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Bifurcation Analysis of a Second-Order Rational Difference Equation

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Abstract This paper investigates the bifurcation structure of a second-order rational difference equation, given by

$$x_{n+1} = \frac{\gamma + \beta x_n + \alpha x_{n-1}}{\rho + A x_n + B x_{n-1}},$$

where all parameters $\gamma, \beta, \alpha, \rho, A$, and B are positive. The initial conditions satisfy $x_i \geq 0$ for $i = -k, -k+1, \ldots, 0$. The system models nonlinear interactions influenced by both current and delayed states. We derive analytical conditions for the existence of bifurcations in the system. Specifically, we study saddle-node, Neimark–Sacker, and period-doubling bifurcations. In addition, we analyze the direction of the period-doubling bifurcation. Numerical examples are provided to validate the theoretical results, and bifurcation diagrams are generated to illustrate the transitions that occur.

1 Introduction

Difference equations are fundamental in analyzing discrete-time dynamical systems, serving as tools to approximate differential equations and as independent models. They are widely used to represent processes that change in discrete time intervals in fields such as biology, physics, economics, and ecology. They represent processes that change in discrete time intervals [7, 10, 11]. Equation (1.3), introduced by E. Camouzis and G. Ladas in [4, 11], was studied for local stability and global attractivity of its equilibrium point, as well as the existence of a singular, locally stable prime period-two solution.

X. S. Luo et al. [13] developed a new hybrid control strategy that combines state feedback and parameter perturbation to control period-doubling bifurcations and chaos in discrete nonlinear dynamical systems.

The dynamic behavior and sufficient conditions for the occurrence of bifurcations were studied by B. Raddad and M. Saleh in [16, 17].

There are numerous recent studies that focus on the role of difference equations, such as [14, 18]. Additionally, other research specifically explores bifurcation analysis and its significance in understanding the dynamical behavior of difference equations, as seen in [3, 19].

In 2018, the authors in [5] investigated the qualitative behavior of a second-order rational difference equation. They discussed parametric conditions for the existence and direction of period-doubling and pitchfork bifurcations at the trivial equilibrium point.

M. Saleh and S. Herzallah [8] also studied the dynamics of the second-order rational difference

equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx_n^2 + Cx_{n-1}}, \tag{1.1}$$

Their work focused on stability, invariant intervals, boundedness, and bifurcation. Numerical examples illustrated period-doubling and changes in the system's behavior.
In a related work [9], the same authors investigated the dynamics and bifurcation of the rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}^2}{A + Bx_n^2 + Cx_{n-1}}, \tag{1.2}$$

with positive parameters A, B, C and non-negative initial conditions.
In 2023, the authors [3] analyzed the stability of the unique positive equilibrium of the equation and proved the existence of a Neimark–Sacker bifurcation.
In addition, in 2023, M. Abd El-Moneam [1] studied the stability, global behavior, and convergence rate of a system of nonlinear difference equations with positive parameters. The oscillatory behavior of the solutions was also analyzed.
A. Padder et al. [15] analyzed a tumor-immune interaction model with three cell populations using a system of ODEs. Stability, bifurcation, and sensitivity analyses—supported by numerical simulations—reveal conditions for long-term cell coexistence.
In 2024, Turki D. Alharbi et al. [2] analyzed seventh-order nonlinear difference equations, revealing that their equilibrium points often lack stability.
Recently, S. Shrateh and M. Saleh [20] investigated the dynamics and semi-cycles of a higher-order nonlinear difference equation.
We consider the second-order rational difference equation

$$x_{n+1} = \frac{\gamma + \beta x_n + \alpha x_{n-1}}{\rho + Ax_n + Bx_{n-1}}, \tag{1.3}$$

where the parameters $\gamma, \beta, \alpha, \rho, A$, and B are all positive, and the initial conditions satisfy $x_i \geq 0$ for $i = -k, -k+1, \ldots, 0$.
By altering the variables, the second-order rational difference equation (1.3) with six positive parameters is simplified to

$$z_{n+1} = \frac{w + z_n + ez_{n-1}}{k + z_n + dz_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.4}$$

with four positive parameters: w, k, e, and d.

