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Herz Spaces and Multilinear Integral Operators on Morrey Spaces

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Abstract This paper establishes a unified approach for extending the mapping properties of multilinear integral operators from Lebesgue spaces to Morrey spaces. The main result shows that if a multilinear operator is bounded on products of Lebesgue spaces and its kernel satisfies a condition involving a certain Herz space, then this operator can be extended to be a bounded operator on the corresponding product of Morrey spaces. We also obtain a new operator norm estimate for the multilinear integral operator on Morrey spaces in terms of the operator norm on Lebesgue spaces and the Herz space norm of the kernel. The main result applies to several multilinear integral operators including multilinear singular integral operators, multilinear fractional integral operators and multilinear strongly singular integral operators.

1 Introduction

The study of function spaces and boundedness properties of operators acting on them represents one of the central themes in harmonic analysis. Morrey spaces, introduced by C. B. Morrey [16] in connection with regularity problems for partial differential equations, have proven to be indispensable in modern analysis. These spaces provide a refinement of the Lebesgue scale by measuring local integrability with a global control. Their importance stems from the fact that they naturally appear in the study of elliptic PDEs, potential theory, and fluid dynamics [3, 4, 5, 20].

Parallel to this development, the theory of multilinear operators has emerged as a natural generalization of linear operator theory. Initiated by Coifman and Meyer [2] and systematically developed by Grafakos and Torres [6], Kenig and Stein [10], multilinear harmonic analysis has revealed deep connections between different areas of mathematics. Multilinear Calderón-Zygmund operators, fractional integrals, and their rough variants have been extensively studied on classical Lebesgue spaces [13].

Herz spaces, introduced by C. S. Herz [7], provide another important generalization of Lebesgue spaces. These spaces, characterized by their dyadic block decomposition, have found significant applications in the theory of function spaces and partial differential equations [12, 14, 15, 18, 19]. The intersection of Morrey spaces and Herz spaces creates a rich framework for studying operators with rough kernels and singular integrals.

The main objective of this paper is to bridge these three important theories by developing a comprehensive framework for studying multilinear integral operators on Morrey spaces using Herz space conditions. While the boundedness properties of linear operators on Morrey spaces have been well-established [1, 8, 9, 17], the multilinear case presents additional challenges due to the more complex nature of the kernels and the product space structure.

Notation: Throughout this paper, we use the following notation: - $\mathbb{R}^n$: n-dimensional Euclidean space - $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$: open ball centered at x with radius r - $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$: collection of all balls in $\mathbb{R}^n$ - L^p: classical Lebesgue space

for $1 < p < \infty$ - L_c^p: space of L^p functions with compact support - M_p^q: Morrey space for $1 < p \leq q < \infty$ - $\mathcal{K}_{p,q}^{\alpha}$: Herz space for $1 \leq p, q < \infty$, $\alpha \in \mathbb{R}$ - χ_E: characteristic function of set E - $\chi_j = \chi_{B(0,2^j)\setminus B(0,2^{j-1})}$: dyadic annuli characteristic functions

This paper is organized as follows: Section 2 provides the necessary preliminaries and definitions of Morrey spaces, Herz spaces, and multilinear integral operators. Section 3 contains our main results, including the definition of multilinear operators on Morrey spaces, boundedness theorems, and operator norm estimates. We also present applications to multilinear singular integrals, fractional integrals, and strongly singular operators.

2 Preliminaries and Definitions

We begin by recalling the fundamental definitions and properties of the function spaces that will be used throughout this paper. The following definitions are standard in harmonic analysis and can be found in [20, 15, 6].

Definition 2.1 (Morrey Space [20, 17]). Let $1 < p \leq q < \infty$. The Morrey space M_p^q consists of all locally integrable functions f satisfying

$$\|f\|_{M_p^q} = \sup_{B \in \mathbb{B}} \frac{1}{|B|^{\frac{1}{p}-\frac{1}{q}}} \|\chi_B f\|_{L^p} < \infty.$$

The space M_p^q generalizes the Lebesgue space L^p (when $p = q$) and provides a finer scale for measuring local integrability with global control.

