) < c$, the equation has no solution.
- If $f(x) = c$, the equation has exactly one solution (a positive real, double root).
- If $f(x) > c$, the equation has two solutions x_1, x_2 satisfying $0 < x_1 < x_0 < x_2$.

Therefore, assuming $f(x_0) > c$, the following is done:

- The larger root x_2 is determined by an affine change of variable $x = x_0 + X$, which transforms the equation into another equation in X:

$$X^3 + aX^2 = c.$$

This is an equation that has already been solved, and any equation of this type has exactly one positive root.

- The smaller root x_1 is determined, except in equation (3), by an affine change of variable $x = x_0 - X$. Then, equation (E) is transformed into an equation in X of type (3):

$$aX^2 - X^3 = c.$$

Here, if X_1 is the smallest root of this equation, then $x_1 = x_0 - X_1$.

However, applying such an affine change of variable to equation (3) transforms it into an equation of the same type (3). This is what made al-Tusi solve equation (3) differently, by reducing its solution to solving an equation of the type:

$$x^2 + ax = c,$$

which is another type that had already been solved.

In practice, x_2 is calculated using a numerical algorithm for an equation that has already been solved, while x_1 is calculated using a special algorithm for equation (3). These two steps effectively form a single stage, and the determination of the smaller root actually precedes the determination of the larger root in equation (3).

To clarify what al-Tusi did, he announces the value of x_0 and establishes that $f(x_0)$ is the maximum value, without explicitly stating the methods he used to determine x_0. This is particularly significant because the use of a function's derivative, setting it to zero to find its maximum, and even the concept of a function of a variable, are ideas associated with 18th-century mathematics.

It is therefore remarkable and thought-provoking that a 12th-century mathematician could apply such reasoning systematically. Indeed, al-Tusi directly provided the value of x_0 for equations (2) and (3), which are:

$$x_0 = \frac{2a}{3}, \quad x_0 = \sqrt{\frac{b}{3}}.$$

However, when dealing with equations (1), (4), and (5), he presented x_0 as a root of an equation equivalent to $f'(x) = 0$. Therefore, the analytical aspect of al-Tusi's reasoning in this context represents a form of genuine mathematical analysis.

In brief, we can summarize his method as follows:

To solve an equation such as

$$x^3 + a = bx,$$

al-Tusi first determined the maximum of the curve

$$y = bx - x^3.$$

He used the derivative of the function to find that the maximum occurs at

$$x = \sqrt{\frac{b}{3}}.$$

He then computed the maximum value of y, which is

$$y_{\max} = 2\left(\frac{b}{3}\right)^{3/2},$$

by substituting $x = \sqrt{b/3}$ back into $y = bx - x^3$. From this, al-Tusi derived the equation:

$$bx - x^3 = a.$$

which has a solution if

$$2\left(\frac{b}{3}\right)^{3/2} \geq a.$$

Al-Tusi also concluded that the equation has a positive root if

$$D = \frac{b^3}{27} - \frac{a^2}{4} \geq 0,$$

where D is the discriminant of the equation.

There have been attempts to explain the appearance of the derivative expression in this section of al-Tusi's treatise. After two studies of this issue, it can be concluded that, in solving each of the equations mentioned above, the derivative expression appeared in al-Tusi's work during his algebraic and analytical investigation of a real number x_0 that satisfies the system of relations:

$$\begin{cases} x < x_0 \Rightarrow f(x) < f(x_0), \\ x > x_0 \Rightarrow f(x) < f(x_0). \end{cases}$$

This means that x_0 is the value of x at which $f(x)$ reaches its maximum. It indicates that al-Tusi calculated x_0 based on the determination (1) that he provided at the beginning of his study of equations (1)–(5).

Rashid noted that the presence of certain concepts and tools from the field of mathematical analysis in al-Tusi's mathematics is neither accidental nor incidental, but intentional.

A specialized study by Nicolas Faris confirmed that the two sections of al-Tusi's treatise are components of a unified work created for a single algebraic purpose: solving cubic equations. The second study emphasized that al-Tusi used algebraic calculations alongside numerical solutions and approximate computations in both sections of the treatise, and that his use of geometry (conic sections, geometric representations) in the first section was aimed at the same algebraic objectives [14]. It should be noted that al-Tusi was far from the modern concept of a function of a variable $f(x)$, and from the notion of the derivative of that function, which we denote by $f'(x)$. The era in which al-Tusi lived was very distant from the time when the concept of a function was formally developed.

