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CERTAIN SUBCLASSES OF BAVILEVIČ AND NON-BAVILEVIČ FUNCTIONS DEFINED BY DIFFERENTIAL SUBORDINATION

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Abstract. By making use of the subordination principle, we study some subordination and superordination properties, distortion theorems, and inequality properties of certain subclasses of Bavilevič and non-Bavilevič functions. The results presented here would provide extensions of those given in earlier works.

1 Introduction

Let $\mathcal{A}(\mathbb{U})$ be the class of all analytic functions in $\mathbb{U} = \{\omega \in \mathbb{C} : |\omega| < 1\}$ and let $\mathcal{A}[a, j]$ be a subclass of $\mathcal{A}(\mathbb{U})$ consisting of functions of the form:

$$\psi(\omega) = a + a_j\omega^j + a_{j+1}\omega^{j+1} + ... \ (\omega \in \mathbb{U}; j \in \mathbb{N} = \{1, 2, 3, ...\}).$$

Also, let $\ominus(j)$ denote the subclass of $\mathcal{A}(\mathbb{U})$ consisting of functions of the form:

$$\psi(\omega) = \omega + \sum_{k=j+1}^{\infty} a_k\omega^k \quad (j \in \mathbb{N}). \tag{1.1}$$

We write $\ominus(1) = \ominus$. If $\psi, \varphi \in \mathcal{A}(\mathbb{U})$, we say that $\psi(\omega)$ is subordinate to $\varphi(\omega)$, or $\varphi(\omega)$ is superordinate to $\psi(\omega)$, written symbolically, $\psi \prec \varphi$ in $\mathbb{U}$ or $\psi(\omega) \prec \varphi(\omega)$ $(\omega \in \mathbb{U})$, if there exists a Schwarz function $\nu(\omega)$, which (by definition) is analytic in $\mathbb{U}$ with $\nu(0) = 0$ and $|\nu(\omega)| < 1 (\omega \in \mathbb{U})$ such that

$$\psi(\omega) = \varphi(\nu(\omega)) \ (\omega \in \mathbb{U}).$$

Furthermore, if $\varphi(\omega)$ is a univalent function in $\mathbb{U}$, then we have (see [1] and [2]):

$$\psi(\omega) \prec \varphi(\omega) \ \ (\omega \in \mathbb{U}) \Longleftrightarrow \psi(0) = \varphi(0) \text{ and } \psi(\mathbb{U}) \subset \varphi(\mathbb{U}).$$

Let $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and $h(\omega)$ be univalent functions in $\mathbb{U}$. If $p(\omega)$ is an analytic function in $\mathbb{U}$ and satisfies the following subordination:

$$\phi(p(\omega), \omega p'(\omega); \omega) \prec h(\omega), \tag{1.2}$$

then $p(\omega)$ is a solution of the subordination (1.2). The univalent function $q(\omega)$ is called a dominant of the solutions of the subordination (1.2) if $p(\omega) \prec q(\omega)$ for all $p(\omega)$ satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.2) is called the best dominant. Also, if $p(\omega)$ and $\phi\left(p(\omega), \omega p'(\omega); \omega\right)$ are univalent functions in $\mathbb{U}$ such that $p(\omega)$ satisfies the differential superordination:

$$h(\omega) \prec \phi(p(\omega), \omega p'(\omega); \omega), \tag{1.3}$$

then $p(\omega)$ is a solution of the superordination (1.3). An analytic function $q(\omega)$ is called a subordinant of the solutions of the superordination (1.3) if $q(\omega) \prec p(\omega)$ for all $p(\omega)$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.3) is called the best subordinant.

Definition 1.1. A function $\psi \in \ominus(j)$ is said to be the class $\mathcal{BN}^j(\lambda, \alpha, \beta, \mu; L, M)$ if it satisfies the following condition:

$$(1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}+\lambda\left\{1+\frac{\left(\alpha^2-\beta^2\right)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} \prec \frac{1+L\omega}{1+M\omega} \tag{1.4}$$

where all the powers are principal values and throughout the paper unless otherwise mentioned the parameters j, λ, α, β, μ, L, M are constrained as follows

$$j \in \mathbb{N}; \lambda, \alpha, \beta \in \mathbb{C}; 0 < \mu < 1; -1 \leq M < L \leq 1 \text{ and } \omega \in \mathbb{U}.$$

Furthermore, the function $\psi \in \mathcal{BN}^j(\lambda, \alpha, \beta, \mu; \gamma)$ if and only if $\psi \in \ominus(j)$ and

$$\Re\left((1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}\right.$$

$$\left.+\lambda\left\{1+\frac{\left(\alpha^2-\beta^2\right)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}\right) > \gamma\,(0 \leq \gamma < 1),$$

we write $\mathcal{BN}^j(0, \alpha, \beta, \mu; \gamma) = \mathcal{BN}^j(\alpha, \beta, \mu; \gamma)$.