Definition 2.2 (Herz Space [7, 15]). Let $1 \leq p, q < \infty$ and $\alpha \in \mathbb{R}$. The Herz space $\mathcal{K}_{p,q}^{\alpha}$ on $\mathbb{R}^n \setminus B(0,1)$ consists of all locally integrable functions f on $\mathbb{R}^n \setminus B(0,1)$ satisfying

$$\|f\|_{\mathcal{K}_{p,q}^{\alpha}} = \left(\sum_{j=1}^{\infty} (2^{j\alpha} \|\chi_j f\|_{L^p})^q \right)^{\frac{1}{q}} < \infty,$$

where $\chi_j = \chi_{B(0,2^j)\setminus B(0,2^{j-1})}$. Herz spaces provide a natural framework for studying functions with certain decay properties at infinity.

We now introduce the class of multilinear operators studied in this paper, following the framework established in [6, 10].

Definition 2.3 (Multilinear Integral Operator [6]). Let $1 < p_1, \ldots, p_m, p < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. A bounded multilinear operator $T : L^{p_1} \times \cdots \times L^{p_m} \to L^p$ belongs to the class $\mathcal{L}_{(p_1,\ldots,p_m),p}$ if there exists a measurable function $K : (\mathbb{R}^n)^{m+1} \to \mathbb{R}$ such that for any $f_i \in L_c^{p_i}$, $i = 1, \ldots, m$, and for $x \notin \bigcap_{i=1}^m \operatorname{supp}(f_i)$,

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i) d\vec{y}.$$

We say T is associated with the kernel K.

Following the approach in [8, 9], we introduce a scaled version of the kernel that will be crucial for our analysis:

Definition 2.4 (Scaled Kernel). Let $r > 0$ and $\theta \in \mathbb{R}$. For a kernel $K : (\mathbb{R}^n)^{m+1} \to \mathbb{R}$, define its scaled version:

$$K_r^{\theta}(x, \vec{v}) = r^{\theta} K(rx, r(x + v_1), \ldots, r(x + v_m)), \quad x, v_i \in \mathbb{R}^n,$$

where $\vec{v} = (v_1, \ldots, v_m)$. This scaling captures the behavior of the kernel under dilations and will be essential for our Herz space conditions.

Remark 2.5. The scaled kernel K_r^θ captures the behavior of K under dilations. This scaling is natural in the context of Morrey and Herz spaces because it respects the scaling invariance properties of these spaces. In particular, the parameter θ is chosen to balance the homogeneity of the kernel with the scaling properties of Morrey space norms.

Remark 2.6. The Herz space condition $\mathcal{K}^{\alpha}_{p',1}$ with the ℓ^1-sum over dyadic annuli quantifies the "off-diagonal" decay of the kernel. The ℓ^1-sum ensures absolute convergence of the dyadic decomposition, which is crucial for controlling the tail part of multilinear operators in Morrey spaces.

3 Main Results

3.1 Multilinear Estimates and Boundedness

We begin with a fundamental multilinear estimate that extends the linear results in [8] to the multilinear setting.

Lemma 3.1 (Multilinear Estimate). *Let $\theta \in \mathbb{R}$, $1 < p_i \le q_i < \infty$ for $i = 1, \dots, m$, and $1 < p \le q < \infty$ with $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$, $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i}$. Let $r > 0$ and $K : (\mathbb{R}^n)^{m+1} \to \mathbb{R}$ be measurable. If for any $x \in \mathbb{R}^n$, $K_r^\theta(x/r, \cdot) \in \mathcal{K}^{\alpha}_{p',1}$ with $\alpha = \sum_{i=1}^m (\frac{n}{p_i} - \frac{n}{q_i})$, then there exists $C > 0$ independent of K such that for any $f_i \in M^{q_i}_{p_i}$, $x \in \mathbb{R}^n$ and $r > 0$,*

$$\int_{(\mathbb{R}^n)^m} \prod_{i=1}^m \chi_{\{|x-y_i|>r\}} |K(x,\vec{y}) \prod_{i=1}^m f_i(y_i)| d\vec{y} \le C r^{-\theta+mn-\sum_{i=1}^m \frac{n}{q_i}} \|K_r^\theta(x/r,\cdot)\|_{\mathcal{K}^{\alpha}_{p',1}} \prod_{i=1}^m \|f_i\|_{M^{q_i}_{p_i}}.$$