Nevertheless, this did not prevent him from setting the derivative expressions of five algebraic forms to zero in the same chapter, based on the determination of their maximum values (relation 1), and using algebraic calculations, analytical concepts, and reasoning [14].

The Role of the West in the Development of the Concept of the Derivative

Descartes and Fermat introduced analytic geometry around 1630 AD. This approach showed that curves could be represented by equations and, conversely, that equations could describe curves. While Muslims and Greeks had studied curves, their work was largely limited to circles, conic sections, and certain problems related to tangents and areas. However, by the 17th century, students encountered a vast number of curves that needed to be analyzed, as any equation could now represent a new curve. Consequently, the ancient Greek methods and the contributions of Muslim mathematicians were no longer sufficient. Although they could compute tangents to circles, conic sections, and some more complex curves using classical geometry, they did not possess a general method to solve all these problems. In the sixteenth century, in his book *A History of Calculus*, Boyer (1949) noted that the development of the algorithms of calculus was fundamentally aided by the systematic introduction of symbols representing quantities in algebraic relations.

The introduction of symbols into algebra, largely due to the French mathematician François Viète, enabled rapid progress in analytic geometry and allowed the concepts of average rate of change and functions to enter algebraic reasoning through continuous and refined observations.

In the century preceding the work of Newton and Leibniz, new methods were developed, which were essentially modifications of the classical geometric procedures of the time (such as the method of exhaustion extensively used by Euclid) but easier to apply [31]. By applying the known algebraic procedures, such as those discussed by Boyer, Cavalieri (1598–1647) nearly arrived at a preliminary form of one of the important integral theorems. However, the sums envisioned by Cavalieri were not "classical" sums of finite terms; rather, they represented a collection of infinitely many lines. Cavalieri's proposal, which appeared in 1639, was independently formulated by several other mathematicians between 1635 and 1655, including Torricelli, Roberval, Pascal, Fermat, and Wallis. This theory played an important role in the development of infinitesimal methods. His idea began from the observation that the pages of a book are infinitely thin, meaning that:

(i) The pages can be made so thin that, no matter how many pages you stack (in a finite number), they would never reach the thickness of a real book. This implies that you would need an infinite number of such infinitesimally thin pages.

(ii) The pages can be made so thin that it is impossible to cut them into thinner slices, no matter how you try; they are indivisible. They are parts of the book, but if their number is infinite, whether they make up the entire book becomes a matter of question.

There is an important distinction between these concepts: indivisible elements usually have fewer dimensions than the original object. For example, in the book illustration, the pages are considered two-dimensional, even though the book itself is three-dimensional. In contrast, infinitesimals do not have to be indivisible; one can imagine three-dimensional pages that are already infinitely thin, yet can still be cut in half to become even thinner. In this case, these elements remain homogeneous, retaining the same dimensions as the page, but with a smaller size than ever [22]. To understand how this approach could be used to solve "pre-calculus" problems, see Figure 9. Using this type of reasoning, one can relate the area of an ellipse to the area of a circle by observing that each indivisible "slice" of the ellipse is half the height of the circle. Consequently, the total area of the ellipse equals half the area of the circle. Cavalieri himself made use of indivisible quantities, which he described in his book *Geometria indivisibilibus continuorum*. Out of necessity, he was extremely meticulous, making his work difficult to read.

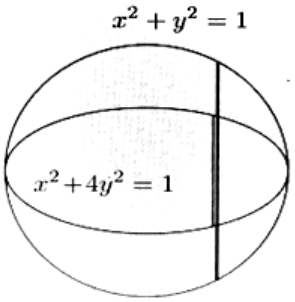


Figure 9. The area of the ellipse using Cavalieri's principle

Boyer wrote: "Perhaps the first theory of infinitesimals arose from the suggestion that there might exist a more general algebraic procedural rule, such as those formulated by the generation that came after Newton and Leibniz and which became fundamental to calculus" [27].

