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New congruences and infinite families of congruences for overcubic partition quadruples

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Abstract Let $\bar{b}_k(n)$ denote the number of overcubic partitions of n with $k-$ tuples. In this paper, we study congruence properties of the overcubic partitions of n with $4-tuples$. We shall establish some new Ramanujan-type congruences and infinite families of congruences satisfied by $\bar{b}_k(n)$ for $k = 4$.

1 Introduction

Throughout the paper, we assume $|q| < 1$ and for positive integers n, we use the standard notation

$$(a;q)_\circ := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

A partition of a nonnegative integer n is a non-increasing sequence of positive integers known as its parts, whose sum is n. $p(n)$ denotes the number of partitions of n. For example $p(5) = 7$ and the seven partitions of 5 are

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)$$

The study of the arithmetic properties of the number $p(n)$ was introduced by Ramanujan [18, 19]. He proved the following three congruences for all $n \geq 0$

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5 \\ p(7n+5) &\equiv 0 \pmod 7 \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n) = \prod_{n\geq 1} \frac{1}{(1-q^n)} = \frac{1}{(q;q)_\infty} = \frac{1}{f_1}$$

$$\text{where} \quad f_k = (q^k;q^k)_\infty = \prod_{m=0}^{\infty}(1-q^{km}) \quad \text{for integers} \quad k \geq 0.$$

A cubic partition of n is a partition wherein the even parts can appear in two colours. It is denoted by $a(n)$ and define $a(0) = 1$. For example $a(3) = 4$
An overpartition of an integer n is a non-increasing sequence of positive integers whose sum is n,

in which the first occurrence of a number may be overlined. It is denoted by $\overline{p}(n)$. For example $\overline{p}(3) = 8$ given by

$$3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1$$

Corteel and Lovejoy [20] defined the generating function for the overpartition function as follows

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + ...$$

The theory of overpartitions has been extensively studied in the literature; see, for example [24, 25, 26, 22, 21, 23]
In 2010, H. C. Chan[4, 5, 6] introduced the cubic partition function $a(n)$. The generating function of $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty} = \frac{1}{f_1 f_2}$$

Following H. C. Chan, B. Kim [10] studied the overpartition analogue of the cubic partition in which the overcubic partition function $\bar{a}(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{a}(n)q^n = \frac{f_4}{f_1^2 f_2}$$

Hirschhorn [8] gave an elementary proof of the results satisfied by $\bar{a}(n)$, which appeared in Kim's paper[10]. Sellers [15] proved sevaral properties satisfied by $\bar{a}(n)$. Zhao and Zhong [16] studied the number of cubic partition pairs, denoted by $b(n)$. We can interpret $b(n)$ as the number of 4-color partitions of n with colors r, y, o, and b subject to the restriction that the colors o and b appear only in even parts. The generating function of $b(n)$ is defined as

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{f_1^2 f_2^2}$$

Zhao and Zhong [16] established congruences modulo 5, 7, and 9 of $b(n)$.
Recently Kim [9] studied congruence properties of $\bar{b}(n)$. We can interpret $\bar{b}(n)$ as the number of 4-color partitions of n with colors r, y, o, and b subject to the restriction that the colors o and b appear only in even parts, in which the first occurrence parts may be overlined. $\bar{b}(n)$ is known as the overcubic partition pair, and the generating function of $\bar{b}(n)$ is defined by

$$\sum_{n=0}^{\infty} \bar{b}(n)q^n = \frac{f_4^2}{f_1^4 f_2^2}$$

If $\lambda_i,\ i = 1, 2, 3.....k$ are overcubic partition of n with $\sum_{i=1}^{k} \lambda_i = n$ then$(\lambda_1, \lambda_2.....\lambda_k)$ is said to be overcubic partition of n with $k - tuples$. If $\bar{b}_k(n)$ denotes the number of overcubic partition of n with with $k-$ tuples, then the generating function is given by

$$\sum_{n=0}^{\infty} \bar{b}_k(n)q^n = \frac{f_4^k}{f_1^{2k} f_2^k} \tag{1.1}$$

Kim[9] proved congruences modulo 3 and 64 for $\bar{b}_2(n)$. Lin [11] proved infinite families of congruence modulo 3 and 5 for $\bar{b}_2(n)$. Ray and Barman [14] found congruence modulo 256 for $\bar{b}_2(n)$. Naika and Shivashankar [12] proved some Ramanujan-type and infinite families of congruences modulo powers of 2 for $\bar{b}_2(n)$. In 2024, Buragohain and Saikia [3] established new congruences modulo 8, 16, 32.