Proof. The proof extends the linear argument in [8, Lemma 2.1] to the multilinear case. For each $j \ge 1$, define the multilinear annulus:

$$A_j = \prod_{i=1}^m \{y_i \in \mathbb{R}^n : 2^{j-1} r \le |x - y_i| < 2^j r\}.$$

By Hölder's inequality and the definition of Morrey spaces [20], we have:

$$\begin{aligned}
&\int_{A_j} |K(x,\vec{y}) \prod_{i=1}^m f_i(y_i)| d\vec{y} \\
&\le \left(\int_{A_j} |K(x,\vec{y})|^{p'} d\vec{y} \right)^{\frac{1}{p'}} \prod_{i=1}^m \left(\int_{2^{j-1} r \le |x-y_i| < 2^j r} |f_i(y_i)|^{p_i} dy_i \right)^{\frac{1}{p_i}} \\
&\le \left(\int_{A_j} |K(x,\vec{y})|^{p'} d\vec{y} \right)^{\frac{1}{p'}} \prod_{i=1}^m \left(C(2^j r)^{n(1-\frac{p_i}{q_i})} \|f_i\|^{p_i}_{M^{q_i}_{p_i}} \right)^{\frac{1}{p_i}}.
\end{aligned}$$

Now we apply the change of variables technique from [9]: let $w = x/r$, $z_i = y_i/r$, and $v_i = z_i - w$. Then:

$$\begin{aligned}
&\left(\int_{A_j} |K(x,\vec{y})|^{p'} d\vec{y} \right)^{\frac{1}{p'}} \\
&= \left(\int_{\prod_{i=1}^m \{2^{j-1} \le |v_i| < 2^j\}} |K(rw, r(w+v_1), \dots, r(w+v_m))|^{p'} r^{mn} d\vec{v} \right)^{\frac{1}{p'}} \\
&= r^{\frac{mn}{p'}} \left(\int_{\prod_{i=1}^m \{2^{j-1} \le |v_i| < 2^j\}} |K_r^\theta(w,\vec{v})|^{p'} r^{-\theta p'} d\vec{v} \right)^{\frac{1}{p'}} \\
&= r^{\frac{mn}{p'} - \theta} \|\chi_j K_r^\theta(w,\cdot)\|_{L^{p'}}.
\end{aligned}$$

Combining these estimates and summing over j gives:

$$\begin{aligned}
&\int_{|x-y_i|>r} |K(x,\vec{y})\prod_{i=1}^m f_i(y_i)|d\vec{y}\\
&\le C\sum_{j=1}^{\infty} r^{\frac{mn}{p'}-\theta}\|\chi_j K_r^\theta(w,\cdot)\|_{L^{p'}}(2^j r)^{\sum_{i=1}^m \frac{n}{p_i}-\frac{n}{q_i}}\prod_{i=1}^m \|f_i\|_{M_{p_i}^{q_i}}\\
&= Cr^{-\theta+mn-\sum_{i=1}^m \frac{n}{q_i}}\left(\sum_{j=1}^{\infty} 2^{j\alpha}\|\chi_j K_r^\theta(w,\cdot)\|_{L^{p'}}\right)\prod_{i=1}^m \|f_i\|_{M_{p_i}^{q_i}}\\
&= Cr^{-\theta+mn-\sum_{i=1}^m \frac{n}{q_i}}\|K_r^\theta(x/r,\cdot)\|_{\mathcal{K}^\alpha_{p',1}}\prod_{i=1}^m \|f_i\|_{M_{p_i}^{q_i}}.
\end{aligned}$$

This completes the proof of the multilinear estimate. □

Remark 3.2. Lemma 3.1 will be used to control the "tail" part of the operator $\mathcal{T}$ in Definition 3.3, which involves integration over the region where $|x-y_i|>r$.

Now we define the extension of multilinear operators to Morrey spaces, following the approach in [8, 9] for the linear case.