Boyer wrote: "Perhaps the first theory of infinitesimals arose from the suggestion that there might exist a more general algebraic procedural rule, such as those formulated by the generation that came after Newton and Leibniz and which became fundamental to calculus" [27]. Nevertheless, all expectations regarding the methods of calculus developed during those years remained closely tied to geometry, and the role of algebra was still weak. Moreover, infinitesimal lines and planar surfaces appeared, but not infinitesimal numbers [2]. Algebra and analytic geometry in the sixteenth and seventeenth centuries constituted an essential step in this direction, namely, uprooting the Aristotelian idea that in arithmetic no number can be smaller than one. Boyer noted that there was nothing contradictory in the idea of infinitesimal constants or numbers, since they were compatible with the geometric indivisibles that had been successfully used [27].

In the seventeenth century, European mathematicians such as Isaac Barrow, Pierre de Fermat, Blaise Pascal, John Wallis, and others discussed the idea of the derivative, particularly in relation to determining maxima and minima and tangent points of various curves. Fermat developed a method for finding maxima, minima, and tangents to different curves, which was equivalent to what is now called differentiation. Subsequently, Isaac Newton later formulated his early ideas on calculus, which arose directly from Fermat's method of drawing tangents [28].

For example, Fermat (1601–1665) followed a method that led him to determine maximum and minimum values, which first appeared in 1637. The problem concerned dividing a line segment of length b into two parts, a and $(b - a)$ in such a way that their product, $a(b - a)$, is maximized. The steps of his argument were as follows:

Assume that the first segment becomes $a + E$; consequently, the second segment becomes $b - a - E$. The product of the new segments is then:

$$(a + E)(b - a - E) = ba - a^2 + bE - 2aE - E^2$$

Fermat's computational and algebraic method is based on an observation made by Pappus, which corresponds to the assumption that a function remains constant in the neighborhood of a value at which it attains a maximum or a minimum. Fermat then compared the two expressions obtained and referred to their equality as a false equality (*adaequalitatem*). For simplicity, the symbol $\sim$ will be used, corresponding to Fermat's verbal expression "adaeguabitur" [26].

$$ba - a^2 + bE - 2aE - E^2 \sim ab - a^2$$

Which gives:

$$bE \sim 2aE + E^2$$

This means that the smaller the interval, the closer the false equality is to a true equation. Fermat therefore divided the equation by E and then set $E = 0$, obtaining the result $a = b/2$. According to his assumptions, his procedure can be summarized as follows:

$$\frac{a(b-a) - (a+E)(b-a-E)}{E} \to 0 \quad \text{as } E \to 0$$

Today this is written as $E \to 0$, but it appears that Fermat interpreted the disappearance of the term E as meaning that E actually equals zero. Fermat also used his method to determine tangents to curves. The following are the steps he followed to find the tangent to a parabola:

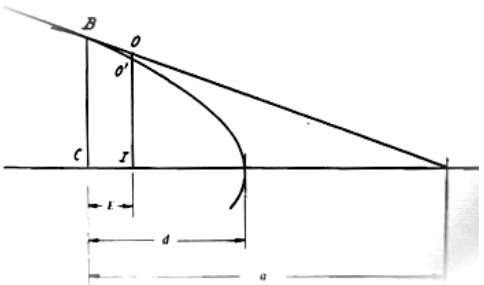


Figure 10. Fermat's method to determine the tangent to a parabola

By examining the figure drawn by Fermat and according to what he called the "defining property" of the curve—what we today refer to as its equation—and using the similarity property of the two triangles in Figure 10, we obtain the following:

$$\frac{d}{d-E} = \frac{(CB)^2}{(IO')^2} > \frac{(CB)^2}{IO^2} = \frac{a^2}{(a-E)^2}$$

$$\frac{d}{d-E} > \frac{a^2}{(a-E)^2}$$

$$dE^2 - 2daE > -a^2E$$

As E is taken to be smaller and smaller, the point O can be regarded as lying on the curve; consequently, the inequality becomes a false equality, that is:

$$dE^2 - 2daE \sim -a^2E$$

By dividing the resulting expression by E and then assuming that $E = 0$, we obtain:

$$dE + a^2 \sim 2ad$$

$$a^2 = 2ad$$

$$a = 2d$$

Fermat considered his method in terms of equations and infinitesimals, treating algebra as the fundamental element. However, based on subsequent criticisms, concerns began to arise regarding the apparent futility of applying algebraic rules to quantities such as E. The most significant criticism of this procedure was that all the algebraic operations performed with the quantity E included division by E as if it were a regular positive quantity, which becomes problematic when E is ultimately set equal to zero [39].