2 Preliminaries

Ramanujan's general theta function is defined by

$$f(a,b)=\sum_{n=0}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2},\quad |ab|<1.$$

The function $f(a,b)$ satisfies the well-known Jacobi's Triple product identity, which can be expressed as

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.1}$$

Three special cases of $f(a,b)$ are defined for $|q|<1$, by [[2], p.36, Entry 22]

$$\phi(q)=f(q,q)=\sum_{k=0}^{\infty}q^{k^2}=\frac{f_2^5}{f_1^2f_4^2} \tag{2.2}$$

$$\psi(q)=f(q,q^3)=\sum_{k=0}^{\infty}q^{k(k+1)/2}=\frac{f_2^2}{f_1}, \tag{2.3}$$

$$f(-q)=f(-q,-q^2)=\sum_{k=-\infty}^{\infty}(-1)^kq^{k(3k-1)/2}=f_1 \tag{2.4}$$

where the product representations in (2.2)-(2.4) arise from (2.1).

Furthermore,the q-product representations of $\varphi(-q)$ and $\psi(-q)$ are given by

$$\varphi(-q)=\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}\quad\text{and}\quad\psi(-q)=\frac{(q;q)_\infty(q^4;q^4)_\infty}{(q^2;q^2)_\infty}.$$

We also define

$$\chi(q)=(-q;q^2)_\infty.$$

In this section, we present several 2- and 3- dissection formulas that were needed to derive our results. we also present $p-$ dissection of $\psi(q)$ and f_1

Lemma 2.1 ([17], Lemma 2.1). *The following 2-dissections hold*

$$\frac{1}{f_1^2}=\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8^8} \tag{2.5}$$

$$\frac{1}{f_1^8}=\frac{f_4^{28}}{f_2^{28}f_8^8}+8q\frac{f_4^{16}}{f_2^{24}}+16q^2\frac{f_4^4f_8^8}{f_2^{20}} \tag{2.6}$$

$$\frac{1}{f_1^{12}}=\frac{f_4^{42}}{f_2^{42}f_8^{12}}+12q\,\frac{f_4^{30}}{f_2^{38}f_8^4}+48q^2\,\frac{f_4^{18}f_8^4}{f_2^{34}}+64q^3\,\frac{f_4^6f_8^{12}}{f_2^{30}} \tag{2.7}$$

$$\frac{1}{f_1^{16}}=\frac{f_4^{56}}{f_2^{56}f_8^{16}}+16q\,\frac{f_4^{44}}{f_2^{52}f_8^8}+96q^2\,\frac{f_4^{32}}{f_2^{48}}+256q^3\,\frac{f_4^{20}f_8^8}{f_2^{44}}+256q^4\,\frac{f_4^8f_8^{16}}{f_2^{40}} \tag{2.8}$$

Lemma 2.2 ([2], p, 345). *The following 3-dissections hold:*

$$f_1^3=\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3 \tag{2.9}$$

Lemma 2.3 ([7], Theorem 2.1). *For any odd prime p, we have*

$$\psi(q)=\sum_{k=0}^{\frac{p-3}{2}}q^{\frac{k^2+k}{2}}f\left(q^{\frac{p^2+(2k+1)p}{2}},\,q^{\frac{p^2-(2k+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}).$$

Moreover, for $0\le k\le\frac{p-3}{2}$,

$$\frac{k^2+k}{2}\not\equiv\frac{p^2-1}{8}\pmod 8.$$

Lemma 2.4 ([7], Theorem 2.2). *For any prime $p \geq 5$, we have*

$$f_1 = \sum_{\substack{k=-(p-1)/2 \\ k\neq\pm(p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right) + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f_{p^2}.$$

where

$$\frac{\pm p-1}{6} = \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \frac{-(p-1)}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

Furthermore, if

$$-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}, \quad \text{and } k \neq \frac{\pm p-1}{6},$$

then

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

Lemma 2.5 ([1], Lemma 2.3). *For any prime $p \geq 3$, we have*

$$f_1^3 = \sum_{\substack{k=0 \\ k\neq\frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn\cdot\frac{pn+2k+1}{2}} + (-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} f_{p^2}^3.$$

Furthermore, if $k \neq \frac{p-1}{2}$ and $0 \leq k \leq p-1$, then

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod p.$$

Lemma 2.6 ([13], p. 212). *We have the following 5-dissection formula:*

$$f_1 = f_{25}\left(a(q^5) - q - \frac{q^2}{a(q^5)}\right),$$

where

$$a := a(q) := \frac{(q^2, q^3, q^5; q^5)_\infty}{(q, q^4, q^5; q^5)_\infty}.$$

Lemma 2.7 ([2], p. 303, entry 17(v)). *We have*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right),$$

where $A(q) = f(-q^3, -q^4)$, $B(q) = f(-q^2, -q^5)$ and $C(q) = f(-q, -q^6)$.