This paper is organized as follows. In Section 2, we analyze the fixed points of Equation (1.4) and convert it into a two-dimensional system. Section 3 investigates the occurrence of a *saddle-node bifurcation* by treating w as the bifurcation parameter. In Section 4, we explore the conditions under which a *Neimark-Sacker bifurcation* arises. Section 5 focuses on the *period-doubling bifurcation* and derives the conditions under which it occurs. In Section 6, we determine the *direction* of the period-doubling bifurcation using normal form analysis. Numerical examples and simulations for each case are provided using the MATLAB package to support the theoretical findings.

2 Dynamics of $z_{n+1} = \frac{w+z_n+ez_{n-1}}{k+z_n+dz_{n-1}}$

In this section, we analyze the fixed point of equation (1.4) and convert it into a two-dimensional system.
A fixed point $\bar{z}$ of equation (1.4) satisfies

$$\bar{z} = \frac{w + \bar{z} + e\bar{z}}{k + \bar{z} + d\bar{z}}. \tag{2.1}$$

Hence,

$$(1+d)\bar{z}^2 = w - (k - e - 1)\bar{z}. \tag{2.2}$$

The fixed points are obtained by solving the corresponding quadratic equation (2.2) are

$$\bar{z} = \frac{(e+1-k) \pm \sqrt{(k-e-1)^2 + 4w(1+d)}}{2(1+d)}. \tag{2.3}$$

As we are concerned with the unique positive fixed point, we select the positive root accordingly

$$\bar{z} = \frac{(e+1-k) + \sqrt{(k-e-1)^2 + 4w(1+d)}}{2(1+d)}. \tag{2.4}$$

To convert Equation (1.4) into a two-dimensional system involving the four positive parameters e, d, w, and k, we introduce new variables $r_n = z_{n-1}$ and $s_n = z_n$. This transformation yields the following system

$$\begin{aligned} r_{n+1} &= s_n, \\ s_{n+1} &= \frac{w + s_n + e r_n}{k + s_n + d r_n}, \end{aligned} \qquad n = 0, 1, 2, \ldots \tag{2.5}$$

The corresponding Jacobian matrix $J(u, v)$ at the positive fixed point is

$$JF(r,s)\big|_{(\bar{z},\bar{z})} = \begin{pmatrix} 0 & 1 \\ \frac{ek - dw + \bar{z}(e-d)}{(k+\bar{z}+d\bar{z})^2} & \frac{k - w + (d-e)\bar{z}}{(k+\bar{z}+d\bar{z})^2} \end{pmatrix}. \tag{2.6}$$

Note that the determinant of the Jacobian matrix is given by

$$\|J(g(\bar{z},\bar{z}))\| = \frac{dw - ek - \bar{z}(e-d)}{(k+\bar{z}+d\bar{z})^2}, \tag{2.7}$$

and the trace is

$$\mathrm{Tr}(J(g(\bar{z},\bar{z}))) = \frac{k - w + (d-e)\bar{z}}{(k+\bar{z}+d\bar{z})^2}. \tag{2.8}$$

Using equation (2.4), we can simplify the determinant and the trace of the Jacobian at the fixed point as follows

$$\det(J(g(\bar{z},\bar{z}))) = \frac{d\bar{z} - e}{k + \bar{z} + d\bar{z}}, \tag{2.9}$$

and

$$\mathrm{Tr}(J(g(\bar{z},\bar{z}))) = \frac{1 - \bar{z}}{k + \bar{z} + d\bar{z}}. \tag{2.10}$$

3 Analysis of Bifurcation Behavior

In this section, we investigate the bifurcation behavior of Equation (1.4) applying the lemma stated below.

Lemma 3.1 ([6]). *Suppose*

$$\mathbf{z} \mapsto h(\mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^2, \quad h : \mathbb{R}^2 \to \mathbb{R}^2.$$

Let $B = Jh(\mathbf{z}^*)$, where $\mathbf{z}^*$ is a fixed point of $h(\mathbf{z})$. Then

(i) When $\|B\| = -1 - Trace(B)$, the eigenvalues are $\lambda_1 = -\|B\|$ and $\lambda_2 = -1$.

(ii) When $\|B\| = Trace(B) - 1$, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \|B\|$.