Definition 3.3 (Multilinear Operator on Morrey Spaces). Let $1<p_i<q_i<\infty$, $i=1,\dots,m$, and $1<p<q<\infty$ with $\frac{1}{p}=\sum_{i=1}^m \frac{1}{p_i}$, $\frac{1}{q}=\sum_{i=1}^m \frac{1}{q_i}$. Suppose $T\in\mathcal{L}_{(p_1,\dots,p_m),p}$. For any $\vec{f}=(f_1,\dots,f_m)\in\prod_{i=1}^m M_{p_i}^{q_i}$ and $x\in B(z,r)\in\mathbb{B}$, define

$$\mathcal{T}(\vec{f})(x)=T(\chi_{B(z,2r)}f_1,\dots,\chi_{B(z,2r)}f_m)(x)+\int_{(\mathbb{R}^n)^m\setminus\prod_{i=1}^m B(z,2r)} K(x,\vec{y})\prod_{i=1}^m f_i(y_i)d\vec{y}.$$

The choice of $2r$ (rather than Cr for some other constant C) is for convenience; the definition is independent of this specific constant factor, as shown in the proof of Theorem 3.5.

Remark 3.4. The proof of Theorem 3.5 follows a natural strategy: we split the operator into near and far parts. The near part is controlled using the Lebesgue space boundedness of T, while the far part is estimated using Lemma 3.1 and the Herz space condition on the kernel.

Our main boundedness result extends [8, Theorem 3.1] to the multilinear setting:

Theorem 3.5 (Boundedness on Morrey Spaces). *Let $1<p_i<q_i<\infty$, $i=1,\dots,m$, and $1<p<q<\infty$ with $\frac{1}{p}=\sum_{i=1}^m\frac{1}{p_i}$, $\frac{1}{q}=\sum_{i=1}^m\frac{1}{q_i}$. Suppose*

$$\theta=mn-\sum_{i=1}^m\frac{n}{q_i}+\frac{n}{q}.$$

Let $T\in\mathcal{L}_{(p_1,\dots,p_m),p}$ be associated with kernel K. If K satisfies

$$\sup_{x\in\mathbb{R}^n}\sup_{r>0}\|K_r^\theta(x/r,\cdot)\|_{\mathcal{K}^\alpha_{p',1}}<\infty,\quad \text{where } \alpha=\sum_{i=1}^m\left(\frac{n}{p_i}-\frac{n}{q_i}\right),$$

then $\mathcal{T}$ is a well-defined bounded multilinear operator from $\prod_{i=1}^m M_{p_i}^{q_i}$ to M_p^q and satisfies

$$\|\mathcal{T}\|_{\prod M_{p_i}^{q_i}\to M_p^q}\le\|T\|_{\prod L^{p_i}\to L^p}+C\sup_{x\in\mathbb{R}^n}\sup_{r>0}\|K_r^\theta(x/r,\cdot)\|_{\mathcal{K}^\alpha_{p',1}}$$

for some $C>0$ independent of K.

Proof. We first show that $\mathcal{T}$ is well-defined. Let $B(z,R)\in\mathbb{B}$. Since $T\in\mathcal{L}_{(p_1,\dots,p_m),p}$ and $\chi_{B(z,2R)}f_i\in L^{p_i}$, the first term is well-defined by the definition of $\mathcal{L}_{(p_1,\dots,p_m),p}$. Lemma 3.1 guarantees the absolute integrability of the second term.

To show independence of the ball choice, let $x \in B(z,R) \cap B(w,S)$ with $B(z,R) \cap B(w,S) \neq \emptyset$. Choose $B(u,M)$ such that $B(z,2R) \cup B(w,2S) \subset B(u,M)$. Following the approach in [8, Theorem 3.1], we have:

$$\begin{aligned}
&\mathcal{T}(\vec{f})(x) \text{ defined via } B(z,R) \\
&= T(\chi_{B(z,2R)} f_1, \dots)(x) + \int_{(\mathbb{R}^n)^m \setminus \prod B(z,2R)} K(x,\vec{y}) \prod f_i(y_i) d\vec{y} \\
&= T(\chi_{B(u,M)} f_1, \dots)(x) + \int_{(\mathbb{R}^n)^m \setminus \prod B(u,M)} K(x,\vec{y}) \prod f_i(y_i) d\vec{y}.
\end{aligned}$$

A similar computation holds for the definition via $B(w,S)$, hence $\mathcal{T}$ is well-defined.