Lemma 2.8. *For all primes p and all $k, m, j \geq 1$, we have*

$$f_{pm}^{p^{k-1}j} \equiv f_m^{p^k j} \pmod{p^k} \tag{2.10}$$

3 Main Results

Theorem 3.1. *Let p be an odd prime then for any $\alpha \geq 0$ and for $1 \leq i \leq p-1$ we have*

$$\sum_{n=0}^{\infty} \overline{b}_4(8 \cdot p^{2\alpha} n + p^{2\alpha}) q^n \equiv 8\psi(q) \pmod{16} \tag{3.1}$$

$$\text{and} \quad \overline{b}_4(8 \cdot p^{2\alpha+2} n + (8i+p) \cdot p^{2\alpha+1}) = 0 \pmod{16} \tag{3.2}$$

Proof. Thanks to [3, Theorem 3.9, Eq. (3.68)], we have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{b}_4(8n+1) q^n &\equiv 8 f_1 f_2 \pmod{16} \\ &\equiv 8 f_1^3 \pmod{16} \qquad (3.3) \\ &\equiv 8\psi(q) \pmod{16}. \qquad (3.4) \end{aligned}$$

which is the $\alpha = 0$ case of (3.1). Suppose (3.1) holds for some integer $\alpha \geq 0$. Then we have

$$\sum_{n=0}^{\infty} \overline{b}_4(8 \cdot p^{2\alpha} n + p^{2\alpha}) q^n \equiv 8\psi(q) \pmod{16} \tag{3.5}$$

By employing Lemma 2.3, we have

$$\sum_{n=0}^{\infty} \overline{b}_4(8 \cdot p^{2\alpha} n + p^{2\alpha}) q^n \equiv 8 \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}) \pmod{16} \tag{3.6}$$

Consider the congruence

$$\frac{k^2+k}{2} \equiv \frac{p^2-1}{8} \pmod{p}$$

The above relation is equivalent to $(2k+1)^2 \equiv 0 \pmod{p}$, then it implies that $k = \frac{p-1}{2}$. Extracting the terms involving $q^{pn+\frac{p^2-1}{8}}$ from (3.6) and then dividing by $q^{\frac{p^2-1}{8}}$, we get

$$\sum_{n=0}^{\infty} \overline{b}_4(8 \cdot p^{2\alpha}(pn + \frac{p^2-1}{8}) + p^{2\alpha}) q^{pn} \equiv 8\psi(q^p) \pmod{16} \tag{3.7}$$

which yields

$$\sum_{n=0}^{\infty} \overline{b}_4\big(8 \cdot p^{2\alpha+1} n + p^{2\alpha+2}\big) q^{pn} \equiv 8\psi(q^p) \pmod{16} \tag{3.8}$$

Extracting the terms that involve q^{pn} from (3.8), we obtain

$$\sum_{n=0}^{\infty} \overline{b}_4(8 \cdot p^{2\alpha+2} n + p^{2\alpha+2}) q^n \equiv 8\psi(q) \pmod{16} \tag{3.9}$$

which is the $\alpha+1$ case of (3.1). Thus by induction (3.1) is true for all $\alpha \geq 0$.
Also from (3.8) by extracting the terms that involve q^{pn+i} where $1 \leq i \leq p-1$, we obtain (3.2) □

Theorem 3.2. *For all $n \geq 0$, and $\alpha, \beta, \gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{b}_4(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 2\gamma) q^n \equiv 8 f_1^3 \pmod{16} \tag{3.10}$$

Proof. We first observe that (3.10) holds for $\alpha = \beta = \gamma = 0$ by virtue of (3.3). Now assume that (3.10) holds for some $\alpha \geq 0$ with $\beta = \gamma = 0$. Applying (2.9) and extracting all terms of the form q^{3n+1}, we obtain

$$\sum_{n=0}^{\infty} \overline{b}_4\big(8 \cdot 3^{2\alpha+1}n + 3^{2\alpha+2}\big)q^n \equiv 8f_3^3 \pmod{16} \tag{3.11}$$

Extracting all the terms involving q^{3n} and replacing q^3 by q, from both side of (3.11) we get

$$\sum_{n=0}^{\infty} \overline{b}_4\big(8 \cdot 3^{2\alpha+2}n + 3^{2\alpha+2}\big)q^n \equiv 8f_1^3 \pmod{16} \tag{3.12}$$

(3.12) is the $\alpha + 1$ case of (3.10) with $\beta = \gamma = 0$. Suppose (3.10) holds for $\alpha, \beta \geq 0$, with $\gamma = 0$.
Employing Lemma 2.6 in (3.10) with $\gamma = 0$, then extracting the terms with q^{5n+3}, we get

$$\sum_{n=0}^{\infty} \overline{b}_4\big(8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1}n + 3^{2\alpha} \cdot 5^{2\beta+2}\big)q^n \equiv 8f_5^3 \pmod{16} \tag{3.13}$$