(iii) When $\|B\| = 1$ and $Trace(B) - 1 < \|B\|$, the matrix B has complex eigenvalues $\lambda_{1,2} = e^{\pm i\theta}$, where $\theta = \cos^{-1}\left(\frac{Trace(B)}{2}\right)$.

Based on the above lemma, we state the following corollary

Corollary 3.2 ([6]). *Consider a one-parameter family of planar maps given by*

$$\mathbf{z} \mapsto h_\alpha(\mathbf{x}), \quad \mathbf{z} \in \mathbb{R}^2, \quad h_\alpha : \mathbb{R}^2 \to \mathbb{R}^2,$$

with a fixed point $\mathbf{z}^*$, and let $B = Jh_\alpha(\mathbf{z}^*)$. Then:

(i) *When* $\|B\| = -1 - Trace(B)$*, the system undergoes a* ***flip bifurcation****.*

(ii) *When* $\|B\| = -1 + Trace(B)$*, the system undergoes a* ***saddle-node bifurcation****.*

(iii) *When* $\|B\| = 1$ *and* $Trace(B) - 1 < \|B\|$*, the system undergoes a* ***Neimark–Sacker bifurcation****.*

4 Occurrence of Saddle-Node Bifurcation

In this section, we study the case of a saddle-node bifurcation for Equation (1.4) applying Corollary 3.2.
This type of bifurcation occurs in the trace-determinant plane when

$$\|J\| = \text{Trace}(J) - 1. \tag{4.1}$$

Substituting the expressions for the determinant and the trace, we have

$$\frac{d\bar{z} - e}{k + \bar{z} + d\bar{z}} = \frac{1 - \bar{z}}{k + \bar{z} + d\bar{z}} - 1. \tag{4.2}$$

Simplifying, we obtain the following.

$$\bar{z} = \frac{-k + e + 1}{2(1 + d)}. \tag{4.3}$$

Recall that the fixed point is given by equation (2.4). Equating equation (2.4) and equation (4.3) for $\bar{z}$, we get

$$w = -\frac{(k - e - 1)^2}{4(1 + d)}. \tag{4.4}$$

This result is not feasible, as w is a positive parameter. Therefore, we conclude that system (2.5) cannot undergo a saddle-node bifurcation.

5 Occurrence of Neimark–Sacker Bifurcation

In this section, we examine the conditions under which a Neimark–Sacker bifurcation occurs in Equation (1.4), using Corollary 3.2 as the basis for our analysis.
A Neimark–Sacker bifurcation arises if the Jacobian matrix possesses two complex conjugate eigenvalues of modulus 1. This occurs when

$$\|J\| = 1 \quad \text{and} \quad -2 < \text{Trace}(J) < 2,$$

in the trace-determinant plane.
Start with the condition

$$\|J\| = \frac{d\bar{z} - e}{k + \bar{z} + d\bar{z}} = 1. \tag{5.1}$$

This simplifies to

$$\bar{z} = -e - k. \tag{5.2}$$

Using equation 2.4, we obtain

$$\sqrt{(k - e - 1)^2 + 4w(1 + d)} = -(3e + 2ed + k + 2kd + 1). \tag{5.3}$$

This leads to a contradiction, since the left-hand side is non-negative while the right-hand side is strictly negative (as $k, d, e > 0$). Therefore, the system cannot exhibit a Neimark–Sacker bifurcation at the equilibrium point $(\bar{z}, \bar{z})$.

6 Occurrence of Period-Doubling Bifurcation

In this section, we examine the occurrence of a period-doubling (flip) bifurcation in Equation (1.4), using Corollary 3.2.

