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Expanding the Concept of Picture Fuzzy Vector Spaces and Their Fundamental Properties

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Abstract This study introduces the concept of a picture fuzzy vector space, extending the framework of fuzzy and intuitionistic fuzzy vector spaces. The proposed model integrates topological elements to enable a richer representation of imprecise information. Fundamental properties such as absolute convexity, balance, and absorbency are formally defined within this setting, and the notion of a convex balanced hull is established. A series of propositions is developed to illustrate the relationships among these properties. Theoretical analysis demonstrates how this structure enhances the modeling of uncertainty and partial truth. Potential applications include neural networks, image processing, machine learning, and artificial intelligence, highlighting the framework's practical and theoretical significance.

1 Introduction

Classical mathematical reasoning often struggles to address uncertainty in concepts that are inherently subjective, such as "the group of attractive women" or "the group of tall men". To capture such imprecision, Zadeh [1] introduced fuzzy set theory in 1965, incorporating membership values to quantify degrees of belonging. This framework became a powerful tool for modeling ambiguity in logic and provided a systematic approach for tackling complex real-world problems. Atanassov [2] enhanced Zadeh's model by adding non-membership functions, thereby improving its capacity to represent uncertainty. Coker [3] further extended the theory by combining membership and non-membership values into intuitionistic fuzzy sets, enabling a more complete representation that accounts for acceptance and rejection degrees. Building upon these advances, B. Cuong and V. Kerinovich [4] proposed picture fuzzy sets (PFS), introducing a third parameter-neutral membership-alongside acceptance and rejection. This richer representation supports more nuanced decision-making in uncertain environments.

In parallel, topology has played a central role in analyzing mathematical structures through concepts such as homeomorphisms and continuity, which formalize the idea of "closeness" in abstract spaces. Chang [6] pioneered fuzzy topological spaces in 1968, extending classical notions like open and closed sets to the fuzzy domain. Lowen [8, 9] refined this framework with new definitions of fuzzy continuity and fuzzy compactness, deepening the theoretical foundation of fuzzy topology. More recently, Nguy'n Xuan [10] advanced picture fuzzy theory by integrating it with rough set concepts in crisp approximation spaces. Subsequent contributions have further enriched the field. Abdul Razaaq et al. [11] investigated the notion of rank in picture fuzzy topological spaces (PFTS) and analyzed its behavior under continuous mappings. Tamilselvan et al. [12, 13] introduced picture fuzzy filters, grills, and ultrafilters, explored their interrelationships, and classified picture fuzzy sets into first and second categories. They also examined picture fuzzy Baire spaces and developed applications to picture fuzzy compact spaces (PFCS) using filter-based methods. Despite these developments, there remains an open gap in integrating topological properties with picture fuzzy vector spaces in a manner that systematically examines their convex and balanced structures. This study addresses that gap by proposing a topological framework for picture fuzzy vector spaces, analyzing key properties such as absolute convexity, balance, absorbency, and convex balanced hulls, and establishing their theoretical

interconnections.

1.1 Literature review

The study of topological vector spaces was enriched in 1967 when Francois Treves [14, 15, 16] formalized core concepts in the field. Katsaras et al. [17] later merged the ideas of fuzzy sets and topological vector spaces, laying the groundwork for fuzzy vector spaces and fuzzy topological vector spaces. In his 1981 work, Katsaras provided a detailed formulation of these structures, enabling the representation of elements with varying degrees of membership a feature absent in their classical counterparts. This extension has proved valuable in decision-making, control theory, and artificial intelligence, where uncertainty is inherent [19]. Subsequent investigations advanced these foundations in multiple directions. Manmekto et al. [20] examined binary convex fuzzy vector spaces and subspaces, formulating their definitions and characterizing their properties. By drawing on concepts from communication theory, the authors applied these structures to message reliability in noisy channels, introducing binary convex fuzzy codes relevant to error correction. Similarly, Bejines et al. [21] extended flag codes a network coding framework through fuzzy vector subspaces, establishing equivalence relations inspired by fuzzy subgroup theory and computing the number of distinct flags in discrete cases. Gereme et al. [22] integrated fuzzy vector space theory with fuzzy subsets to define fuzzy codes in such spaces, refining the Hamming distance concept for binary fuzzy codes and connecting these measures to error detection in binary symmetric channels.

Other strands of research expanded the algebraic and topological frameworks further. Kungumaraj et al. [23] explored the evolution from classical algebraic structures (groups, rings, fields, vector spaces) toward fuzzy, intuitionistic fuzzy, and neutrosophic systems, including neutrosophic linear transformations with applications to neural networks and image processing. Moumita Chiney and Samanta [24] proposed intuitionistic fuzzy topological vector spaces (IFTVS), which combine degrees of membership, non-membership, and hesitation, offering a more nuanced framework for problems involving belief, disbelief, and uncertainty. Recent efforts have extended fuzzy and intuitionistic frameworks into picture fuzzy and related higher-order models. Dogra et al. [25] studied picture fuzzy subspaces in crisp vector spaces, defining operations such as intersection, union, and (θ, ϕ, ψ)-cuts, along with direct sums and linear transformations. Further works [26] expanded these ideas to picture fuzzy sub-hyperspaces in hyper vector spaces, establishing closure properties and applying them to decision-making. The same group also introduced Fermatean fuzzy subspaces [27] as a generalization of intuitionistic and Pythagorean models, demonstrating applications in career placement. Ahmad et al. [28] applied picture fuzzy theory to B-spline interpolation, using open uniform knot vectors for improved curve modeling.