Extracting the terms involving q^{5n} and the replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} \overline{b}_4\big(8 \cdot 3^{2\alpha} \cdot 5^{2\beta+2}n + 3^{2\alpha} \cdot 5^{2\beta+2}\big)q^n \equiv 8f_1^3 \pmod{16} \tag{3.14}$$

This is the $\beta+1$ case with $\gamma = 0$. Thus (3.10) is true for all $\alpha, \beta \geq 0$ with $\gamma = 0$. Now we assume (3.10) is true all $\alpha, \beta \geq 0$ and some $\gamma \geq 0$. Employing Lemma 2.7 in (3.10) and extracting all the terms involving q^{7n+6}, we get

$$\sum_{n=0}^{\infty} \overline{b}_4\big(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1}n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}\big)q^n \equiv 8f_7^3 \pmod{16} \tag{3.15}$$

Finally extracting the terms involving q^{7n} ,we get

$$\sum_{n=0}^{\infty} \overline{b}_4\big(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}\big)q^n \equiv 8f_1^3 \pmod{16} \tag{3.16}$$

Above is the $\gamma + 1$ case of (3.10). This completes the proof by mathematical induction. □

Corollary 3.3. *For all $n \geq 0$, and $\alpha, \beta, \gamma \geq 0$,we have*

$$\sum_{n=0}^{\infty} \overline{b}_4(8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma}n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma})q^n$$

$$\equiv \begin{cases} 8 \pmod{16}, & \text{if } n = \dfrac{k(3k-1)}{2}, \\ 0 \pmod{16}, & \text{otherwise}, \end{cases} \tag{3.17}$$

$$\sum_{n=0}^{\infty} \overline{b}_4(8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma}n + 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma})q^n \equiv 8f_3^3 \pmod{16} \tag{3.18}$$

and $$\sum_{n=0}^{\infty} \overline{b}_4(8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma}n + 17 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma})q^n \equiv 0 \pmod{16} \tag{3.19}$$

Proof. By substituting (2.9) into (3.10),

$$\sum_{n=0}^{\infty} \overline{b}_4(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma})q^n \equiv 8(f_3 - 3qf_9^3) \pmod{16} \tag{3.20}$$

Now extracting the terms involving q^{3n}, we obtain

$$\sum_{n=0}^{\infty} \overline{b}_4(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n + 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma})q^n \equiv 8f_1 \pmod{16} \tag{3.21}$$

Using (2.4) in (3.21), we obtain (3.17)

Again extracting the terms involving q^{3n+1} and q^{3n+2} from both sides of (3.20) we arrive at equations (3.18), and (3.19). □

Corollary 3.4. *For all $n \geq 0$, and $\alpha, \beta, \gamma \geq 0$,we have*

$$\sum_{n=0}^{\infty} \overline{b}_4(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}(3n+i) + 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma})q^n \equiv 0 \pmod{16}, \text{ for } i = 1,2 \tag{3.22}$$

Proof. We see that the right-hand side of the congruence in (3.18) is a function of q^3. Hence , by extracting the terms involving q^{3n+i} for $i = 1, 2$, we obtain the congruence (3.22). □

Corollary 3.5. *For all $n \geq 0$, and $\alpha, \beta, \gamma \geq 0$,we have*

$$\sum_{n=0}^{\infty} \overline{b}_4(8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n + \cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma})q^n \equiv 8f_5^3 \pmod{16} \tag{3.23}$$

$$\sum_{n=0}^{\infty} \overline{b}_4(8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}(5n+i) + 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma})q^n \equiv 0 \pmod{16} \text{ for } i = 2,4 \tag{3.24}$$

$$\text{and} \quad \sum_{n=0}^{\infty} \overline{b}_4(8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}(5n+j) + 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma})q^n \equiv 0 \pmod{16}$$

$$\text{for } j \in \{1,2,4,3\} \tag{3.25}$$

Proof. Using Lemma 2.6 into (3.10) and extracting the terms involving q^{5n+3}, we get (3.23)

Again, following a similar argument as above, by extracting the terms involving q^{5n+2} and q^{5n+4}, we arrive at the corresponding congruences given in (3.24)

Right hand side of (3.23) is a function of q^5, therefore extracting the terms that contains q^{5n+j} where $j \in \{1,2,3,4\}$, we arrive at (3.25). □

Corollary 3.6. *For all $n \geq 0$, and $\alpha, \beta, \gamma \geq 0$,we have*

$$\overline{b}_4(8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}(7n+j) + 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2})q^n \equiv 0 \pmod{16} \text{ for } j = 1,2,4,3,5,6 \tag{3.26}$$