For boundedness, let $B(z,R) \in \mathbb{B}$ and write:

$$\|\chi_{B(z,R)} \mathcal{T}(\vec{f})\|_{L^p} \leq \|T(\chi_{B(z,2R)} f_1, \dots, \chi_{B(z,2R)} f_m)\|_{L^p} + \|\chi_{B(z,R)} F\|_{L^p},$$

where $F(x) = \int_{(\mathbb{R}^n)^m \setminus \prod B(z,2R)} K(x,\vec{y}) \prod f_i(y_i) d\vec{y}$.

The first term is bounded by the Lebesgue space boundedness of T:

$$\|T\|_{\prod L^{p_i} \to L^p} \prod_{i=1}^m \|\chi_{B(z,2R)} f_i\|_{L^{p_i}} \leq \|T\|_{\prod L^{p_i} \to L^p} \prod_{i=1}^m (2R)^{\frac{n}{p_i} - \frac{n}{q_i}} \|f_i\|_{M^{q_i}_{p_i}}.$$

For the second term, by Lemma 3.1 and the condition on K:

$$|F(x)| \leq C R^{-\theta + mn - \sum_{i=1}^m \frac{n}{q_i}} \sup \|K_r^\theta\| \prod_{i=1}^m \|f_i\|_{M^{q_i}_{p_i}}.$$

Since $\theta = mn - \sum_{i=1}^m \frac{n}{q_i} + \frac{n}{q}$, we have:

$$R^{-\theta + mn - \sum_{i=1}^m \frac{n}{q_i}} = R^{-\frac{n}{q}}.$$

Therefore:

$$\|\chi_{B(z,R)} F\|_{L^p} \leq C R^{\frac{n}{p}} \cdot R^{-\frac{n}{q}} \sup \|K_r^\theta\| \prod_{i=1}^m \|f_i\|_{M^{q_i}_{p_i}} = C R^{\frac{n}{p} - \frac{n}{q}} \sup \|K_r^\theta\| \prod_{i=1}^m \|f_i\|_{M^{q_i}_{p_i}}.$$

Combining both estimates and taking supremum over all balls gives the desired boundedness result. □

3.2 Extension Properties and Additional Results

Theorem 3.6 (Extension Property). *Under the assumptions of Theorem 3.5, for any* $\vec{f} \in \prod_{i=1}^m (L^{p_i} \cap M^{q_i}_{p_i})$, *we have* $\mathcal{T}(\vec{f}) = T(\vec{f})$.

Proof. Let $\vec{f} \in \prod_{i=1}^m (L^{p_i} \cap M^{q_i}_{p_i})$. For fixed $B(z,r) \in \mathbb{B}$, define truncations following [8, Theorem 3.2]:

$$\begin{aligned}
f_i^0 &= \chi_{B(z,2r)} f_i, \\
f_i^j &= \chi_{B(z,2^{j+1}r) \setminus B(z,2^j r)} f_i, \quad j \geq 1.
\end{aligned}$$

Since $|f_i - \sum_{j=0}^N f_i^j| \leq |f_i|$ and $\lim_{N\to\infty} |f_i - \sum_{j=0}^N f_i^j| = 0$, the dominated convergence theorem gives:

$$\lim_{N\to\infty} \|f_i - \sum_{j=0}^N f_i^j\|_{L^{p_i}} = 0.$$

By the boundedness of T on Lebesgue spaces [6]:

$$\lim_{N\to\infty} \left\| T(\vec{f}) - T\left(\sum_{j=0}^N f_1^j, \dots, \sum_{j=0}^N f_m^j \right) \right\|_{L^p} = 0.$$

Hence, for almost every x:

$$T(\vec{f})(x) = T(f_1^0, \ldots, f_m^0)(x) + \sum_{j=1}^{\infty} T(f_1^j, \ldots, f_m^j)(x).$$

For $x \in B(z,r)$ and $j \geq 1$, since $\operatorname{supp} f_i^j \cap B(z,r) = \emptyset$, we have by the kernel representation:

$$T(f_1^j, \ldots, f_m^j)(x) = \int_{(\mathbb{R}^n)^m} K(x, \vec{y}) \prod_{i=1}^{m} f_i^j(y_i) d\vec{y}.$$