Beyond picture fuzzy sets, other generalizations have emerged. Ozlu [29] proposed picture type-2 hesitant fuzzy sets, formalizing set operations and similarity measures, and validating them through numerical studies. Hagery et al. [30] employed neutrosophic topological vector spaces to design an automated diabetic retinopathy detection system, integrating image preprocessing, deep feature extraction, and metaheuristic optimization for accurate diagnosis. Further extensions include Kungumaraj et al. [23] on neutrosophic topological vector spaces with generalized transformations, and Sathinathan et al. [31] on refined neutrosophic vector spaces (r-NVS), whose algebraic operations were demonstrated in an e-commerce ranking scenario. Collectively, these works demonstrate a steady progression from classical and fuzzy topological vector spaces to richer models like intuitionistic, picture fuzzy, Fermatean, and neutrosophic spaces-broadening both theoretical foundations and practical applicability in uncertainty modeling. Despite these developments, there remains a gap in systematically integrating picture fuzzy structures with topological vector space theory in a way that rigorously addresses convexity, balance, and related structural properties. The present study aims to fill this gap.

1.2 Motivation and contribution

The picture fuzzy vector space framework provides a richer and more adaptable approach for modeling uncertainty and vagueness. Extending classical fuzzy set theory, this study integrates topological concepts into picture fuzzy vector spaces, enabling a deeper structural analysis. Fun-

damental characteristics including absolute convexity, balance, and absorbency are formally examined, and the notion of a convex balanced hull is established in this context. A series of propositions is presented to demonstrate the interdependence of these properties. In addition to its theoretical contributions, the work underscores the applicability of picture fuzzy topological vector spaces in diverse domains such as neural networks, image analysis, machine learning, and artificial intelligence.

1.3 Organization of this study

Section 2 introduces the foundational notions of picture fuzzy sets and picture fuzzy topological spaces. Building on this groundwork, Section 3 explores specialized categories of picture fuzzy spaces including convex, balanced, and absorbing types together with their core properties. Sections 4 and 5 focus on the concept of balance within picture fuzzy vector spaces, framed within the theoretical model illustrated in Figure 1.

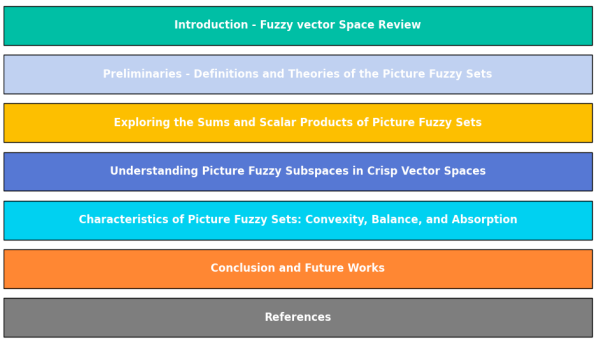


Figure 1. The workflow of this study.

2 Preliminaries

This section presents key concepts and propositions that are crucial for this research.

Definition 2.1. [7]A picture fuzzy set (PFS) $\mathcal{G}$ defined on a universe $\mathcal{X}$ is represented as

$$\mathcal{G} = \{(y, \gamma_{\mathcal{G}}(y), \eta_{\mathcal{G}}(y), \nu_{\mathcal{G}}(y)) \mid y \in \mathcal{X}\}.$$

Here, $\gamma_{\mathcal{G}}(y)$, $\eta_{\mathcal{G}}(y)$, and $\nu_{\mathcal{G}}(y)$ denote the degrees of positive, neutral, and negative membership of y in $\mathcal{X}$, respectively. These values satisfy the condition

$$0 \leq \gamma_{\mathcal{G}}(y) + \eta_{\mathcal{G}}(y) + \nu_{\mathcal{G}}(y) \leq 1, \quad \forall y \in \mathcal{X}.$$

The degree of refusal (indeterminacy) associated with y is given by

$$\rho_{\mathcal{G}}(y) = 1 - \big(\gamma_{\mathcal{G}}(y) + \eta_{\mathcal{G}}(y) + \nu_{\mathcal{G}}(y)\big).$$

Definition 2.2. [7]For any two picture fuzzy sets $\mathcal{G}$ and $\mathcal{H}$ over $\mathcal{X}$, the following hold:

(i) $\mathcal{G} \subseteq \mathcal{H}$ if and only if, for all $y \in \mathcal{X}$,

$$\gamma_{\mathcal{G}}(y) \leq \gamma_{\mathcal{H}}(y), \quad \eta_{\mathcal{G}}(y) \leq \eta_{\mathcal{H}}(y), \quad \nu_{\mathcal{G}}(y) \geq \nu_{\mathcal{H}}(y);$$