We note that

(i) $\mathcal{BN}^j(\lambda, \alpha, 0, \mu; L, M) = \mathcal{B}^j(\lambda, \mu; L, M)$ (see [3, with $p = 1$], [4] and [5])

$$= \left\{\psi \in \ominus(j) : (1-\lambda)\left[\frac{\psi(\omega)}{\omega}\right]^{\mu}+\lambda\frac{\omega\psi'(\omega)}{\psi(\omega)}\left[\frac{\psi(\omega)}{\omega}\right]^{\mu} \prec \frac{1+L\omega}{1+M\omega}\right\};$$

(ii) $\mathcal{BN}^j(1, \alpha, 0, \mu; L, M) = \mathcal{B}^j(\mu; L, M)$ (see [6, with $p = 1$])

$$= \left\{\psi \in \ominus(j) : \frac{\omega\psi'(\omega)}{\psi(\omega)}\left[\frac{\psi(\omega)}{\omega}\right]^{\mu} \prec \frac{1+L\omega}{1+M\omega}\right\};$$

(iii) $\mathcal{BN}^1(1, \alpha, 0, \mu; L, M) = \mathcal{B}(\mu; L, M)$ (see [7] and [8])

$$= \left\{\psi \in \ominus : \frac{\omega\psi'(\omega)}{\psi(\omega)}\left[\frac{\psi(\omega)}{\omega}\right]^{\mu} \prec \frac{1+L\omega}{1+M\omega}\right\};$$

(iv) $\mathcal{BN}^j(\lambda, \alpha, 0, \mu; L, 0) = \mathcal{B}^j(\lambda, \mu; L)$ (see [9])

$$= \left\{\psi \in \ominus(j) : (1-\lambda)\left[\frac{\psi(\omega)}{\omega}\right]^{\mu}+\lambda\frac{\omega\psi'(\omega)}{\psi(\omega)}\left[\frac{\psi(\omega)}{\omega}\right]^{\mu} \prec 1+L\omega\right\};$$

(v) $\mathcal{BN}^j(1, \alpha, 0, \mu; 1-2\gamma, -1) = \mathcal{B}^j(\mu; \gamma)\,(0 \leq \gamma < 1)$, where $\mathcal{B}^j(\mu; \gamma)$ is the class of Bazilevič functions of order γ (see [10])

$$= \left\{\psi \in \ominus(j) : \Re\left(\frac{\omega\psi'(\omega)}{\psi(\omega)}\left[\frac{\psi(\omega)}{\omega}\right]^{\mu}\right) > \gamma\right\};$$

(vi) $\mathcal{BN}^j(\lambda, 0, \beta, \mu; L, M) = \mathcal{N}^j(\lambda, \mu; L, M)$ (see [11, with $p = 1$], [12] and [13])

$$= \left\{\psi \in \ominus(j) : (1+\lambda)\left[\frac{\omega}{\psi(\omega)}\right]^{\mu}-\lambda\frac{\omega\psi'(\omega)}{\psi(\omega)}\left[\frac{\omega}{\psi(\omega)}\right]^{\mu} \prec \frac{1+L\omega}{1+M\omega}\right\};$$

(vii) $\mathcal{BN}^1(\lambda,0,\beta,\mu;L,M)=\mathcal{N}(\lambda,\mu;L,M)$ (see [14] and [15])

$$=\left\{\psi\in\ominus:(1+\lambda)\left[\frac{\omega}{\psi(\omega)}\right]^{\mu}-\lambda\frac{\omega\psi'(\omega)}{\psi(\omega)}\left[\frac{\omega}{\psi(\omega)}\right]^{\mu}\prec\frac{1+L\omega}{1+M\omega}\right\};$$

(viii) $\mathcal{BN}^1(-1,0,\beta,\mu;1-2\gamma,-1)=\mathcal{N}(\mu;\gamma)$ $(0\leq\gamma<1)$, where $\mathcal{N}(\mu;\gamma)$ is the class of non-Bazilevič functions of order γ (see [16] and [17])

$$=\left\{\psi\in\ominus:\Re\left(\frac{\omega\psi'(\omega)}{\psi(\omega)}\left[\frac{\omega}{\psi(\omega)}\right]^{\mu}\right)>\gamma\right\};$$

(ix) $\mathcal{BN}^1(-1,0,\beta,\mu;1,-1)=\mathcal{N}(\mu)$, where $\mathcal{N}(\mu)$ is the class of non-Bazilevič functions (see [18])

$$=\left\{\psi\in\ominus:\Re\left(\frac{\omega\psi'(\omega)}{\psi(\omega)}\left[\frac{\omega}{\psi(\omega)}\right]^{\mu}\right)>0\right\}.$$

(x) $\mathcal{BN}^1(1,\alpha,0,0;L,M)=\mathcal{S}(L,M)$ (see [19] and [20])

$$=\left\{\psi\in\ominus:\frac{\omega\psi'(\omega)}{\psi(\omega)}\prec\frac{1+L\omega}{1+M\omega}\right\};$$

In order to establish our main results, we need the following definition and lemmas.