In England, the mathematician John Wallis (1616–1703) conducted similar studies using the concepts of infinity and infinitesimals, although he did not provide a formal definition for them. He was largely inspired by reading Cavalieri's works *On the Indivisibles*. While Cavalieri's approach was primarily geometric, Wallis adopted an essentially algebraic method. He used algebraic symbols in his representations to achieve brevity and clarity.

Following Cavalieri's approach, Wallis considered a plane figure to consist of an infinite number of parallel lines. However, he preferred to describe them as parallelograms with a height of $1/\infty$ or an extremely small fraction of the figure's height. Here, the symbol ∞ for infinity appears for the first time. Wallis stated that $1/\infty$ represents an infinitesimal quantity.

Using this approach, he demonstrated that the area of a triangle is equal to the product of its base and half its height. He assumed that the triangle was divided into an infinite number of parallelograms of height $1/\infty$, considered as lines parallel to the base. The areas of these parallelograms, starting from the vertex to the base, form an arithmetic sequence beginning at zero. Wallis then applied a known algebraic rule, according to which the sum of an arithmetic sequence is equal to the product of the last term and half the number of terms. In applying this formula, Wallis did not distinguish between finite and infinite numbers. He designated the triangle's base as A and its height as B, so that the last term of the sequence is $AB \cdot 1/\infty$. The total area is then given by the formula: The total area of the triangle is given by

$$\frac{1}{\infty} \cdot AB \cdot \frac{\infty}{2} = \frac{1}{2}AB$$

Wallis's work illustrates the independence of arithmetic from geometry. However, his use of infinitesimals was no clearer than in Fermat's work. Indeed, Fermat made E disappear by setting it equal to zero, and similarly, Wallis expressed the infinitesimal height as $1/\infty = 0$. Nevertheless, thirty years later, Wallis made an interesting remark about the infinitesimal nature of $1/\infty$, stating:

> "One may observe a great difference between the ratio of the infinite to the finite, and that of the finite to nothing. For $1/\infty$ is merely an infinitesimal part which, when multiplied by infinity, may equal the whole; but $0/1$, which equals nothing, can in no way, by multiplication, become equal to something" [7].

Isaac Newton and **Gottfried Leibniz** are considered the independent founders of calculus in the late seventeenth century. Although they developed almost the same ideas, they did so in different ways and using different terminology. Newton developed calculus in the context of his studies on motion and gravity. He was concerned with continuous change, calculating areas under curves, and the velocities of moving bodies. He used the concept of fluxions to describe derivatives and fluents to describe integrals, in addition to applying these concepts practically in physics. Notably, he did so in his work *Philosophiæ Naturalis Principia Mathematica* (1687), where he pioneered the use of calculus to understand the laws of motion and gravity [17].

Leibniz, on the other hand, contributed to the development of calculus independently of Newton. He introduced a symbolic system to simplify mathematical operations, such as dx, dy for derivatives and $\int$ for integrals. He was also interested in the philosophical and algebraic ideas underlying calculus, providing much of the theoretical foundation that became central in mathematics [3].

Despite their independent work, a long-standing dispute arose between the followers of Newton and those of Leibniz over the priority of discovery, with each side accusing the other of scientific plagiarism. Today, however, both scientists are recognized as pioneers of calculus, and each made invaluable contributions to the development of this branch of mathematics.

Newton (1642–1727) worked on expanding power series. One of his most well-known results is what is called Newton's Binomial Theorem, but it can be said that Newton went further in

expanding the formula:

$$(a+b)^n = \sum_{h=0}^{n} \binom{n}{h} a^{n-h} b^h$$

For any integer n, he extended it to a more general expanded form:

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$$

For any integer or rational number r, Newton arrived at results like these through several methods, such as division, extracting roots, and so on. For example, by expanding a power series, the value

$$\frac{a^2}{b+x}$$

was obtained by dividing a^2 by $b+x$.