(ii) $\mathcal{G} = \mathcal{H}$ iff $\mathcal{G} \subseteq \mathcal{H}$ and $\mathcal{H} \subseteq \mathcal{G}$;

(iii) The union is defined as

$$\mathcal{G} \cup \mathcal{H} = \{(y, \max(\gamma_{\mathcal{G}}(y), \gamma_{\mathcal{H}}(y)), \min(\eta_{\mathcal{G}}(y), \eta_{\mathcal{H}}(y)), \min(\nu_{\mathcal{G}}(y), \nu_{\mathcal{H}}(y))) \mid y \in \mathcal{X}\};$$

(iv) The intersection is given by

$$\mathcal{G} \cap \mathcal{H} = \{(y, \min(\gamma_{\mathcal{G}}(y), \gamma_{\mathcal{H}}(y)), \min(\eta_{\mathcal{G}}(y), \eta_{\mathcal{H}}(y)), \max(\nu_{\mathcal{G}}(y), \nu_{\mathcal{H}}(y))) \mid y \in \mathcal{X}\};$$

(v) The complement is defined as

$$\mathcal{G}^c = \{(y, \nu_{\mathcal{G}}(y), \eta_{\mathcal{G}}(y), \gamma_{\mathcal{G}}(y)) \mid y \in \mathcal{X}\}.$$

Definition 2.3. [7]Some special classes of picture fuzzy sets are:

(i) A constant PFS $\widehat{(\gamma, \eta, \nu)} = \{(y, \gamma, \eta, \nu) \mid y \in \mathcal{X}\}$;

(ii) The universal set $1_{\mathcal{X}} = \widehat{(1, 0, 0)} = \{(y, 1, 0, 0) \mid y \in \mathcal{X}\}$;

(iii) The empty set $\phi = 0_{\mathcal{X}} = \widehat{(0, 0, 1)} = \{(y, 0, 0, 1) \mid y \in \mathcal{X}\}$.

Definition 2.4. [17] Consider fuzzy sets $\mathcal{G}_1, \ldots, \mathcal{G}_n$ defined on a universe $\mathfrak{E}$. We construct the product $\mathcal{G}_1 \times \cdots \times \mathcal{G}_n$ as the fuzzy set $\mathcal{G}$ on $\mathfrak{E}^n$ with membership function

$$\gamma_{\mathcal{G}}(x_1, \ldots, x_n) = \min\{\gamma_{\mathcal{G}_1}(x_1), \ldots, \gamma_{\mathcal{G}_n}(x_n)\}.$$

Next, define the mapping $f : \mathfrak{E}^n \to \mathfrak{E}$ by $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. Then the fuzzy sum $\mathcal{G}_1 + \cdots + \mathcal{G}_n$ is given as $f(\mathcal{G})$.

Finally, for any scalar λ and a fuzzy set $\mathcal{H}$ on $\mathfrak{E}$, the scalar multiple $\lambda\mathcal{H}$ as $g(\mathcal{H})$, where $g : \mathfrak{E} \to \mathfrak{E}$ is the map $g(x) = \lambda x$.

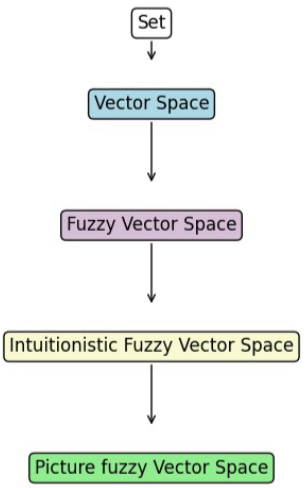


Figure 2. The workflow of the proposed model.

3 Exploring the Sum and Scalar Product of PFSs

Throughout this work, we let $\mathfrak{E}$ denote a vector space over a field $\mathcal{K}$, where $\mathcal{K}$ is taken to be either $\mathbb{R}$ (the real numbers) or $\mathbb{C}$ (the complex numbers).

Definition 3.1. Let $\mathcal{G}_1, \ldots, \mathcal{G}_n$ be picture fuzzy sets (PFSs) on a universe $\mathfrak{E}$. We construct the product set $\mathcal{G}_1 \times \cdots \times \mathcal{G}_n$, denoted by $\mathcal{G}$ in $\mathfrak{E}^n$, with the membership degrees defined as follows:

$$\gamma_{\mathcal{G}}(x_1, \ldots, x_n) = \max\{\gamma_{\mathcal{G}_1}(x_1), \ldots, \gamma_{\mathcal{G}_n}(x_n)\},$$
$$\eta_{\mathcal{G}}(x_1, \ldots, x_n) = \min\{\eta_{\mathcal{G}_1}(x_1), \ldots, \eta_{\mathcal{G}_n}(x_n)\},$$
$$\nu_{\mathcal{G}}(x_1, \ldots, x_n) = \min\{\nu_{\mathcal{G}_1}(x_1), \ldots, \nu_{\mathcal{G}_n}(x_n)\}.$$

Now, consider the mapping $f : \mathfrak{E}^n \to \mathfrak{E}$ defined by $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. We then define the sum of these sets as $\mathcal{G}_1 + \cdots + \mathcal{G}_n := f(\mathcal{G})$. Further, for a scalar λ and a PFS $\mathcal{H}$ in $\mathfrak{E}$, scalar multiplication is defined as $\lambda\mathcal{H} := g(\mathcal{H}), \quad$ where $g : \mathfrak{E} \to \mathfrak{E},\ g(x) = \lambda x$.