Proof. Extracting all the terms involving q^{7n+j} for $j \in \{1,2,3,4,5,6\}$ from both sides of the equation (3.15), we obtain (3.26) □

Theorem 3.7. *For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have*

$$\overline{b}_4(8n+7) \equiv 0 \pmod{2^6} \tag{3.27}$$

$$\sum_{n=0}^{\infty} \overline{b}_4(8n+3)q^n \equiv 32f_1^9 \pmod{2^6} \tag{3.28}$$

$$\sum_{n=0}^{\infty} \overline{b}_4(8\cdot 3^{4\alpha}\cdot n + 3^{4\alpha+1})q^n \equiv 32f_1^9 \pmod{2^6} \tag{3.29}$$

Proof. We recall from [3, Theorem.3.9, Eq.(3.65).], that

$$\sum_{n=0}^{\infty} \bar{b}_4(2n+1)q^n = 8\frac{f_2^{20}}{f_1^{28}} \tag{3.30}$$

$$= 8f_2^{20}\left(\frac{1}{f_1^{16}}\right)\left(\frac{1}{f_1^{12}}\right) \tag{3.31}$$

Utilising (2.7) and (2.8) in (3.31), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_4(2n+1)q^n \equiv 8\frac{f_4^{98}}{f_2^{78}f_8^8} + 96q\frac{f_4^{86}}{f_2^{74}f_8^{20}} \pmod{2^6} \tag{3.32}$$

Extracting the terms involving q^{2n+1} from (3.32) and dividing both sides by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \bar{b}_4(4n+3)q^n \equiv 32\frac{f_2^{86}}{f_1^{74}f_4^{20}} \pmod{2^6}$$

$$\equiv 32f_2^9 \pmod{2^6} \qquad \text{Thanks to Lemma.2.8} \tag{3.33}$$

By extracting the odd and even terms from (3.33), we obtain (3.27) and (3.28) respectively. (3.28) is the $\alpha = 0$ case of (3.29). Thus (3.29) is true fro $\alpha = 0$. Assume that the result holds for some $\alpha \geq 0$, then we have

$$\sum_{n=0}^{\infty} \bar{b}_4\left(8\cdot 3^{4\alpha}n + 3^{4\alpha+1}\right)q^n \equiv 32f_1^9 \pmod{2^6} \tag{3.34}$$

Employing Lemma 2.2 and Lemma 2.8, we obtain

$$\sum_{n=0}^{\infty} \bar{b}_4\left(8\cdot 3^{4\alpha}n + 3^{4\alpha+1}\right)q^n \equiv \quad \equiv 32f_3^3 + 32qf_3^2f_9^3 + 32q^2f_3f_9^6 + 32q^3f_9^9 \pmod{2^6} \tag{3.35}$$

By extracting the terms containing q^{3n}, we get

$$\sum_{n=0}^{\infty} \bar{b}_4\left(8\cdot 3^{4\alpha+1}n + 3^{4\alpha+1}\right)q^n \equiv 32f_1^3 + 32qf_3^9 \pmod{2^6} \tag{3.36}$$

Employing (2.9) in (3.36), we have

$$\sum_{n=0}^{\infty} \bar{b}_4\left(8\cdot 3^{4\alpha+1}n + 3^{4\alpha+1}\right)q^n \equiv 32f_3 + 32qf_9^3 + 32qf_3^9 \pmod{2^6}$$

Now extracting the terms that contains q^{3n+1}, we have

$$\sum_{n=0}^{\infty} \bar{b}_4\left(8\cdot 3^{4\alpha+1}(3n+1) + 3^{4\alpha+1}\right)q^n \equiv 32f_3^3 + 32f_1^9 \pmod{2^6}$$

Using (2.9), we have

$$\sum_{n=0}^{\infty} \bar{b}_4\left(8\cdot 3^{4\alpha+2}n + 3^{4\alpha+3}\right)q^n \equiv 32f_3^3 + 32f_3^3 + 32qf_3^2f_9^3 + 32q^2f_3f_9^6 + 32q^3f_9^9 \pmod{2^6}$$

$$\equiv 32qf_3^2f_9^3 + 32q^2f_3f_9^6 + 32q^3f_9^9 \pmod{2^6} \tag{3.37}$$

Extracting the terms involving q^{3n} from (3.37) gives

$$\sum_{n=0}^{\infty} \bar{b}_4\left(8\cdot 3^{4\alpha+3}n + 3^{4\alpha+3}\right)q^n \equiv 32qf_3^9 \pmod{2^6} \tag{3.38}$$