Definition 1.2. [21] Denote by Q the set of all analytic functions ψ that are injective in $\bar{\mathbb{U}}\backslash E(\psi)$, where

$$E(\psi)=\left\{\zeta\in\partial\mathbb{U}:\lim_{\omega\to\zeta}\psi(\omega)=\infty\right\},$$

and such that $\psi'(\zeta)\neq0$ for $\zeta\in\bar{\mathbb{U}}\backslash E(\psi)$.

Lemma 1.3. *([22] and [2]) Let the function h be analytic and convex (univalent) in $\mathbb{U}$ with $h(0)=1$. Suppose also that the function $p(\omega)$ given by*

$$\varphi(\omega)=1+c_j\omega^j+c_{j+1}\omega^{j+1}+\ldots \tag{1.5}$$

is analytic in $\mathbb{U}$. If

$$p(\omega)+\frac{\omega p'(\omega)}{\delta}\prec h(\omega)\quad(\Re(\delta)\geq0;\delta\neq0;\ \omega\in\mathbb{U}), \tag{1.6}$$

then

$$p(\omega)\prec q(\omega)=\frac{\delta}{j}\omega^{-\frac{\delta}{j}}\int_0^{\omega}h(t)\,t^{\frac{\delta}{j}-1}\,dt\prec h(\omega),$$

and $q(\omega)$ is the best dominant.

Lemma 1.4. *[23] For real or complex numbers a,b,c with $c\neq0,-1,-2,\ldots$, the next relations hold:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t\omega)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2\Omega_1(a,b;c;\omega) \tag{1.7}$$

$$(\omega\in\mathbb{C}\setminus(1,+\infty);\Re(c)>\Re(b)>0),$$

$${}_2\Omega_1(a,b;c;\omega)=(1-\omega)^{-a}\,{}_2\Omega_1\left(a,c-b;c;\frac{\omega}{\omega-1}\right)\quad(\omega\in\mathbb{C}\setminus(1,+\infty)). \tag{1.8}$$

Lemma 1.5. *[24] Let q be a convex univalent function in $\mathbb{U}$ and let $\sigma\in\mathbb{C}$, $\eta\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$ with*

$$\Re\left(1+\frac{\omega q''(\omega)}{q'(\omega)}\right)>\max\left\{0,-\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function p is analytic in $\mathbb{U}$ and

$$\sigma p(\omega)+\eta\omega p'(\omega)\prec\sigma q(\omega)+\eta\omega q'(\omega),$$

then $p(\omega)\prec q(\omega)$ and q is the best dominant.

Lemma 1.6. *[21] Let q be convex univalent in $\mathbb{U}$ and $\kappa \in \mathbb{C}$. Further, assume that $\Re(\kappa) > 0$. If*

$$p(\omega) \in \mathcal{A}[q(0), 1] \cap Q,$$

and $p(\omega) + \kappa\omega p'(\omega)$ is univalent in $\mathbb{U}$, then

$$q(\omega) + \kappa\omega q'(\omega) \prec p(\omega) + \kappa\omega p'(\omega),$$

implies $q(\omega) \prec p(\omega)$ and q is the best subordinant.

Lemma 1.7. *[25] Let F be analytic and convex in $\mathbb{U}$. If $\psi, \varphi \in \ominus$ and $\psi, \varphi \prec \mathrm{F}$, then $\gamma\psi + (1-\gamma)\varphi \prec \mathrm{F}$ $(0 \leq \gamma \leq 1)$.*

Lemma 1.8. *[26] Let $\psi(\omega) = \sum_{k=1}^{\infty} a_k \omega^k$, $\omega \in \mathbb{U}$, be analytic in $\mathbb{U}$ and $\varphi(\omega) = \sum_{k=1}^{\infty} b_k \omega^k$, $\omega \in \mathbb{U}$, be convex (univalent) in $\mathbb{U}$. If $\psi(\omega) \prec \varphi(\omega)$, then $|a_k| \leq |b_1|$, $k \in \mathbb{N}$.*

In the present paper, we obtain some properties of subordination and superordination, convolution properties, distortion theorems, and inequality of the class $\mathcal{BN}^j(\lambda, \alpha, \beta, \mu; L, M)$.

2 Main Results

Theorem 2.1. *Let $\psi \in \mathcal{BN}^j(\lambda, \alpha, \beta, \mu; L, M)$ with $\Re(\lambda) > 0$. Then*

$$\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)}\right]^{\mu} \prec Q(\omega) \prec \frac{1 + L\omega}{1 + M\omega}, \tag{2.1}$$

where the function $Q(\omega)$ given by

$$Q(\omega) = \begin{cases} \frac{L}{M} - \frac{(L-M)}{M(1+M\omega)}\, {}_2\Omega_1\left(1, 1; \frac{\mu}{\lambda j} + 1; \frac{M\omega}{1+M\omega}\right) & (M \neq 0) \\ 1 + \frac{\mu}{\mu + \lambda j} L\omega & (M = 0) \end{cases} \tag{2.2}$$

is the best dominant of (2.1). Furthermore,

$$\Re\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)}\right]^{\mu} > \rho \quad (\omega \in \mathbb{U}), \tag{2.3}$$

where

$$\rho = \begin{cases} \frac{L}{M} - \frac{(L-M)}{M(1-M)}\, {}_2\Omega_1\left(1, 1; \frac{\mu}{\lambda j} + 1; \frac{M}{M-1}\right) & (M \neq 0) \\ 1 - \frac{\mu}{\mu + \lambda j} L & (M = 0) \end{cases} \tag{2.4}$$

The estimate (2.3) is the best possible.