In his book *De analysi per aequationes numero terminorum infinitas*, written in 1669 but published later in 1711, we can appreciate Newton's method of integration, which was designed to avoid explicit differentiation. This method can be illustrated as follows:

Suppose z represents the area ABD, where $AB = x$, $BD = y$, and AD is part of a curve. By giving x a small positive and finite increment (analogous to Fermat's E), denoted as $B\beta$ in Figure 11, the area z increases by the region $B\beta\delta D$. Considering the corresponding rectangle $B\beta HK$ with O as the base and V as the height, the area z increases by OV.

Thus, we observe that Newton introduced two different increments: one for the independent variable and another for the dependent variable. This approach allowed him to avoid invoking a false equality.

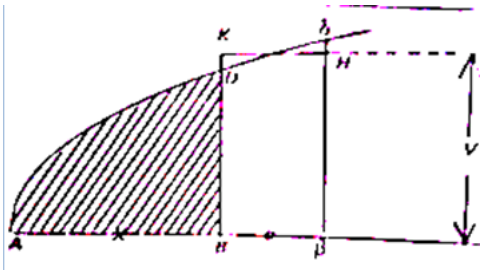


Figure 11. : Diagram supporting Newton's argument

And Newton illustrates his method with a specific example:

$$z = \frac{2}{3}x^{3/2}$$

which he rewrites in the following form:

$$z^2 = \frac{4}{9}x^3$$

The increment O in the variable x corresponds to an increase in the area z, as follows:

$$(z+Ov)^2 = \frac{4}{9}(x+O)^3$$

By expanding the powers and noting that $z^2 = \frac{4}{9}x^3$, we find that:

$$z^2 + 2Ovz + O^2v^2 = \frac{4}{9}x^3 + \frac{4}{3}x^2O + \frac{4}{3}xO^2 + \frac{4}{9}O^3$$

Simplifying, we have:

$$2Ovz + O^2v^2 = \frac{4}{3}x^2O + \frac{4}{3}xO^2 + \frac{4}{9}O^3$$

And by dividing by O:

$$2vz + Ov^2 = \frac{4}{3}x^2 + \frac{4}{3}xO + \frac{4}{9}O^2$$

Referring back to Figure (11), he writes: "If we assume that $B\beta$ vanishes to infinity, or that O becomes zero, then v and y in this case will be equal, and the parts multiplied by O will disappear." Here, Newton lets O approach zero, and by observing that $O = 0$, this equality leads us to

$$2yz = \frac{4}{3}x^2,$$

which means

$$2y \cdot \frac{2}{3}x^{3/2} = \frac{4}{3}x^2,$$

$$y = x^{1/2}.$$

At this point, Newton mistakenly believed the converse to be true as well; that is, if $y = x^{1/2}$, then the area corresponds to

$$z = \frac{2}{3}x^{3/2}.$$

Consequently, we know that it lacks the constant of integration, and Newton had indeed already found a specific antiderivative.

In his book *Methods of Fluxions*, written in 1667 but published posthumously, Newton introduced fluxions for the values x and y that depend on time (a fluxion being the instantaneous rate of change or the slope at a given point). He then denoted the fluxions as $\dot{x}$ and $\dot{y}$. Considering a small increment of time o, the flux of x and y becomes, respectively, $x + \dot{x}o$ and $y + \dot{y}o$.

From here, the fundamental problem of calculus arose: giving the relationship between the fluxions and, conversely, giving the relationship between their rates of change. Newton's method relied on replacing the values x, y, z with the incremented quantities, $x + \dot{x}o$, $y + \dot{y}o$, $z + \dot{z}o$, and so on.