Again extracting the terms involving q^{3n+1}, we obtain

$$\sum_{n=0}^{\infty}\overline{b}_4\big(8\cdot 3^{4\alpha+4}n+3^{4\alpha+5}\big)q^n\equiv 32f_1^9 \pmod{2^6} \tag{3.39}$$

Thus (3.29) is true for $\alpha+1$. Hence, by mathematical induction, the result is true for all $\alpha\geq 0$ □

Theorem 3.8. *For all $n\geq 0$ and $\alpha,\ \beta.\ \gamma\ \geq 0$, we have*

$$\sum_{n=0}^{\infty}\overline{b}_4\big(8\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\big)\equiv 32f_1^9 \pmod{2^6} \tag{3.40}$$

Proof. We observe that, (3.29) implies that (3.40) holds for all $\alpha\geq 0$ when $\beta=\gamma=0$. We assume that (3.40) holds for all $\alpha\geq 0$ and for some $\beta\geq 0$ with $\gamma=0$

$$\sum_{n=0}^{\infty}\overline{b}_4\big(8\cdot 3^{4\alpha}\cdot 5^{2\beta}n+3^{4\alpha+1}\cdot 5^{2\beta}\big)\equiv 32f_1^9 \pmod{2^6} \tag{3.41}$$

Applying Lemma 2.6 and extracting the terms of the form q^{5n+4}, we obtain

$$\sum_{n=0}^{\infty}\overline{b}_4\big(8\cdot 3^{4\alpha}\cdot 5^{2\beta+1}n+7\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\big)\equiv 32qf_5^9 \pmod{2^6} \tag{3.42}$$

Extracting the terms of the form q^{5n+1} from (3.42), we have

$$\sum_{n=0}^{\infty}\overline{b}_4\big(8\cdot 3^{4\alpha}\cdot 5^{2\beta+2}n+3^{4\alpha+1}\cdot 5^{2\beta+2}\big)\equiv 32f_1^9 \pmod{2^6} \tag{3.43}$$

which shows the result is true for $\beta+1$. Thus the result is true for all $\alpha,\beta\geq 0$ with $\gamma=0$ Assume that the congruence (3.40) is true for all $\alpha,\beta\geq 0$ and for some $\gamma\geq 0$.
Utilizing (2.7) in (3.40) and extracting the terms of the form q^{7n+4}, we arrive at

$$\sum_{n=0}^{\infty}\overline{b}_4\big(8\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}\big)q^n\equiv 32q^2f_7^9 \pmod{2^6} \tag{3.44}$$

Now extracting the terms of the form q^{7n+2}, we obtain

$$\sum_{n=0}^{\infty}\overline{b}_4\big(8\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\big)q^n\equiv 32f_1^9 \pmod{2^6} \tag{3.45}$$

which is the $\gamma+1$ case of (3.40). Hence, by induction (3.40) is true for all $\alpha,\beta,\gamma\geq 0$. □

Theorem 3.9. *For all $n\geq 0$ and $\gamma\geq 0$ we have*

$$\sum_{n=0}^{\infty}\overline{b}_4(24n+19)q^n\equiv 32f_1f_3^6 \pmod{2^6} \tag{3.46}$$

$$\sum_{n=0}^{\infty}\overline{b}_4\big(24\cdot 5^{2\gamma}n+19\cdot 5^{2\gamma}\big)q^n\equiv 32f_1f_6^3 \pmod{2^6} \tag{3.47}$$

$$\sum_{n=0}^{\infty}\overline{b}_4\big(24\cdot 5^{2\gamma+1}n+23\cdot 5^{2\gamma+1}\big)q^n\equiv 32q^3f_5f_{30}^3 \pmod{2^6} \tag{3.48}$$

and $$\sum_{n=0}^{\infty}\overline{b}_4\big(24\cdot 5^{2\gamma+1}(5n+j)+23\cdot 5^{2\gamma+1}\big)q^n\equiv 0 \pmod{2^6} \quad \textit{for} \quad j=0,1,2,4 \tag{3.49}$$

Proof. Using Lemma 2.2 in (3.28) we arrive at

$$\sum_{n=0}^{\infty} \overline{b}_4(8n+3)q^n \equiv 32(f_3^3 + qf_3^2 f_9^3 + q^2 f_3 f_9^6 + q^3 f_9^9) \pmod{2^6} \tag{3.50}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we get

$$\sum_{n=0}^{\infty} \overline{b}_4(24n+3)q^n \equiv 32(f_1^3 + qf_3^9) \pmod{2^6} \tag{3.51}$$

$$\sum_{n=0}^{\infty} \overline{b}_4(24n+11)q^n \equiv 32f_1^2 f_3^3 \pmod{2^6} \tag{3.52}$$

and $$\sum_{n=0}^{\infty} \overline{b}_4(24n+19)q^n \equiv 32f_1 f_3^6 \pmod{2^6} \tag{3.53}$$