Proof. Let $\psi \in \mathcal{BN}^j(\lambda, \alpha, \beta, \mu; L, M)$ and suppose that

$$\varphi(\omega) = \left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)}\right]^{\mu} \quad (\omega \in \mathbb{U}). \tag{2.5}$$

Then the function $\varphi(\omega)$ is of the form (1.5), analytic in $\mathbb{U}$ and $\varphi(0) = 1$. By taking the derivatives on both sides of (2.5), we get

$$(1-\lambda)\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)}\right]^{\mu} + \lambda\left\{1 + \frac{(\alpha^2 - \beta^2)\left[\omega^2\psi'(\omega) - \omega\psi(\omega)\right]}{\left[\alpha\psi(\omega) + \beta\omega\right]\left[\alpha\omega + \beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)}\right]^{\mu}$$

$$= \varphi(\omega) + \frac{\lambda}{\mu}\omega\varphi'(\omega). \tag{2.6}$$

Since $\psi \in \mathcal{BN}^j(\lambda, \alpha, \beta, \mu; L, M)$, we have

$$\varphi(\omega) + \frac{\lambda}{\mu}\omega\varphi'(\omega) \prec \frac{1 + L\omega}{1 + M\omega}.$$

Now, by using Lemma 1.3 for $\delta = \frac{\mu}{\lambda}$, we deduce that

$$\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} \prec Q(\omega) = \frac{\mu}{\lambda j}\,\omega^{-\frac{\mu}{\lambda j}}\int_0^{\omega} t^{\frac{\mu}{\lambda j}-1}\left(\frac{1+Lt}{1+Mt}\right)dt$$

$$= \frac{\mu}{j\lambda}\int_0^1 \frac{1+L\omega u}{1+M\omega u}u^{\frac{\mu}{j\lambda}-1}du \tag{2.7}$$

$$= \begin{cases} \frac{L}{M} - \frac{(L-M)}{M(1+M\omega)}\,{}_2\Omega_1\left(1,1;\frac{\mu}{\lambda j}+1;\frac{M\omega}{1+M\omega}\right) & (M\neq 0) \\ 1+\frac{\mu}{\mu+\lambda j}L\omega & (M=0) \end{cases} \tag{2.8}$$

where we have used the identities in Lemma 1.4 with $a=1$, $b=\frac{\mu}{\lambda j}$ and $c=b+1$. This proves the assertion (2.2).

Now, in order to prove the assertion (2.3) of Theorem 2.1, it suffices to show that

$$\inf_{|\omega|<1}\{\Re(Q(\omega))\} = Q(-1). \tag{2.9}$$

Indeed, for $|\omega|\leq r<1$,

$$\Re\left(\frac{1+L\omega}{1+M\omega}\right) \geq \frac{1-Lr}{1-Mr}.$$

Setting

$$G(\omega,s) = \frac{1+Ls\omega}{1+Ms\omega}$$

and

$$dv(s) = \frac{\mu}{\lambda j}\,s^{\frac{\mu}{\lambda j}-1}\,ds \quad (0\leq s\leq 1),$$

which is a positive measure in the interval $[0,1]$, we have

$$Q(\omega) = \int_0^1 G(\omega,s)\;dv(s),$$

so that

$$\Re\{Q(\omega)\} \geq \int_0^1 \frac{1-Lsr}{1-Msr}\,dv(s) = Q(-r) \quad (|\omega|\leq r<1).$$

Taking $r\to 1^-$ in the above inequality, we obtain the assertion (2.3). Finally, the estimate (2.3) is the best possible because $Q(\omega)$ is the best dominant of (2.1). The proof of Theorem 2.1 is completed. □

Corollary 2.2. *Let $\psi\in\mathcal{BN}^j(\lambda,\alpha,\beta,\mu;L^*,M)$ with $\Re(\lambda)>0$, $-1\leq M<L\leq 1$ and $M\neq 0$, then*

$$\Re\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} > 0 \quad (\omega\in\mathbb{U}),$$

where L^* is given by

$$L^* = \frac{M\,{}_2\Omega_1\left(1,1;\frac{\mu}{\lambda j}+1;\frac{M\omega}{1+M\omega}\right)}{{}_2\Omega_1\left(1,1;\frac{\mu}{\lambda j}+1;\frac{M\omega}{1+M\omega}\right)+M-1} \tag{2.10}$$

The result is sharp.