Theorem 6.1. *The equilibrium point of system (2.5) undergoes a flip bifurcation when*

$$w = \frac{1}{4(1+d)}\left(\frac{(-2d-1-k+e)^2}{d^2} - (k-e-1)^2\right), \tag{6.1}$$

provided that

$$\frac{(-2d-1-k+e)^2}{d^2} > (k-e-1)^2. \tag{6.2}$$

Proof. According to Corollary 3.2, a flip bifurcation occurs if

$$\|J\| = -1 - \text{Trace}(J). \tag{6.3}$$

Substituting the expressions for the determinant and trace of the Jacobian, we obtain

$$\frac{d\bar{z}-e}{k+\bar{z}+d\bar{z}} = \left(\frac{\bar{z}-1}{k+\bar{z}+d\bar{z}}\right) - 1. \tag{6.4}$$

This simplifies to

$$\bar{z} = \frac{-1-k+e}{2d}. \tag{6.5}$$

Since $\bar{z} > 0$, it follows that

$$e - k > 1. \tag{6.6}$$

Substituting the expression for $\bar{z}$ into the equilibrium equation, we get

$$\frac{-1-k+e}{2d} = \frac{(e+1-k)+\sqrt{(k-e-1)^2+4w(1+d)}}{2(1+d)}. \tag{6.7}$$

Solving for w, we obtain

$$w = \frac{1}{4(1+d)}\left(\frac{(-2d-1-k+e)^2}{d^2} - (k-e-1)^2\right). \tag{6.8}$$

Since $w > 0$, we require

$$\frac{(-2d-1-k+e)^2}{d^2} > (k-e-1)^2. \tag{6.9}$$

□

7 Direction of the Flip Bifurcation

In this section, we determine the path of the flip bifurcation of system (2.5) at equation (6.1). First, we present the following theorem.

Theorem 7.1. *[18] Consider the flip bifurcation case for an n-dimensional map*

$$\tilde{z} = Rz + E(z), \quad z \in \mathbb{R}^n, \tag{7.1}$$

where $E(z) = \mathcal{O}(\|z\|^2)$ is a smooth function. In addition, its Taylor expansion is

$$E(z) = \frac{1}{2}C(z,z) + \frac{1}{6}D(z,z,z) + \mathcal{O}(\|z\|^4). \tag{7.2}$$

The components of the bilinear and trilinear forms C and D are defined by

$$C_i(x,y) = \sum_{k,j=1}^{n} \frac{\partial^2 Z_i(\varrho)}{\partial \varrho_k \partial \varrho_j}\bigg|_{\varrho=0} x_k y_j, \tag{7.3}$$

$$D_i(x,y,z) = \sum_{l,k,j=1}^{n} \frac{\partial^3 Z_i(\varrho)}{\partial \varrho_l \partial \varrho_k \partial \varrho_j}\Big|_{\varrho=0} x_l y_k z_j. \tag{7.4}$$

Assume that the Jacobian matrix R has an eigenvalue $\lambda = -1$, and the corresponding critical eigen-space T^c is one-dimensional, spanned by an eigenvector $u^* \in \mathbb{R}^n$ such that

$$Ru^* = \lambda u^*. \tag{7.5}$$

Let $v^* \in \mathbb{R}^n$ be the adjoint eigenvector, i.e.,

$$R^T v^* = \lambda v^*. \tag{7.6}$$

We impose the normalization condition on v^* and u^* such that

$$\langle v^*, u^* \rangle = 1.$$

Define T^s to be the $(n-1)$-dimensional eigen-space of the matrix R associated with all eigenvalues distinct from λ. Since R and $R - \lambda I_n$ possess the same invariant subspaces, it follows that a vector $z \in \mathbb{R}^n$ lies in T^s if and only if

$$\langle v^*, z \rangle = 0.$$

To determine the direction in which the period-doubling bifurcation unfolds, we evaluate the critical normal form coefficient $p(0)$, given by the invariant expression

$$p(0) = \frac{1}{6}\langle v^*, D(u^*,u^*,u^*)\rangle - \frac{1}{2}\left\langle v^*, C\left(u^*, (R-I_n)^{-1}C(u^*,u^*)\right)\right\rangle. \tag{7.7}$$

If $p(0) > 0$, a unique, stable period-two orbit branches from the equilibrium state at the bifurcation point

We initiate the analysis by relocating the fixed point to the origin through the following change of variables

$$x_n = r_n - \bar{z}, \quad y_n = s_n - \bar{z}.$$

After this change of variables, System (2.5) then becomes:

$$x_{n+1} = z_n, \tag{7.8}$$

$$y_{n+1} = \frac{w + (y_n + \bar{z}) + e(x_n + \bar{z})}{k + (y_n + \bar{z}) + d(x_n + \bar{z})} - \bar{z}. \tag{7.9}$$