Example 3.2. Let $\mathcal{G}$, $\mathcal{H}$ be any two PFS in VS $\mathfrak{E}$ over the field $\mathcal{K}$ and $\lambda \in \mathcal{K}$ with [0,1]. $\lambda = (1, 0, 0)$

Table 1. Membership values

		$\mathcal{G}$	$\mathcal{H}$	$\mathcal{A}+\mathcal{B}$	$\lambda\mathcal{G}$	$\lambda\mathcal{H}$	$1_{\mathcal{X}}$	$0_{\mathcal{X}}$
	γ	0.3	0.3	0.3	0.3	0.3	1	0
x	η	0.2	0.3	0.2	0.2	0.3	0	0
	ν	0.3	0.4	0.3	0.3	0.4	0	1
	γ	0.6	0.4	0.6	0.6	0.4	1	0
y	η	0.2	0.4	0.2	0.2	0.4	0	0
	ν	0.1	0.2	0.1	0.1	0.2	0	1

Lemma 3.3. (a) Let $\mathcal{A}, \mathcal{B}$ be any two PFSs in a vector space $\mathfrak{E}$ over a field $\mathcal{K}$, and $\lambda \in \mathcal{K}$. Then we define $\mathcal{G} + \mathcal{H}$ and $\lambda\mathcal{G}$ as picture fuzzy sets in $\mathfrak{E}$ defined by
$(\mathcal{G} + \mathcal{H})(z) = \langle z, \vee_{x+y=z}\{\gamma_{\mathcal{G}}(x) \wedge \gamma_{\mathcal{H}}(y)\}, \wedge_{x+y=z}\{\eta_{\mathcal{G}}(x) \vee \eta_{\mathcal{H}}(y)\}, \wedge_{x+y=z}\{\nu_{\mathcal{G}}(x) \vee \nu_{\mathcal{H}}(y)\}\rangle$, and

$$\lambda\mathcal{G} = \left\langle x, \gamma_{\lambda\mathcal{G}(x)}, \eta_{\lambda\mathcal{G}(x)}, \nu_{\lambda\mathcal{G}(x)} \right\rangle.$$

where

$$\gamma_{\lambda\mathcal{G}(x)} = \begin{cases} \gamma_{\mathcal{G}}\left(\frac{1}{\lambda}x\right) & \text{if } \lambda \neq 0, \forall x \in \mathfrak{E}, \\ 0 & \text{if } \lambda = 0 \text{ and } x \neq 0, \\ \sup_{y \in \mathcal{X}} \gamma_{\mathcal{G}}(y) & \text{if } \lambda = 0 \text{ and } x = 0, \end{cases}$$

$$\eta_{\lambda\mathcal{G}(x)} = \begin{cases} \eta_{\mathcal{G}}\left(\frac{1}{\lambda}x\right) & \text{if } \lambda \neq 0, \forall x \in \mathfrak{E}, \\ 0 & \text{if } \lambda = 0 \text{ and } x \neq 0, \\ \inf_{y \in X} \eta_{\mathcal{G}}(y) & \text{if } \lambda = 0 \text{ and } x = 0. \end{cases}$$

and

$$\nu_{\lambda\mathcal{G}(x)} = \begin{cases} \nu_{\mathcal{G}}\left(\frac{1}{\lambda}x\right) & \text{if } \lambda \neq 0, \forall x \in \mathfrak{E}, \\ 0 & \text{if } \lambda = 0 \text{ and } x \neq 0, \\ \inf_{y \in X} \nu_{\mathcal{G}}(y) & \text{if } \lambda = 0 \text{ and } x = 0. \end{cases}$$

(b) For all scalars λ and all $x \in \mathfrak{E}$: $\gamma_{\lambda\mathcal{G}}(x) \leq \gamma_{\mathcal{G}}(x), \eta_{\lambda A}(x) \leq \eta_{\mathcal{G}}(x), \nu_{\lambda\mathcal{G}}(x) \geq \nu_{\mathcal{G}}(x)$.

Proposition 3.4. Let $\mathfrak{E}$ and $\mathfrak{F}$ be vector spaces over a field $\mathcal{K}$, and let $f : \mathfrak{E} \to \mathfrak{F}$ be a linear transformation. Then, for any PFSs $\mathcal{G}, \mathcal{H}$ in $\mathfrak{E}$ and for every scalar $\lambda \in \mathcal{K}$, the following hold:

(a) $f(\mathcal{G} + \mathcal{H}) = f(\mathcal{G}) + f(\mathcal{H})$,

(b) $f(\lambda\mathcal{G}) = \lambda f(\mathcal{G})$.