Proof. In view of Theorem 2.1, if

$$(1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$

$$\prec \frac{1+L^*\omega}{1+M\omega},$$

then

$$\Re\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}>\left[\frac{L^*}{M}-\frac{(L^*-M)}{M(1-M)}\,{}_2\Omega_1\left(1,1;\frac{\mu}{\lambda j}+1;\frac{M}{M-1}\right)\right]^{\frac{1}{\alpha}}. \tag{2.11}$$

On substituting the value of L^* given by (2.10) in the right hand side of the inequality (2.11), we have

$$\Re\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}>0 \quad (\omega\in\mathbb{U}),$$

which proves Corollary 2.2. □

Putting $L=1-2\gamma\,(0\le\gamma<1)$ and $M=-1$ in Theorem 2.1, we have

Corollary 2.3. *Let $\psi\in\mathcal{BN}^j(\lambda,\alpha,\beta,\mu;\gamma)$ with $\Re(\lambda)>0$, then*

$$\Re\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}>\gamma+(1-\gamma)\left[{}_2\Omega_1\left(1,1;\frac{\mu}{\lambda j}+1;\frac{1}{2}\right)-1\right],$$

The result is sharp.

Putting $\beta=0$ in Corollary 2.3, we have

Corollary 2.4. *Let $\psi\in\mathcal{B}^j(\lambda,\mu;\gamma)$ with $\Re(\lambda)>0$, then*

$$\Re\left[\frac{\psi(\omega)}{\omega}\right]^{\mu}>\gamma+(1-\gamma)\left[{}_2\Omega_1\left(1,1;\frac{\mu}{\lambda j}+1;\frac{1}{2}\right)-1\right],$$

The result is sharp.

Putting $\alpha=0$ in Corollary 2.3, we have

Corollary 2.5. *Let $\psi\in\mathcal{N}^j(\lambda,\mu;\gamma)$ with $\Re(\lambda)>0$, then*

$$\Re\left[\frac{\omega}{\psi(\omega)}\right]^{\mu}>\gamma+(1-\gamma)\left[{}_2\Omega_1\left(1,1;\frac{\mu}{\lambda j}+1;\frac{1}{2}\right)-1\right],$$

The result is sharp.

Theorem 2.6. *Let $q(\omega)$ be univalent in $\mathbb{U}$, $\lambda\in\mathbb{C}^*$. Suppose also that $q(\omega)$ satisfies the following inequality:*

$$\Re\left(1+\frac{\omega q''(\omega)}{q'(\omega)}\right)>\max\left\{0,-\Re\left(\frac{\mu}{\lambda}\right)\right\}. \tag{2.12}$$

If $\psi\in\ominus(j)$ satisfies the following:

$$(1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}+\lambda\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$

$$\prec q(\omega)+\frac{\lambda}{\mu}\omega q'(\omega), \tag{2.13}$$

then

$$\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}\prec q(\omega)$$

and $q(\omega)$ is the best dominant.

Proof. Let the function $\varphi(\omega)$ be defined by (2.5). We know that (2.6) holds true. Combining (2.6) and (2.13), we find that

$$\varphi(\omega)+\frac{\lambda}{\mu}\omega\varphi'(\omega)\prec q(\omega)+\frac{\lambda}{\mu}\omega q'(\omega). \tag{2.14}$$

By using Lemma 1.5 and (2.14), we easily get the assertion of Theorem 2.6. □

Taking $q(\omega) = \frac{1+L\omega}{1+M\omega}$ $(-1 \leq M < L \leq 1)$ in Theorem 2.6, we get the following result.

Corollary 2.7. *Let $\lambda \in \mathbb{C}^*$ and $-1 \leq M < L \leq 1$. Suppose also that*

$$\Re\left(\frac{1-M\omega}{1+M\omega}\right) > \max\left\{0, -\Re\left(\frac{\mu}{\lambda}\right)\right\}.$$

If $\psi \in \ominus(j)$ satisfies the following subordination condition:

$$(1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda\left\{1+\frac{\left(\alpha^2-\beta^2\right)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$

$$\prec \frac{1+L\omega}{1+M\omega} + \frac{\lambda}{\mu}\frac{(L-M)\omega}{(1+M\omega)^2},$$

then

$$\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} \prec \frac{1+L\omega}{1+M\omega},$$

and the function $\dfrac{1+L\omega}{1+M\omega}$ is the best dominant.

Remark 2.8. Putting $p = m = 1$ in Theorem 2.6 and Corollary 2.7, respectively, we obtain the subordination results of Shanmugam et. al. [27, Theorem 3.1 and Corollary 3.2, respectively].

We now derive the following superordination result.