Alternatively, we can represent the system equivalently as

$$Z_{n+1} = RZ_n + E(Z_n), \tag{7.10}$$

where

$$R = \begin{bmatrix} 0 & 1 \\ \frac{e(k+\bar{z}) - d(w+\bar{z})}{(k+\bar{z}+d\bar{z})^2} & \frac{(k-w)+\bar{z}(d-e)}{(k+\bar{z}+d\bar{z})^2} \end{bmatrix}.$$

Using Theorem 7.1, we obtain

$$C(z,z) = \begin{bmatrix} 0 \\ C_2(z,z) \end{bmatrix}, \tag{7.11}$$

where

$$C_2(\phi,\psi) = \frac{2d(d\bar{z}-e)}{(k+\bar{z}+d\bar{z})^2}\phi_1\psi_1 + \frac{(-e-d+2d\bar{z})}{(k+\bar{z}+d\bar{z})^2}(\phi_1\psi_2 + \phi_2\psi_1) + \frac{-2(1-\bar{z})}{(k+\bar{z}+d\bar{z})^2}\phi_2\psi_2.$$

Moreover,

$$D(z,z,z)=\begin{bmatrix}0\\ D_2(z,z,z)\end{bmatrix}, \tag{7.12}$$

where

$$\begin{aligned}D_2(\phi,\psi,\eta)=&\frac{6d^2(e-d\bar{z})}{(k+\bar{z}+d\bar{z})^3}\,\phi_1\psi_1\eta_1\\&+\frac{2d(2e+d-3d\bar{z})}{(k+\bar{z}+d\bar{z})^3}\left[\phi_1\psi_1\eta_2+\phi_1\psi_2\eta_1+\phi_2\psi_1\eta_1\right]\\&+\frac{2(e+2d-3d\bar{z})}{(k+\bar{z}+d\bar{z})^3}\left[\phi_1\psi_2\eta_2+\phi_2\psi_1\eta_2+\phi_2\psi_2\eta_1\right]\\&+\frac{6(1-\bar{z})}{(k+\bar{z}+d\bar{z})^3}\,\phi_2\psi_2\eta_2.\end{aligned}$$

We proceed to compute the eigenvectors of the matrices R and R^T associated with the eigenvalue $\lambda=-1$ at the bifurcation point (6.8), with the additional condition $2d\bar{z}=-2d-1-k+e$. Assume v^* and u^* denote the eigenvectors of R^T and R, respectively, associated with $\lambda=-1$. These satisfy the following relations

$$Ru^*=-u^*,\quad R^Tv^*=-v^*. \tag{7.13}$$

$$u^*=\begin{bmatrix}1\\-1\end{bmatrix}, \tag{7.14}$$

and

$$v^*=\begin{bmatrix}-\frac{e(k+\bar{z})-d(w+\bar{z})}{(k+\bar{z}+d\bar{z})^2}\\1\end{bmatrix}. \tag{7.15}$$

Standardize u^* and v^*

$$\langle v^*,u^*\rangle=\sum_{i=1}^{2}u_i^*v_i^*=-\frac{e(k+\bar{z})-d(w+\bar{z})}{(k+\bar{z}+d\bar{z})^2}-1. \tag{7.16}$$

Take

$$v=\xi\cdot\begin{pmatrix}-\frac{e(k+\bar{z})-d(w+\bar{z})}{(k+\bar{z}+d\bar{z})^2}\\1\end{pmatrix}, \tag{7.17}$$

such that

$$\xi=\frac{1}{-1-\frac{e(k+\bar{z})-d(w+\bar{z})}{(k+\bar{z}+d\bar{z})^2}}=\frac{(k+\bar{z}+d\bar{z})^2}{-(k+\bar{z}+d\bar{z})^2-(e(k+\bar{z})-d(w+\bar{z}))}.$$