Proof. (a) Let $\mathtt{M} = \{f(x) : x \in \mathfrak{E}\}$ and fix $w \in \mathfrak{F}$. We compare

$$a = \left(\gamma_{f(\mathcal{A}+\mathcal{B})}(w),\ \eta_{f(\mathcal{A}+\mathcal{B})}(w),\ \nu_{f(\mathcal{A}+\mathcal{B})}(w)\right)$$

with

$$b = \left(\gamma_{f(\mathcal{G})+f(\mathcal{H})}(w),\ \eta_{f(\mathcal{G})+f(\mathcal{H})}(w),\ \nu_{f(\mathcal{G})+f(\mathcal{H})}(w)\right).$$

If $w \notin \mathtt{M}$, then clearly $a = 0$. Since any decomposition $w = x + y$ with $x, y \in \mathfrak{E}$ forces at least one of x, y outside $\mathtt{M}$, it follows that the values of γ, η, ν vanish, hence $b = 0 = a$.

Now assume $w \in \mathtt{M}$. For any $\epsilon > 0$, we can select $x \in \mathfrak{E}$ with $f(x) = w$ such that

$$\gamma_{f(\mathcal{A}+\mathcal{B})}(w),\ \eta_{f(\mathcal{A}+\mathcal{B})}(w),\ \nu_{f(\mathcal{A}+\mathcal{B})}(w) > a - \epsilon.$$

For any $x, y \in \mathfrak{E}$ satisfying $x + y = w$, one has

$$\max\{\gamma_{\mathcal{G}}(x), \gamma_{\mathcal{H}}(y)\},\ \min\{\eta_{\mathcal{G}}(x), \eta_{\mathcal{H}}(y)\},\ \min\{\nu_{\mathcal{G}}(x), \nu_{\mathcal{H}}(y)\} > a - \epsilon.$$

Because $f(x) + f(y) = w$, we obtain

$$\begin{aligned} b &\geq \max\{\gamma_{f(\mathcal{G})}(f(x)), \gamma_{f(\mathcal{H})}(f(y))\}, \\ &\quad \min\{\eta_{f(\mathcal{G})}(f(x)), \eta_{f(\mathcal{H})}(f(y))\}, \\ &\quad \min\{\nu_{f(\mathcal{G})}(f(x)), \nu_{f(\mathcal{H})}(f(y))\} \\ &\geq a - \epsilon. \end{aligned}$$

Since ϵ was arbitrary, $b \geq a$. By reversing the argument and considering decompositions $w = f(x_1) + f(x_2)$, one also obtains $a \geq b$. Thus $a = b$.

(b) Take $w \in \mathfrak{F}$ and compare

$$c = \big(\gamma_{f(\lambda\mathcal{G})}(w), \eta_{f(\lambda\mathcal{G})}(w), \nu_{f(\lambda\mathcal{G})}(w)\big), \quad d = \big(\gamma_{\lambda f(\mathcal{G})}(w), \eta_{\lambda f(\mathcal{G})}(w), \nu_{\lambda f(\mathcal{G})}(w)\big).$$

If $w \notin \mathtt{M}$, then $c = d = 0$. If $\lambda \neq 0$, then

$$c = \sup_{f(x)=\frac{1}{\lambda}w} \gamma_{\mathcal{G}}(x), \qquad \inf_{f(x)=\frac{1}{\lambda}w} \eta_{\mathcal{G}}(x), \qquad \inf_{f(x)=\frac{1}{\lambda}w} \nu_{\mathcal{G}}(x) = d.$$

If $\lambda = 0$ and $w \neq 0$, then $c = d = 0$. For $w = 0$, one has

$$c = \sup_{y\in\mathfrak{E}} \gamma_{\mathcal{G}}(y), \quad \inf_{y\in\mathfrak{E}} \eta_{\mathcal{G}}(y), \quad \inf_{y\in\mathfrak{E}} \nu_{\mathcal{G}}(y) = d.$$

□

Corollary 3.5. $\lambda(\mathcal{G} + \mathcal{H}) = \lambda\mathcal{G} + \lambda\mathcal{H}$ for all PFSs $\mathcal{A}, \mathcal{B} \in \mathcal{E}$ and all scalars λ.