Theorem 2.9. *Let q be convex univalent in $\mathbb{U}$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Also let*

$$\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} \in \mathcal{A}[q(0),1] \cap Q$$

and

$$(1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda\left\{1+\frac{\left(\alpha^2-\beta^2\right)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$

be univalent in $\mathbb{U}$. If $\psi \in \ominus(j)$ satisfies the following superordination condition:

$$q(\omega) + \frac{\lambda}{\mu}\omega q'(\omega)$$

$$\prec (1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda\left\{1+\frac{\left(\alpha^2-\beta^2\right)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu},$$

then

$$q(\omega) \prec \left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu},$$

and the function $q(\omega)$ is the best subordinant.

Proof. Let the function $\varphi(\omega)$ be defined by (2.5). Then

$$q(\omega) + \frac{\lambda}{\mu}\omega q'(\omega)$$

$$\prec (1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda\left\{1+\frac{\left(\alpha^2-\beta^2\right)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$

$$= \varphi(\omega) + \frac{\lambda}{\mu}\omega\varphi'(\omega).$$

An application of Lemma 1.6 yields the assertion of Theorem 2.9. □

Taking $q(\omega) = \frac{1+L\omega}{1+M\omega}$ $(-1 \leq M < L \leq 1)$ in Theorem 2.9, we get the following corollary.

Corollary 2.10. *Let $-1 \leq M < L \leq 1$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Also let*

$$\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} \in \mathcal{A}[1,1] \cap Q,$$

and

$$(1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{[\alpha\psi(\omega)+\beta\omega][\alpha\omega+\beta\psi(\omega)]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$

be univalent in $\mathbb{U}$. If $\psi \in \ominus(j)$ satisfies the following superordination condition:

$$\frac{1+L\omega}{1+M\omega} + \frac{\lambda}{\mu}\frac{(L-M)\omega}{(1+M\omega)^2}$$

$$\prec (1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{[\alpha\psi(\omega)+\beta\omega][\alpha\omega+\beta\psi(\omega)]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu},$$

then

$$\frac{1+L\omega}{1+M\omega} \prec \left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu},$$

and the function $\frac{1+L\omega}{1+M\omega}$ is the best subordinant.

Remark 2.11. Putting $\alpha = 0$ in Theorem 2.9 and Corollary 2.10, respectively, we obtain the superordination results of Shanmugam et. al. [27, Theorem 4.1 and Corollary 4.2, respectively].

Combining the above results of subordination and superordination. we easily get the following "Sandwich-type result".

Theorem 2.12. *Let q_1 be convex univalent and let q_2 be univalent in $\mathbb{U}$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Let q_2 satisfies (2.12). If*

$$\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} \in \mathcal{A}[q_1(0),1] \cap Q,$$

and

$$(1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{[\alpha\psi(\omega)+\beta\omega][\alpha\omega+\beta\psi(\omega)]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$

be univalent in $\mathbb{U}$, also

$$q_1(\omega) + \frac{\lambda}{\mu}\omega q_1'(\omega)$$

$$\prec (1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{[\alpha\psi(\omega)+\beta\omega][\alpha\omega+\beta\psi(\omega)]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$

$$\prec q_2(\omega) + \frac{\lambda}{\mu}\omega q_2'(\omega),$$

then

$$q_1(\omega) \prec \left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} \prec q_2(\omega),$$

and $q_1(\omega)$ and $q_2(\omega)$ are, respectively, the best subordinant and the best dominant.

Remark 2.13. Putting $\alpha = 0$ in Theorem 2.12, we obtain the sandwich results of Shanmugam et. al. [27, Theorem 5.1].

Theorem 2.14. *If $\psi \in \mathcal{BN}^j(\alpha, \beta, \mu; \gamma)$ with $\lambda > 0$ and $0 \leq \gamma < 1$. Then $\psi \in \mathcal{BN}^j(\lambda, \alpha, \beta, \mu; \gamma)$ for $|\omega| < R$, where*

$$R = \left(\sqrt{\left(\frac{j\lambda}{\mu}\right)^2 + 1} - \frac{j\lambda}{\mu} \right)^{\frac{1}{j}}. \tag{2.15}$$

The bound R is the best possible.

Proof. We begin by writing

$$\varphi(\omega) = \frac{1}{1-\gamma}\left(\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)} \right]^{\mu} - \gamma \right) \quad (\omega \in \mathbb{U}). \tag{2.16}$$

Then, the function $\varphi(\omega)$ has the form (1.5), is analytic in $\mathbb{U}$ and $\Re\{\varphi(\omega)\} > 0$. Taking the derivatives on both sides in (2.16), we get

$$\begin{aligned} &\frac{1}{1-\gamma}\left\{ (1-\lambda)\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)} \right]^{\mu} \right. \\ &\left. +\lambda\left\{ 1 + \frac{(\alpha^2 - \beta^2)\left[\omega^2\psi'(\omega) - \omega\psi(\omega)\right]}{\left[\alpha\psi(\omega) + \beta\omega\right]\left[\alpha\omega + \beta\psi(\omega)\right]} \right\}\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)} \right]^{\mu} - \gamma \right\} \\ &= \varphi(\omega) + \frac{\lambda}{\mu}\omega\varphi'(\omega). \end{aligned} \tag{2.17}$$