Let the critical eigen-space associated with the eigenvalue $\lambda=-1$ be denoted by T^c, and suppose it is one-dimensional, spanned by the eigenvector u^*. We define T^s as the invariant subspace associated with all other eigenvalues of the matrix R. Since $R-\lambda I_n$ preserves the same invariant subspaces as R, it follows that a vector $z\in\mathbb{R}^n$ lies in the stable subspace T^s if and only if the inner product $\langle v^*,z\rangle=0$.
To evaluate the quantity $p(0)$, we use the expression

$$p(0)=\frac{1}{6}\left\langle v,D(u^*,u^*,u^*)\right\rangle-\frac{1}{2}\left\langle n,\,C\left(u^*,\,(R-I)^{-1}C(u^*,u^*)\right)\right\rangle, \tag{7.18}$$

where $D(\cdot,\cdot,\cdot)$ and $C(\cdot,\cdot)$ are multi-linear forms, and $(R-I)^{-1}$ denotes the inverse of the shifted matrix. We evaluate

$$C(u^*,u^*)=\begin{bmatrix}0\\ \dfrac{2(d-1)\left[(d-1)\bar{z}-e+1\right]}{(k+\bar{z}+d\bar{z})^2}\end{bmatrix}, \tag{7.19}$$

$$D(u^*,u^*,u^*)=\begin{bmatrix}0\\ D_2(u^*,u^*,u^*)\end{bmatrix}, \tag{7.20}$$

where

$$D_2(u^*,u^*,u^*)=\frac{6(d-1)^2\,(e-(d-1)\bar{z}-1)}{(k+\bar{z}+d\bar{z})^3}$$

$$\langle v,D(u^*,u^*,u^*)\rangle=\frac{6(d-1)^2\big(e-(d-1)\bar{z}-1\big)}{(k+\bar{z}+d\bar{z})\left[-(k+\bar{z}+d\bar{z})^2-\big(e(k+\bar{z})-d(w+\bar{z})\big)\right]}. \tag{7.21}$$

The matrix $R-I$ is:

$$R-I=\begin{bmatrix}-1 & 1\\ \frac{e(k+\bar{z})-d(w+\bar{z})}{(k+\bar{z}+d\bar{z})^2} & \frac{(k-w)+\bar{z}(d-e)}{(k+\bar{z}+d\bar{z})^2}-1\end{bmatrix}. \tag{7.22}$$

$$(R-I)^{-1}=\frac{1}{1-\frac{k-w+ek-dw}{(k+\bar{z}+d\bar{z})^2}}\begin{bmatrix}\frac{(k-w)+\bar{z}(d-e)}{(k+\bar{z}+d\bar{z})^2}-1 & -1\\ -\frac{e(k+\bar{z})-d(w+\bar{z})}{(k+\bar{z}+d\bar{z})^2} & -1\end{bmatrix}. \tag{7.23}$$

$$(R-I)^{-1}C(u^*,u^*)=-\frac{2(d-1)\big[(d-1)\bar{z}-e+1\big]}{(k+\bar{z}+d\bar{z})^2-\big[k(1+e)-w(1+d)\big]}\begin{bmatrix}1\\1\end{bmatrix}. \tag{7.24}$$

$$C(u^*,(R-I)^{-1}C(u^*,u^*))=\begin{pmatrix}0\\S\end{pmatrix}, \tag{7.25}$$

where

$$S=-\frac{4(d-1)\big[(d-1)\bar{z}-e+1\big][d(d\bar{z}-e)+(1-\bar{z})]}{(k+\bar{z}+d\bar{z})^2\Big((k+\bar{z}+d\bar{z})^2-\big[k(1+e)-w(1+d)\big]\Big)}.$$

Finally, we want to compute

$$\langle v,C(u^*,(R-I)^{-1}C(u^*,u^*))\rangle=\frac{4(d-1)\big[(d-1)\bar{z}-e+1\big]\big[d(d\bar{z}-e)+(1-\bar{z})\big]}{\big(L+e(k+\bar{z})-d(w+\bar{z})\big)\big(L-[k(1+e)-w(1+d)]\big)}, \tag{7.26}$$

where $L=(k+\bar{z}+d\bar{z})^2$.
If $p(0)$ is greater than zero, a unique and stable period-two orbit originates from the fixed point at the bifurcation point, with w as given in equation (6.8).

8 Numerical Results

In this section, we provide a numerical examples that validates our conclusions from the previous sections. The graphs obtained using MATLAB are presented along with discussions to explain the bifurcation.