Example 3.6. Let $\mathcal{G}, \mathcal{H}$ be any two PFS in VS $\mathfrak{E}$ over the field $\mathcal{K}$ and $\lambda \in \mathcal{K}$ with [0,1], $\lambda = 1, 0, 0$

Table 2. Membership values

		$\mathcal{G}$	$\mathcal{H}$	$\mathcal{A}+\mathcal{B}$	$\lambda(\mathcal{A}+\mathcal{B})$	$\lambda\mathcal{G}$	$\lambda\mathcal{H}$	$\lambda\mathcal{G}+\lambda\mathcal{H}$	$1_{\mathcal{X}}$	$0_{\mathcal{X}}$
	γ	0.7	0.3	0.7	0.7	0.7	0.3	0.7	1	0
x	η	0.1	0.3	0.1	0.1	0.1	0.3	0.1	0	0
	ν	0.2	0.3	0.2	0.2	0.2	0.3	0.2	0	1
	γ	0.6	0.5	0.6	0.6	0.6	0.5	0.6	1	0
y	η	0.2	0.3	0.2	0.2	0.2	0.3	0.2	0	0
	ν	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0	1
	γ	0.2	0.1	0.2	0.2	0.2	0.1	0.2	1	0
z	η	0.2	0.1	0.1	0.1	0.2	0.1	0.1	0	0
	ν	0.2	0.5	0.2	0.2	0.2	0.5	0.2	0	1

Lemma 3.7. Consider PFSs $\mathcal{G}_1, \ldots, \mathcal{G}_n, \mathcal{H}_1, \ldots, \mathcal{H}_m$ in $\mathfrak{E}$. Define $\mathcal{G} = \mathcal{G}_1 + \cdots + \mathcal{G}_n$ and $\mathcal{H} = \mathcal{H}_1 + \cdots + \mathcal{H}_m$. Then it follows that $\mathcal{G} + \mathcal{H} = \mathcal{G}_1 + \cdots + \mathcal{G}_n + \mathcal{H}_1 + \cdots + \mathcal{H}_m$. Moreover, a classical subset $\mathcal{G} \subseteq \mathfrak{E}$ may be viewed as a particular case of a PFS, where the membership function is simply the characteristic function of $\mathcal{G}$. For a set $\mathcal{G} \subseteq \mathfrak{E}$, the characteristic function $\chi_{\mathcal{G}} : \mathfrak{E} \to [0, 1]$ is defined by:

$$\chi_{\mathcal{G}}(x) = \begin{cases} 1, & \text{if } x \in \mathcal{G}, \\ 0, & \text{if } x \notin \mathcal{G}. \end{cases}$$

Consequently, sums such as $\mathcal{A} + \mathcal{B}$ remain meaningful even when one of $\mathcal{G}, \mathcal{H} \subseteq \mathfrak{E}$. $\forall\ x \in \mathfrak{E}$ and PFS $\mathcal{H}$, $x + \mathcal{H} := \{x\} + \mathcal{H}$. Define the translation map $f_x : \mathfrak{E} \to \mathfrak{E}$ by $f_x(y) = x + y$. With this, establish the subsequent definition.

Lemma 3.8. *Let $\mathcal{G}, \mathcal{H} \subseteq$ be a PFS $\mathfrak{E}$. Then*

(i) $x + \mathcal{H} = f_x(\mathcal{H})$,

(ii) $\gamma_{x+\mathcal{H}}(z) = \gamma_{\mathcal{H}}(z - x)$, $\eta_{x+\mathcal{H}}(z) = \eta_{\mathcal{H}}(z - x)$, $\nu_{x+\mathcal{H}}(z) = \nu_{\mathcal{H}}(z - x)$,

(iii) $\mathcal{G} + \mathcal{H} = \cup_{x\in\mathcal{G}}(x + \mathcal{H})$.

Proof. **1.** $x + \mathcal{H} = f_x(\mathcal{H})$

By definition, for $x \in \mathfrak{E}$, $x + \mathcal{H}$ is defined as $\{x\} + \mathcal{H}$. The function $f_x : \mathfrak{E} \to \mathfrak{E}$ is defined by $f_x(y) = x + y$ for all $y \in \mathfrak{E}$.

Now, consider the set $\mathcal{H}$ as a PFS. Then:

$$x + \mathcal{H} = \{x\} + \mathcal{H} = \{f_x(y) \mid y \in \mathcal{H}\}.$$

This is precisely the image of the PFS $\mathcal{H}$ under the function f_x, which by definition is $f_x(\mathcal{H})$. Therefore:

$$x + \mathcal{H} = f_x(\mathcal{H}).$$

2. $\gamma_{x+\mathcal{H}}(z) = \gamma_{\mathcal{H}}(z - x)$**,** $\eta_{x+\mathcal{H}}(z) = \eta_{\mathcal{H}}(z - x)$**,** $\nu_{x+\mathcal{H}}(z) = \nu_{\mathcal{H}}(z - x)$