He then extended his calculations by discarding the terms that did not contain o, which satisfied the equality given in the hypothesis, and then divided by zero. Furthermore, since o was assumed to be infinitesimal, the terms containing it could be considered zero relative to the parts that did not depend on it and should be neglected. Using this method, Newton found the inversion theorem (the fundamental theorem of calculus); that is, if z is the area generated by y, then it can be written as: $z^{\cdot}(x)=y$ or equivalently $(z^{\cdot})/(x^{\cdot})=y$

It is worth noting that the fluxions method is one of the two approaches Newton used in his study of infinitesimals. The other method is the method of ratios and ultimate and evanescent quantities, presented in his book *Principia* (published in 1687). In this work, Newton introduced the concept of the "*ultimate ratio in which quantities vanish*" as "*limits in which the ratios of these quantities decrease indefinitely and approach a value which, although it can come arbitrarily close to a certain difference, cannot be exceeded or reached until the quantities are diminished indefinitely*." In this way, Newton overcame the concept of indivisibles and replaced it with an intuitive notion of limits [3].

Around the same period, specifically in Germany, Leibniz (1646–1716) developed his method, which resembles Fermat's approach, and introduced the concept of differentiation. Instead of using E, the increment in the variable x in Leibniz's method is denoted dx, considered as a small arbitrary segment. Given a specific curve, Leibniz considered the corresponding function $v(x)$ and introduced the derivative of v as dv, which represents the segment corresponding to dx as the slope of v relative to the tangent, as shown in Figure 12.

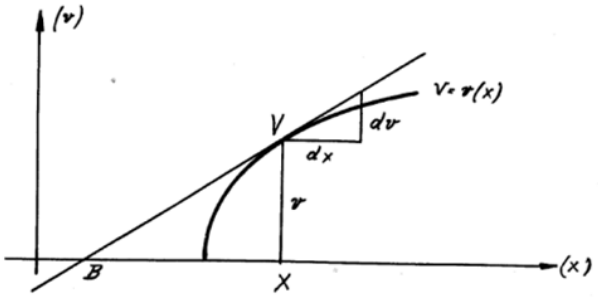


Figure 12. Diagram supporting Leibniz's argument

He also clarified that dv is the amount of change $v(x + dx) - v(x)$. However, we know that this equality does not hold for any finite increment; the two definitions coincide only when the increment is infinitesimal. As noted by Dupont in his 1975 book *Storia del concetto di derivata*, with the introduction of algebraic operations and their application, algebra was no longer considered merely a calculus of finite quantities. Rather, it came to include infinitesimals and relationships involving infinitely small differences, which had to be expressed using new appropriate symbols, capable of conveying the underlying idea [11].
On these principles, Leibniz built his calculus of infinitesimals:

Suppose the derivatives of the functions v, w, y, z are dv, dw, dy, dz, corresponding to $v(x)$, $w(x), y(x), z(x)$, respectively. Leibniz then stated that:

$$\begin{aligned}
&\text{If } a \text{ is a constant, then } da = 0 \text{ and } d(ax) = a\,dx,\\
&\text{If } y = v, \text{ then } dy = dv,\\
&d(z - y + w + x) = dz - dy + dw + dx,\\
&d(vx) = x\,dv + v\,dx,\\
&d\frac{v}{y} = \frac{\pm y\,dv \pm v\,dy}{y^2}.
\end{aligned}$$

These rules were presented without proof, and the derivative dx remained an arbitrary segment. There was no reference to an infinitesimal quantity. If we trace the history of the derivative dx in Leibniz's mathematical work, it must be connected to the differential operator d, which he used to calculate the finite difference between two consecutive terms of a series. In Leibniz's earlier manuscripts (around 1675), we can see that differential calculus emerged as an "*induction*" of calculus on numerical sequences, in the transition from discrete to continuous cases. In these manuscripts, there is evidence of the algebraic rules that were later formalized, and it is clear that Leibniz considered the differential, in essence, as an infinitesimal difference between two consecutive values of the variable. For example, in order to prove the formula

$$d(xy) = x\,dy + y\,dx,$$

Leibniz calculated it based on the difference resulting from substituting x with $x+dx$ and y with $y + dy$, which yields:

$$d(xy) = (x + dx)(y + dy) - xy = x\,dy + y\,dx + dx\,dy.$$

Afterward, Leibniz discarded the term $dx\,dy$. Here, we see that the differences he was considering were infinitesimal. In fact, the product of two infinitesimals, $dx\,dy$, is negligible compared to all the other terms in the sum. This was one of the most discussed points in Leibniz's calculus. It is important to note that algebra alone could not explain how

$$d(xy) = x\,dy + y\,dx + dx\,dy$$

could simultaneously be considered equal to:

$$d(xy) = x\,dy + y\,dx.$$

With regard to integration, Dupont emphasizes that Leibniz insisted on distinguishing his own method from the method of indivisibles. In doing so, Leibniz pointed out that, in his calculations, he obtained quadratures as sums of differentials of area—namely, areas of rectangles with dimensions y and dx—rather than as sums of lines, as in the indivisible approach.