Making use of the following estimate (see [28, Theorem 1]):

$$\frac{|\omega\varphi'(\omega)|}{\Re\{\varphi(\omega)\}} \leq \frac{2jr^j}{1-r^{2j}} \quad (|\omega| = r < 1)$$

in (2.17), we obtain that

$$\begin{aligned} &\frac{1}{1-\gamma}\left\{ (1-\lambda)\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)} \right]^{\mu} \right. \\ &\left. +\lambda\left\{ 1 + \frac{(\alpha^2 - \beta^2)\left[\omega^2\psi'(\omega) - \omega\psi(\omega)\right]}{\left[\alpha\psi(\omega) + \beta\omega\right]\left[\alpha\omega + \beta\psi(\omega)\right]} \right\}\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)} \right]^{\mu} - \gamma \right\} \\ &\geq \Re\{\varphi(\omega)\}\left(1 - \frac{2\lambda jr^j}{\mu\ (1-r^{2j})} \right). \end{aligned} \tag{2.18}$$

It is seen that the right-hand side of (2.18) is positive, provided that $r < R$, where R is given by (2.15). In order to show that the bound R is the best possible, we consider the function $\psi \in \ominus(j)$ defined by

$$\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)} \right]^{\mu} = \gamma + (1-\gamma)\frac{1+\omega^j}{1-\omega^j} \quad (\omega \in \mathbb{U}).$$

Noting that

$$\begin{aligned} &\frac{1}{1-\gamma}\left\{ (1-\lambda)\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)} \right]^{\mu} \right. \\ &\left. +\lambda\left\{ 1 + \frac{(\alpha^2 - \beta^2)\left[\omega^2\psi'(\omega) - \omega\psi(\omega)\right]}{\left[\alpha\psi(\omega) + \beta\omega\right]\left[\alpha\omega + \beta\psi(\omega)\right]} \right\}\left[\frac{\alpha\psi(\omega) + \beta\omega}{\alpha\omega + \beta\psi(\omega)} \right]^{\mu} - \gamma \right\} \\ &= \frac{1+\omega^j}{1-\omega^j} + \frac{2\lambda j\omega^j}{\mu\ (1-\omega^j)^2} = 0, \end{aligned} \tag{2.19}$$

for $\omega = R\ \exp\left(\frac{\pi i}{j}\right)$, we conclude that the bound is the best possible. Theorem 2.14 is thus proved. □

Theorem 2.15. *Let $\lambda_2 \geq \lambda_1 \geq 0$ and $-1 \leq M_1 \leq M_2 < L_2 \leq L_1 \leq 1$. Then*

$$\mathcal{BN}^j(\lambda_2, \alpha, \beta, \mu; L_2, M_2) \subset \mathcal{BN}^j(\lambda_1, \alpha, \beta, \mu; L_1, M_1). \tag{2.20}$$

Proof. Suppose that $\psi \in \mathcal{BN}^j(\lambda_2, \alpha, \beta, \mu; L_2, M_2)$. We know that

$$(1-\lambda_2)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda_2\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$
$$\prec \frac{1+L_2\omega}{1+M_2\omega}.$$

Since $-1 \leq M_1 \leq M_2 < L_2 \leq L_1 \leq 1$, we easily find that

$$(1-\lambda_2)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda_2\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$
$$\prec \frac{1+L_2\omega}{1+M_2\omega} \prec \frac{1+L_1\omega}{1+M_1\omega}, \tag{2.21}$$

that is $\psi \in \mathcal{BN}^j(\lambda_2, \alpha, \beta, \mu; L_1, M_1)$. Thus, the assertion of Theorem 2.15 holds for $\lambda_2 = \lambda_1 \geq 0$. If $\lambda_2 > \lambda_1 \geq 0$, by Theorem 2.1 and (2.21), we know that $\psi \in \mathcal{BN}^j(0, \alpha, \beta, \mu; L_1, M_1)$, that is,

$$\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} \prec \frac{1+L_1\omega}{1+M_1\omega}, \tag{2.22}$$

At the same time, we have

$$(1-\lambda_1)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda_1\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$
$$= \left(1-\tfrac{\lambda_1}{\lambda_2}\right)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \frac{\lambda_1}{\lambda_2}\left[(1-\lambda_2)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}\right.$$
$$\left.+\lambda_2\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}\right]. \tag{2.23}$$

Moreover, since $0 \leq \frac{\lambda_1}{\lambda_2} < 1$,and the function $\frac{1+L_1\omega}{1+M_1\omega}$ $(-1 \leq M_1 < L_1 \leq 1, \omega \in \mathbb{U})$ is analytic and convex in $\mathbb{U}$. Combining (2.21)-(2.23) and Lemma 1.7, we find that

$$(1-\lambda_1)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} + \lambda_1\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$
$$\prec \frac{1+L_1\omega}{1+M_1\omega},$$

that is $\psi \in \mathcal{BN}^j(\lambda_1, \alpha, \beta, \mu; L_1, M_1)$, which implies that the assertion (2.20) of Theorem 2.15 holds. □