Example 8.1. Consider equation (1.4). Fix the parameters L,k,d and assume w as the bifurcation parameter. Take $e=5.0058$, $k=0.9942$, $d=0.4070$, and $0<w\leq 2$. Then, equation (1.4) becomes

$$z_{n+1}=\frac{w+z_n+5.0058z_{n-1}}{0.9942+z_n+0.4070z_{n-1}}. \tag{8.1}$$

The planar form equivalent to equation (8.1) is:

$$\begin{pmatrix}z_1(n+1)\\ z_2(n+1)\end{pmatrix}=\begin{pmatrix}z_2(n)\\ \frac{w+z_n+5.0058z_{n-1}}{0.9942+z_n+0.4070z_{n-1}}\end{pmatrix}. \tag{8.2}$$

The positive fixed point of the system $(\bar{z},\bar{z})$ satisfies

$$\bar{z}=\frac{5.0116+\sqrt{25.116+5.628w}}{2.8140}. \tag{8.3}$$

Theorem (6.1) defines the bifurcation point at (6.8). Thus, the equilibrium point exhibits a flip bifurcation at $w = 0.718$. See Figure 1.

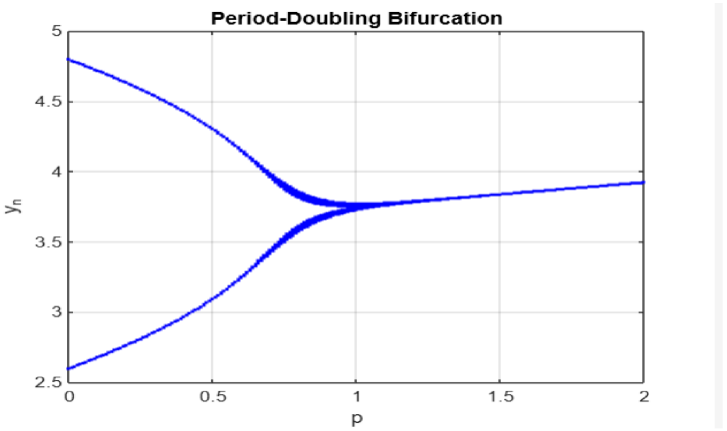


Figure 1. Period-doubling bifurcation of the map $z_{n+1} = \frac{w+z_n+5.0058z_{n-1}}{0.9942+z_n+0.4070z_{n-1}}$, where w is a parameter.

Example 8.2. Consider equation (1.4) with fixed parameters $d = 0.35$, $k = 0.91$, and $e = 4.2$, while treating w as the bifurcation parameter. Assume $0 < w \leq 2$. Then equation (1.4) becomes

$$z_{n+1} = \frac{w + z_n + 4.2z_{n-1}}{0.91 + z_n + 0.35z_{n-1}}. \tag{8.4}$$

The corresponding planar system is

$$\begin{pmatrix} z_1(n+1) \\ z_2(n+1) \end{pmatrix} = \begin{pmatrix} z_2(n) \\ \frac{w+z_2(n)+4.2z_1(n)}{0.91+z_2(n)+0.35z_1(n)} \end{pmatrix}. \tag{8.5}$$

The positive fixed point of the system $(\bar{z}, \bar{z})$ satisfies

$$\bar{z} = \frac{4.29 + \sqrt{18.4041 + 5.4w}}{2.7} \tag{8.6}$$

Theorem (6.1) defines the bifurcation point at (6.8). Thus, the equilibrium point exhibits a flip bifurcation at $w = 0.414$. See Figure 2.