As Dupont (1981) observes, this development marks the emergence of an advanced algebraic process within infinitesimal calculus, distinct from both finite Cartesian algebra and Cavalieri's purely geometric method. Leibniz achieved this by linking algebra to Cavalieri's geometry through the introduction of two-dimensional differentials such as $dx\,dy$, which allowed geometric elements to be expressed symbolically and manipulated through algebraic operations. In doing so, he established a formal calculus of infinitesimals in which infinitesimal quantities were no longer viewed merely as geometric indivisibles, but as entities governed by systematic symbolic rules [12].

$$y\,dx.$$

Leibniz was also the one who introduced the symbol $\int$ to denote summation. For example, he obtained the result:

$$\frac{1}{2}\int (dy)^2 = \int \left(\int dy\right) dy,$$

which can be written in modern notation, while still preserving the structure and meaning that Leibniz attributed to the individual elements of his calculus.

However, Leibniz's *Nova Methodus* proved difficult to read even for leading mathematicians of the time. As Dupont (1981) points out, this work contains ambiguities, puzzles, contradictions, and fundamental rules that were introduced without proof. This may have been due to Leibniz's deliberate attempt to avoid potential criticism associated with the use of infinitesimals [12].

Leibniz made significant contributions to the development of differentiation and integration techniques and played a fundamental role in establishing the theoretical foundations of calculus, including important ideas related to complex integrals [3].

In later periods, calculus developed substantially in Western mathematics through the work of several prominent scientists and mathematicians. Among the most influential figures was Leonhard Euler, who greatly expanded the scope of calculus applications across various fields, particularly physics and engineering. Euler also introduced essential theoretical contributions that remain central to calculus to this day [10].

One of Euler's major achievements was the unification and simplification of mathematical notation in calculus. He introduced several symbols that are still in use today, such as $f(x)$ for functions and the symbol e for the base of the natural logarithm. In addition, Euler investigated the relationship between exponential and logarithmic functions and contributed significantly to the development of the concept of complex functions, which had a profound impact on mathematical analysis [6].

Pierre-Simon Laplace was also among the central figures in the development of calculus. He made major contributions to analytical mathematics, particularly through his work on series theory and differential and integral calculus. In his well-known treatise *Théorie Analytique des Probabilités*, Laplace made extensive use of calculus to advance probability theory and to establish rigorous analytical foundations for it.

In addition, Laplace contributed significantly to the development of partial differential equations and applied them extensively in physics and astronomy. One of his most influential achievements was the development of the Laplace transform, a powerful analytical technique that is widely used to solve differential equations and that continues to play a fundamental role in mathematics, engineering, and applied sciences [16].

3 Conclusion

This study has led to an awareness of the significant milestones in the development of calculus. It has shown how algebra gradually assumed a central role in shaping calculus practices and how algebraic methods were often able to explain certain results more efficiently than purely geometric approaches. At various stages, mathematicians were required to exert considerable

effort to justify their theories and proofs to the mathematical community, and some deliberately avoided explicit references to ideas that might provoke criticism.

Nevertheless, these intellectual struggles played a crucial role in the development of the concept of the derivative, which today constitutes a fundamental pillar in the construction of modern mathematics curricula. In the context of didactic transposition (Yves Chevallard), from secondary education to higher education, the derivative has significantly influenced mathematical thinking among successive generations of learners. This historical and epistemological perspective offers researchers in mathematics and mathematics education valuable opportunities to approach the concept of the derivative from alternative viewpoints, moving beyond rigid, transmission-based teaching methods toward more meaningful and reflective learning practices.

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