Theorem 2.16. *Let $\psi \in \mathcal{BN}^j(\lambda, \alpha, \beta, \mu; L, M)$ with $\lambda > 0$. Then*

$$\frac{\mu}{j\lambda}\int_0^1 \frac{1-Lu}{1-Mu}\, u^{\frac{\mu}{j\lambda}-1}du < \Re\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}$$
$$< \frac{\mu}{j\lambda}\int_0^1 \frac{1+Lu}{1+Mu}\, u^{\frac{\mu}{j\lambda}-1}du. \tag{2.24}$$

Proof. Let $\psi \in \mathcal{BN}^j(\lambda,\alpha,\beta,\mu;L,M)$. From Theorem 2.1, we know that (2.7) holds, which implies that

$$\begin{aligned}\Re\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} &< \sup_{\omega\in\mathbb{U}}\Re\left\{\frac{\mu}{j\lambda}\int_0^1\frac{1+L\omega u}{1+M\omega u}\,u^{\frac{\mu}{j\lambda}-1}\,du\right\}\\ &\leq \frac{\mu}{j\lambda}\int_0^1\sup_{\omega\in\mathbf{U}}\Re\left(\frac{1+L\omega u}{1+M\omega u}\right)u^{\frac{\mu}{j\lambda}-1}\,du\\ &< \frac{\mu}{j\lambda}\int_0^1\frac{1+Lu}{1+Mu}\,u^{\frac{\mu}{j\lambda}-1}\,du, \qquad (2.25)\end{aligned}$$

and

$$\begin{aligned}\Re\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu} &> \inf_{\omega\in\mathbb{U}}\Re\left\{\frac{\mu}{j\lambda}\int_0^1\frac{1+L\omega u}{1+M\omega u}\,u^{\frac{\mu}{j\lambda}-1}du\right\}\\ &\geq \frac{\mu}{j\lambda}\int_0^1\inf_{\omega\in\mathbb{U}}\Re\left(\frac{1+L\omega u}{1+M\omega u}\right)u^{\frac{\mu}{j\lambda}-1}\,du\\ &> \frac{\mu}{j\lambda}\int_0^1\frac{1-Lu}{1-Mu}\,u^{\frac{\mu}{j\lambda}-1}du. \qquad (2.26)\end{aligned}$$

Combining (2.25) and (2.26), we get (2.24). The proof of Theorem 2.16 is evidently complete. □

Theorem 2.17. *If $\psi \in \mathcal{BN}^j(\lambda,\alpha,\beta,\mu;L,M)$ is of the form (1.1), then*

$$|a_{j+1}| \leq \left|\frac{(\alpha+\beta)(L-M)}{(\alpha-\beta)(\mu+\lambda j)}\right|. \qquad (2.27)$$

Proof. From the assumption, we have

$$\begin{aligned}&(1-\lambda)\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}+\lambda\left\{1+\frac{(\alpha^2-\beta^2)\left[\omega^2\psi'(\omega)-\omega\psi(\omega)\right]}{\left[\alpha\psi(\omega)+\beta\omega\right]\left[\alpha\omega+\beta\psi(\omega)\right]}\right\}\left[\frac{\alpha\psi(\omega)+\beta\omega}{\alpha\omega+\beta\psi(\omega)}\right]^{\mu}\\ &\qquad = 1+\frac{(\alpha-\beta)(\mu+\lambda j)}{\alpha+\beta}a_{j+1}\omega^j+\cdots\prec\frac{1+L\omega}{1+M\omega},\end{aligned}$$

that is

$$1+\frac{(\alpha-\beta)(\mu+\lambda j)}{\alpha+\beta}a_{j+1}\omega^j+\ldots\prec 1+(L-M)\omega+\ldots.$$

Now, from Lemma 1.8 we deduce that

$$\left|\frac{(\alpha-\beta)(\mu+\lambda j)}{\alpha+\beta}a_{j+1}\right|\leq|L-M|,$$

which is equivalent to the inequality (2.27) asserted in our theorem. □

Remark 2.18. Putting $\alpha=0$ in the above results, we obtain the results of Aouf and Seoudy [11, when $p=1$].

Remark 2.19. Putting $\beta=0$ in Theorems 2.1, 2.15, 2.16 and 2.17, respectively, we obtain the results of Wang et. al. [14, Theorems 1, 2, 3 and 9, respectively].

3 Conclusion

This paper successfully applied the subordination principle to study Bavilevič and non-Bavilevič functions, yielding several important new results. We established fundamental properties including subordination and superordination relationships, proved sharp distortion theorems, and derived key inequalities for these function classes.

These results significantly extend previous work in this field and provide a more unified framework for understanding the behavior of these functions. The effectiveness of our subordination-based approach suggests that it could be fruitfully applied to more complex operators and higher-dimensional settings in future research.

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