By Lemma 3.1, the tail is absolutely convergent, so:

$$T(\vec{f})(x) = T(f_1^0, \ldots, f_m^0)(x) + \int_{(\mathbb{R}^n)^m \setminus \prod B(z,2r)} K(x, \vec{y}) \prod_{i=1}^{m} f_i(y_i) d\vec{y} = \mathcal{T}(\vec{f})(x).$$

□

Proposition 3.7 (Compact Support Preservation). *Under the assumptions of Theorem 3.5, if $f_i \in M_{p_i}^{q_i}$ have compact support, then $\mathcal{T}(\vec{f})$ also has compact support.*

Proof. Let $\operatorname{supp} f_i \subseteq B(0,R)$ for all i. For $|x| > 2R$, if $|y_i| < R$, then $|x - y_i| > R$. By the decay condition on K from the Herz space assumption and Lemma 3.1, we get rapid decay of $\mathcal{T}(\vec{f})(x)$ as $|x| \to \infty$, implying compact support. More precisely, for $|x| > 2R$, we have:

$$|\mathcal{T}(\vec{f})(x)| \leq C|x|^{-\theta + mn - \sum_{i=1}^{m} \frac{n}{q_i}} \sup \|K_r^\theta\| \prod_{i=1}^{m} \|f_i\|_{M_{p_i}^{q_i}},$$

and the exponent $-\theta + mn - \sum_{i=1}^{m} \frac{n}{q_i} = -\frac{n}{q} < 0$ ensures decay at infinity. □

3.3 Applications and Examples

Theorem 3.8 (Kernels with Homogeneous Singularity). *Let $\gamma \in \mathbb{R}$, $1 < p_i < q_i < \infty$, $i = 1, \ldots, m$, and $1 < p < q < \infty$ with $\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}$, $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$. Suppose*

$$\gamma = mn - \sum_{i=1}^{m} \frac{n}{q_i} + \frac{n}{q}.$$

Let Ω be a homogeneous function satisfying $\|\Omega\|_{L^{p'}(\mathbb{S}^{mn-1})} < \infty$. If $T \in \mathcal{L}_{(p_1,\ldots,p_m),p}$ is associated with kernel K satisfying

$$|K(x, y_1, \ldots, y_m)| \leq \left(\sum_{i=1}^{m} |x - y_i| \right)^{-\gamma} \left| \Omega\left(\frac{(x - y_1, \ldots, x - y_m)}{\|(x - y_1, \ldots, x - y_m)\|} \right) \right|, \quad x \neq y_i,$$

then $\mathcal{T}$ is bounded from $\prod_{i=1}^{m} M_{p_i}^{q_i}$ to M_p^q and

$$\|\mathcal{T}\|_{\prod M_{p_i}^{q_i} \to M_p^q} \leq \|T\|_{\prod L^{p_i} \to L^p} + C\|\Omega\|_{L^{p'}(\mathbb{S}^{mn-1})}.$$

Proof. Define the majorant kernel:

$$H(x, \vec{y}) = \left(\sum_{i=1}^{m} |x - y_i| \right)^{-\gamma} \left| \Omega\left(\frac{(x - y_1, \ldots, x - y_m)}{\|(x - y_1, \ldots, x - y_m)\|} \right) \right|.$$

For any $r > 0$, $x \in \mathbb{R}^n$, $y_i \neq 0$, using the homogeneity of Ω:

$$H_r^\gamma(x/r, \vec{y}) = r^\gamma H(x, x + ry_1, \ldots, x + ry_m) = |\vec{y}|^{-\gamma} |\Omega(\vec{y}/|\vec{y}|)|.$$

Then for any $j \in \mathbb{N}$, following the computation in [8, Theorem 3.3]:

$$\|\chi_j H_r^\gamma(x/r,\cdot)\|_{L^{p'}} = \left(\int_{\prod\{2^{j-1}\le|y_i|<2^j\}} |\Omega(\vec{y}/|\vec{y}|)|^{p'}|\vec{y}|^{-\gamma p'}d\vec{y}\right)^{\frac{1}{p'}}$$
$$\le C2^{-j(\gamma-\frac{mn}{p'})}\|\Omega\|_{L^{p'}(\mathbb{S}^{mn-1})}.$$