We observe that (3.53) is the $\gamma = 0$ case of (3.47). Assume that (3.47) is true for some $\beta \geq 0$. By employing Lemma 2.6 in (3.47) and then extracting the term involving q^{5n+4}, we obtain

$$\sum_{n=0}^{\infty} \overline{b}_4\big(24 \cdot 5^{2\gamma+1}n + 23 \cdot 5^{2\gamma+1}\big)q^n \equiv 32q^3 f_5 f_{30}^3 \pmod{2^6} \tag{3.54}$$

This proves (3.48). Again extracting the terms involving q^{5n+3}, we obtain

$$\sum_{n=0}^{\infty} \overline{b}_4\big(24 \cdot 5^{2\gamma+2}n + 19 \cdot 5^{2\gamma+2}\big)q^n \equiv 32f_1 f_6^3 \pmod{2^6} \tag{3.55}$$

which shows the result is also true for $\gamma+1$. Hence, by mathematical induction, the result is true for all $\gamma \geq 0$.
Again extracting the term involving q^{5n+j} for $j = 0, 1, 2, 4$ from both sides of (3.54) we get (3.49) □

Theorem 3.10. *For all $n \geq 0$ and $\beta \geq 0$ we have*

$$\sum_{n=0}^{\infty} \overline{b}_4(5^{2\beta}(24n+11))q^n \equiv 32f_2 f_3^3 \pmod{2^6} \tag{3.56}$$

$$\sum_{n=0}^{\infty} \overline{b}_4(24 \cdot 5^{2\beta+1}n + 35 \cdot 5^{2\beta})q^n \equiv 32q^2 f_{10} f_{15}^3 \pmod{2^6} \tag{3.57}$$

and $$\overline{b}_4(24 \cdot 5^{2\beta+1}(5n+r) + 35 \cdot 5^{2\beta}) \equiv 0 \pmod{2^6} \quad \textit{for} \quad r = 0, 1, 3, 4 \tag{3.58}$$

Proof. From (3.52) we find that (3.56) is true for $\beta = 0$. Assume that the congruence (3.56) is true for some $\beta > 0$.
Using Lemma 2.6 in (3.56), we obtain

$$\sum_{n=0}^{\infty} \overline{b}_4(24 \cdot 5^{2\beta}n + 11 \cdot 5^{2\beta})q^n \equiv 32f_{50} f_{75}^3 \Big(a(q^{10}) - q^2 - \frac{q^4}{a(q^{10})}\Big) \Big(a(q^{15}) - q^3 - \frac{q^6}{a(q^{15})}\Big)^3 \pmod{2^6} \tag{3.59}$$

Extracting the term involving q^{5n+1}, we obtain

$$\sum_{n=0}^{\infty} \overline{b}_4(24 \cdot 5^{2\beta+1}n + 35 \cdot 5^{2\beta})q^n \equiv 32q^2 f_{10} f_{15}^3 \pmod{2^6} \tag{3.60}$$

which proves (3.57). Again extracting the terms involving q^{5n+2}, we obtain

$$\sum_{n=0}^{\infty} \overline{b}_4(24 \cdot 5^{2\beta+2}n + 11 \cdot 5^{2\beta+2})q^n \equiv 32 f_2 f_3^3 \pmod{2^6} \tag{3.61}$$

Thus, the congruence in (3.56) is true for $\beta+1$. Therefore, by mathematical induction, the result is true for all $\beta \geq 0$. Again extracting the terms of the form q^{5n+r} for $r = 0, 1, 3, 4$ from both sides of (3.60), we obtain (3.58) □

Corollary 3.11. *For all $n \geq 0$ we have*

$$\overline{b}_4(72n + 51) \equiv 0 \pmod{2^6} \tag{3.62}$$

$$\overline{b}_4(216n + 99) \equiv 0 \pmod{2^6} \tag{3.63}$$

$$\textit{and} \quad \overline{b}_4(216n + 171) \equiv 0 \pmod{2^6} \tag{3.64}$$

Proof. Using 2.10 and (2.9) in (3.52) we have

$$\sum_{n=0}^{\infty} \overline{b}_4(24n + 3)q^n \equiv 32 f_3 + 32q f_9^3 + 32 f_3^9 \pmod{2^6} \tag{3.65}$$

Extracting the terms involving q^{3n+1} and q^{3n+2} we obtain

$$\overline{b}_4(72n + 27) \equiv 32 f_3^3 \pmod{2^6} \tag{3.66}$$

$$\text{and} \quad \sum_{n=0}^{\infty} \overline{b}_4(72n + 51)q^n \equiv 0 \pmod{2^6} \tag{3.67}$$