The membership function $\gamma_{x+\mathcal{H}}$ at any element $z \in \mathfrak{E}$ corresponds to the membership function $\gamma_{\mathcal{H}}$ at $z - x$. To prove: Consider the element $z = f_x(y) = x + y$ in $x + \mathcal{H}$. The membership degree of z in $x + \mathcal{H}$ is:

$$\gamma_{x+\mathcal{H}}(z) = \gamma_{\mathcal{H}}(y),$$

where $y = z - x$. Hence:

$$\gamma_{x+\mathcal{H}}(z) = \gamma_{\mathcal{H}}(z - x).$$

Similarly, for the neutral membership function η and the non-membership function ν,

$$\eta_{x+\mathcal{H}}(z) = \eta_{\mathcal{H}}(z - x),$$

$$\nu_{x+\mathcal{H}}(z) = \nu_{\mathcal{H}}(z - x).$$

3. $\mathcal{G} + \mathcal{H} = \bigcup_{x\in\mathcal{G}}(x + \mathcal{H})$

For $\mathcal{G}, \mathcal{H} \subseteq \mathfrak{E}$ a PFS, the sum $\mathcal{G} + \mathcal{H}$ is defined as the union of $x + \mathcal{H}$ over all elements $x \in \mathcal{G}$. Formally:

$$\mathcal{G} + \mathcal{H} = \bigcup_{x\in\mathcal{G}} (x + \mathcal{H}).$$

To prove this:

Take any $z \in \mathcal{G} + \mathcal{H}$. By definition, $z = x + y$ for some $x \in \mathcal{G}$ and $y \in \mathcal{H}$. Since $z = x + y$, this means z belongs to $x + \mathcal{H}$ for that specific x. Hence, $z \in \bigcup_{x\in A}(x + \mathcal{H})$.

Conversely:

If $z \in \bigcup_{x\in\mathcal{G}}(x + \mathcal{H})$, then z belongs to $x + \mathcal{H}$ for some $x \in \mathcal{G}$. This implies $z = x + y$ for some $y \in \mathcal{H}$. Thus, z belongs to $\mathcal{G} + \mathcal{H}$.

Hence proves the equality $\mathcal{G} + \mathcal{H} = \bigcup_{x\in\mathcal{G}}(x + \mathcal{H})$. □

Proposition 3.9. Let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ be PFSs in $\mathfrak{E}$, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be scalars. The following assertions are equivalent:

(i) $\lambda_1\mathcal{G}_1 + \lambda_2\mathcal{G}_2 + \cdots + \lambda_n\mathcal{G}_n \subseteq \mathcal{G}$.

(ii) For all $x_1, x_2, \ldots, x_n$ in $\mathfrak{E}$,

$$\gamma_{\mathcal{G}}(\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_nx_n) \leq \max\{\gamma_{\mathcal{G}_1}(x_1), \gamma_{\mathcal{G}_2}(x_2), \ldots, \gamma_{\mathcal{G}_n}(x_n)\},$$

$$\eta_{\mathcal{G}}(\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_nx_n) \geq \min\{\eta_{\mathcal{G}_1}(x_1), \eta_{\mathcal{G}_2}(x_2), \ldots, \eta_{\mathcal{G}_n}(x_n)\},$$

$$\nu_{\mathcal{G}}(\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_nx_n) \geq \min\{\nu_{\mathcal{G}_1}(x_1), \nu_{\mathcal{G}_2}(x_2), \ldots, \nu_{\mathcal{G}_n}(x_n)\}.$$

Proof. (1) $\Longrightarrow$ (2)

$$\begin{aligned}
\gamma_{\mathcal{G}}(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n) &\leq \gamma_{\lambda_1\mathcal{G}_1+\lambda_2\mathcal{G}_2+\cdots+\lambda_n\mathcal{G}_n}(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n)\\
&\leq \max\{\gamma_{\lambda_1\mathcal{G}_1}(\lambda_1x_1),\gamma_{\lambda_2\mathcal{G}_2}(x_2),\ldots,\gamma_{\lambda_n\mathcal{G}_n}(\lambda_nx_n)\}\\
&\leq \max\{\gamma_{\mathcal{G}_1}(x_1),\gamma_{\mathcal{G}_2}(x_2),\ldots,\gamma_{\mathcal{G}_n}(x_n)\},\\
\eta_{\mathcal{G}}(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n) &\geq \eta_{\lambda_1\mathcal{G}_1+\lambda_2\mathcal{G}_2+\cdots+\lambda_n\mathcal{G}_n}(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n)\\
&\geq \min\{\eta_{\lambda_1\mathcal{G}_1}(\lambda_1x_1),\eta_{\lambda_2\mathcal{G}_2}(x_2),\ldots,\eta_{\lambda_n\mathcal{G}_n}(\lambda_nx_n)\}\\
&\geq \min\{\eta_{\mathcal{G}_1}(x_1),\eta_{\mathcal{G}_2}(x_2),\ldots,\eta_{\mathcal{G}_n}(x_n)\},\\
\nu_{\mathcal{G}}(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n) &\geq \nu_{\lambda_1\mathcal{G}_1+\lambda_2\mathcal{G}_2+\cdots+\lambda_n\mathcal{G}_n}(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n)\\
&\geq \min\{\nu_{\lambda_1\mathcal{G}_1}(\lambda_1x_1),\nu_{\lambda_2\mathcal{G}_2}(x_2),\ldots,\nu_{\lambda_n\mathcal{G}_n}(\lambda_nx_n)\}\\
&\geq \min\{\nu_{\mathcal{G}_1}(x_1),\nu_{\mathcal{G}_2}(x_2),\ldots,\nu_{\mathcal{G}_n}(x_n)\}.
\end{aligned}$$