Therefore:

$$\|H_r^\gamma(x/r,\cdot)\|_{\mathcal{K}^\alpha_{p',1}} \le C\|\Omega\|_{L^{p'}(\mathbb{S}^{mn-1})}\sum_{j=1}^{\infty}2^{j(\alpha-\gamma+\frac{mn}{p'})}.$$

Since $\alpha-\gamma+\frac{mn}{p'} = -\frac{n}{q}$, the series converges. The result follows from Theorem 3.5. □

Example 3.9 (Multilinear Riesz Transforms). Consider the multilinear Riesz transforms studied in [6] with kernel:

$$K(x,y_1,\dots,y_m) = \frac{\prod_{i=1}^m(x-y_i)}{|x-y_1|^{n+1}\cdots|x-y_m|^{n+1}}.$$

This kernel satisfies the conditions of Theorem 3.8 with $\gamma = mn+m$ and Ω bounded. Indeed, we have the pointwise estimate:

$$|K(x,\vec{y})| \le C\left(\sum_{i=1}^m|x-y_i|\right)^{-mn-m},$$

which is consistent with the standard multilinear Calderón-Zygmund kernel size condition. The corresponding operator is bounded on appropriate Morrey spaces with parameters satisfying:

$$mn+m = mn-\sum_{i=1}^m\frac{n}{q_i}+\frac{n}{q} \quad\Rightarrow\quad \frac{n}{q} = m+\sum_{i=1}^m\frac{n}{q_i}.$$

Example 3.10 (Multilinear Fractional Integrals). The multilinear fractional integral operator studied in [10]:

$$I_\alpha(\vec{f})(x) = \int_{(\mathbb{R}^n)^m}\frac{\prod_{i=1}^m f_i(y_i)}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha}}d\vec{y}$$

satisfies the conditions with $\gamma = mn-\alpha$. Theorem 3.8 gives boundedness from $\prod M^{q_i}_{p_i}$ to M^q_p when:

$$\frac{n}{q} = \sum_{i=1}^m\frac{n}{q_i}-\alpha.$$

This extends the classical Hardy–Littlewood–Sobolev theorem to the multilinear Morrey space setting.

Theorem 3.11 (Strongly Singular Multilinear Operators). *Let $\beta\in(0,\infty)$, $\alpha\in[0,\frac{mn\beta}{2})$ and let a,ϕ be smooth functions satisfying the conditions from [11]:*

$$|\partial_x^\mu\partial_y^\nu a(x,\vec{y})| \le C|x-\vec{y}|^{-mn-\alpha-|\mu|-|\nu|},$$
$$|\partial_x^\mu\partial_y^\nu\phi(x,\vec{y})| \le C|x-\vec{y}|^{-\beta-|\mu|-|\nu|},$$
$$|\nabla_x\phi(x,\vec{y})| \ge C|x-\vec{y}|^{-\beta-1},$$

where $|x-\vec{y}| = \sum_{i=1}^m|x-y_i|$. Consider the operator with kernel:

$$K(x,\vec{y}) = e^{i\phi(x,\vec{y})}a(x,\vec{y}).$$

If T is bounded on $\prod L^{p_i}\to L^p$ with $\left|\frac{1}{p}-\frac{1}{2}\right| \le \frac{1}{2}-\frac{\alpha}{mn\beta}$, then its extension $\mathcal{T}$ is bounded on corresponding Morrey spaces.

Proof. The kernel satisfies the pointwise bound:

$$|K(x,\vec{y})| \leq C|x-\vec{y}|^{-mn-\alpha},$$

so we can apply Theorem 3.8 with $\gamma = mn + \alpha$. The boundedness on Lebesgue spaces under the condition $\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{1}{2} - \frac{\alpha}{mn\beta}$ was established in [11] for the linear case and extends to the multilinear setting following [6]. □

Remark 3.12. The results in this paper provide a unified framework that encompasses many classical operators:

- When $m = 1$, we recover the linear results in [8, 9]
- When $p_i = q_i$, we obtain boundedness on products of Lebesgue spaces
- When $\Omega \equiv 1$, we get results for smooth kernels
- The framework applies to rough kernels through the Herz space conditions

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