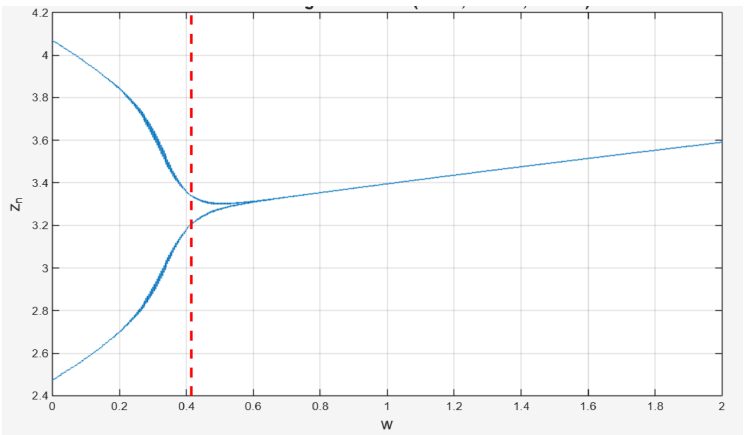


Figure 2. Period-doubling bifurcation of the map $z_{n+1} = \frac{w+z_n+4.2z_{n-1}}{0.91+z_n+0.35z_{n-1}}$, where w is the bifurcation parameter.

Example 8.3. Consider equation (1.4) where the parameters are fixed as $e = 4$, $k = 1$, and $d = 0.5$, and w is treated as the bifurcation parameter.Then equation (1.4) becomes

$$z_{n+1} = \frac{w + z_n + 4z_{n-1}}{1 + z_n + 0.5z_{n-1}}. \tag{8.7}$$

The corresponding planar system is

$$\begin{pmatrix} z_1(n+1) \\ z_2(n+1) \end{pmatrix} = \begin{pmatrix} z_2(n) \\ \frac{w+z_2(n)+4z_1(n)}{1+z_2(n)+0.5z_1(n)} \end{pmatrix}. \tag{8.8}$$

The positive fixed point of the system $(\bar{z}, \bar{z})$ satisfies

$$\bar{z} = \frac{4 + \sqrt{16 + 6w}}{3} \tag{8.9}$$

Returning to Section (4), we obtain $\bar{z} = \frac{2}{3}$. Moreover, from equation (4.4) we obtain $w = -2.67$ which cannot hold since $w > 0$. Therefore, no saddle-node bifurcation occurs.
Returning to Section (5), we obtain $\bar{z} = -5$. However, our analysis is restricted to the positive fixed point only. Moreover, from equation (5.3) we obtain

$$\sqrt{16 + 6w} = -19, \tag{8.10}$$

which cannot hold for $w > 0$. Therefore, no Neimark–Sacker bifurcation occurs.
Theorem (6.1) determines the bifurcation point at (6.8). According to this result, the equilibrium point undergoes a flip bifurcation at $w = -2$, which contradicts our assumption that $w > 0$. See Figure 3 for the corresponding stable behavior.

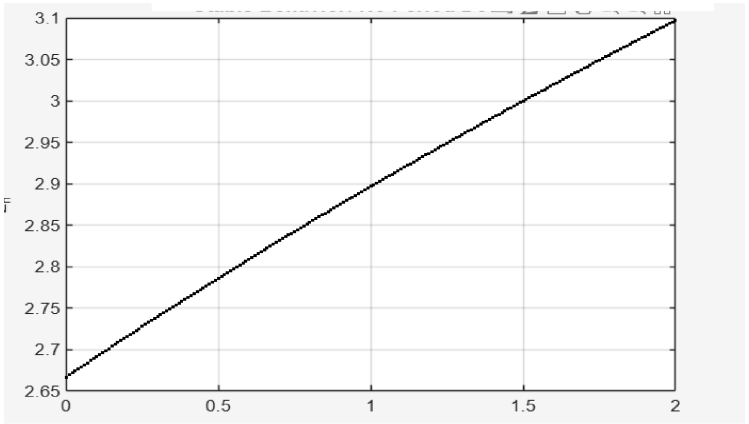


Figure 3. Stable behavior of the map $z_{n+1} = \frac{w+z_n+4z_{n-1}}{1+z_n+0.5z_{n-1}}$.

9 Conclusion

In this paper, we investigated the bifurcation behavior of a second-order rational difference equation (1.3). We derived explicit analytical conditions for the occurrence of saddle-node, Neimark–Sacker, and period-doubling bifurcations. Additionally, we determined the direction of the period-doubling bifurcation through normal form analysis. To validate and illustrate our theoretical findings, we conducted numerical bifurcation analyses using a MATLAB package based on continuation methods. Future work may extend this study by examining other types of bifurcations and their dynamical implications within this framework.

Conflict of Interest: The authors declare that they have no conflict of interest.

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