Extracting the terms involving q^{3n+1} and q^{3n+2} frome (3.66), we obtatin

$$\sum_{n=0}^{\infty} \overline{b}_4(72(3n + 1) + 27)q^n = \sum_{n=0}^{\infty} \overline{b}_4(216n + 99)q^n \equiv 0 \pmod{2^6} \tag{3.68}$$

$$\sum_{n=0}^{\infty} \overline{b}_4(72(3n + 2) + 27)q^n = \sum_{n=0}^{\infty} \overline{b}_4(216n + 171)q^n \equiv 0 \pmod{2^6} \tag{3.69}$$

□

Theorem 3.12. *For any odd prime p with $\left(\frac{-2}{p}\right) = -1$ and for all $1 \leq l \leq p-1$ and $\beta \geq 0$*

$$\sum_{n=0}^{\infty} \overline{b}_4(8n + 3)q^n \equiv 32\psi(q)\psi(q^2) \pmod{2^6} \tag{3.70}$$

$$\sum_{n=0}^{\infty} \overline{b}_4(8 \cdot p^{2\beta}n + 3 \cdot p^{2\beta})q^n \equiv 32\psi(q)\psi(q^2) \pmod{2^6} \tag{3.71}$$

$$\textit{and} \quad \overline{b}_4(8 \cdot p^{2\beta+1}(pn + l) + 3 \cdot p^{2\beta+2})q^n \equiv 0 \pmod{2^6} \tag{3.72}$$

Proof. Using (2.3) in (3.28) we find

$$\sum_{n=0}^{\infty} \overline{b}_4(8n + 3)q^n \equiv 32\frac{f_2^2}{f_1}\frac{f_4^2}{f_2} \pmod{2^6} \tag{3.73}$$

$$\equiv 32\psi(q)\psi(q^2) \pmod{2^6} \tag{3.74}$$

which is the $\beta = 0$ case of the congruence given in (3.71). Assume that the congruence is true

for some integer $\beta \geq 0$, then using Lemma 2.3 in (3.71), we have

$$\sum_{n=0}^{\infty} \bar{b}_4(8 \cdot p^{2\beta} n + 3 \cdot p^{2\beta}) q^n \equiv 32 \Big[\sum_{r=0}^{\frac{p-3}{2}} q^{\frac{r^2+r}{2}} f(q^{\frac{p^2+(2r+1)p}{2}}, \ q^{\frac{p^2-(2r+1)p}{2}}) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}) \Big]$$
$$\cdot \Big[\sum_{s=0}^{\frac{p-3}{2}} q^{s^2+s} f(q^{p^2+(2s+1)p}, \ q^{p^2-(2s+1)p}) + q^{\frac{p^2-1}{4}} \psi(q^{2p^2}) \Big] \pmod{2^6} \tag{3.75}$$

We consider the congruence

$$\frac{(r^2+r)}{2} + (s^2+s) \equiv \frac{3(p^2-1)}{8} \tag{3.76}$$

which yields

$$(4r+2)^2 + 2(2s+1)^2 \equiv 0 \pmod{p} \tag{3.77}$$

Since it is given $\left(\frac{-2}{p}\right) = -1$, the above congruence has only the solution $r = s = \frac{p-1}{2}$.

Now extracting the terms involving $q^{pn+\frac{3(p^2-1)}{8}}$ from the congruence (3.75), dividing both sides by $q^{\frac{3(p^2-1)}{8}}$ and then replacing q^n with q, we obtain

$$\sum_{n=0}^{\infty} \bar{b}_4(8 \cdot p^{2\beta}\big(pn + \frac{3(p^2-1)}{8}\big) + 3 \cdot p^{2\beta}) q^n \equiv 32\psi(q^p)\psi(q^{2p}) \pmod{2^6}$$

$$\text{which implies} \quad \sum_{n=0}^{\infty} \bar{b}_4(8 \cdot p^{2\beta+1} n + 3 \cdot p^{2\beta+2}) q^n \equiv 32\psi(q^p)\psi(q^{2p}) \pmod{2^6} \tag{3.78}$$

Again extracting the terms involving q^{pn} from (3.78), we obtain

$$\sum_{n=0}^{\infty} \bar{b}_4(8 \cdot p^{2\beta+2} n + 3 \cdot p^{2\beta+2}) q^n \equiv 32\psi(q)\psi(q^2) \pmod{2^6}$$

Hence, the congruence in (3.72) is true for $\beta + 1$. This completes the proof of the congruence in (3.72), by induction. Finally, the congruence in (3.72) can be obtained from (3.78) by extracting the terms involving q^{pn+l} for $1 \leq l \leq p-1$ □

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