(2) $\Longrightarrow$ (1)
Assume that $\lambda_i \neq 0$ for each $i = 1,\ldots,k$, while $\lambda_i = 0$ holds for all indices $k < i \leq n$. Let $x_1,\ldots,x_k$ be chosen elements of $\mathfrak{E}$. Then, for every selection of $y_1,\ldots,y_{n-k} \in \mathfrak{E}$,

$$\gamma_{\mathcal{G}}(\lambda_1x_1+\cdots+\lambda_ky_k) \leq \max\{\gamma_{\mathcal{G}}(x_1),\ldots,\gamma_{\mathcal{G}}(x_k),\gamma_{\mathcal{G}_{k+1}}(y_1),\ldots,\gamma_{\mathcal{G}_n}(y_{n-k})\},$$

$$\eta_{\mathcal{G}}(\lambda_1x_1+\cdots+\lambda_ky_k) \geq \min\{\eta_{\mathcal{G}}(x_1),\ldots,\eta_{\mathcal{G}}(x_k),\eta_{\mathcal{G}_{k+1}}(y_1),\ldots,\eta_{\mathcal{G}_n}(y_{n-k})\},$$

$$\nu_{\mathcal{G}}(\lambda_1x_1+\cdots+\lambda_ky_k) \geq \min\{\nu_{\mathcal{G}}(x_1),\ldots,\nu_{\mathcal{G}}(x_k),\nu_{\mathcal{G}_{k+1}}(y_1),\ldots,\nu_{\mathcal{G}_n}(y_{n-k})\}.$$

Since $\gamma_{0\mathcal{G}_j}(0) = \sup_{y\in\mathfrak{E}}\gamma_{\mathcal{G}}(y)$, $\eta_{0\mathcal{G}_j}(0) = \inf_{y\in\mathfrak{E}}\eta_{\mathcal{G}}(y)$, $\nu_{0\mathcal{G}_j}(0) = \inf_{y\in E}\nu_{\mathcal{G}}(y)$, we get

$$\gamma_{\mathcal{G}}(\lambda_1x_1+\cdots+\lambda_ky_k) \leq \max\{\gamma_{\mathcal{G}}(x_1),\ldots,\gamma_{\mathcal{G}}(x_k),\gamma_{0\mathcal{G}_{k+1}}(0),\ldots,\gamma_{0\mathcal{G}_n}(0)\},$$

$$\eta_{\mathcal{G}}(\lambda_1x_1+\cdots+\lambda_ky_k) \geq \min\{\eta_{\mathcal{G}}(x_1),\ldots,\eta_{\mathcal{G}}(x_k),\eta_{0\mathcal{G}_{k+1}}(0),\ldots,\eta_{0\mathcal{G}_n}(0)\},$$

$$\nu_{\mathcal{G}}(\lambda_1x_1+\cdots+\lambda_ky_k) \geq \min\{\nu_{\mathcal{G}}(x_1),\ldots,\nu_{\mathcal{G}}(x_k),\nu_{0\mathcal{G}_{k+1}}(0),\ldots,\nu_{0\mathcal{G}_n}(0)\}.$$

Now,

$$\begin{aligned}
\gamma_{\lambda_1\mathcal{G}_1+\cdots+\lambda_n\mathcal{G}_n}(z) &= \sup_{x_1+\cdots+x_k=z}[\max\{\gamma_{\lambda_1\mathcal{G}_1}(x_1),\ldots,\gamma_{\lambda_n\mathcal{G}_n}(x_n)\}]\\
&= \sup_{x_1+\cdots+x_k=z}\left[\max\{\gamma_{\lambda_1\mathcal{G}_1}(x_1),\ldots,\gamma_{\lambda_k\mathcal{G}_k}(x_k),\gamma_{0\mathcal{G}_{k+1}}(0),\ldots,\gamma_{0\mathcal{G}_n}(0)\}\right]\\
&= \sup_{x_1+\cdots+x_k=z}\left[\max\{\gamma_{\lambda_1\mathcal{G}_1}(\frac{1}{\lambda_1}x_1),\ldots,\gamma_{\lambda_k\mathcal{G}_k}(\frac{1}{\lambda_k}x_k),\gamma_{0\mathcal{G}_{k+1}}(0),\ldots,\gamma_{0\mathcal{G}_n}(0)\}\right]\\
&\leq \sup_{x_1+\cdots+x_k=z}\gamma_{\mathcal{G}}\left(\lambda_1\frac{1}{\lambda_1}x_1+\cdots+\lambda_k\frac{1}{\lambda_k}x_k\right)\\
&= \gamma_{\mathcal{G}}(z),
\end{aligned}$$

$$\begin{aligned}
\eta_{\lambda_1\mathcal{G}_1+\cdots+\lambda_n\mathcal{G}_n}(z) &= \inf_{x_1+\cdots+x_k=z}[\min\{\eta_{\lambda_1\mathcal{G}_1}(x_1),\ldots,\eta_{\lambda_n\mathcal{G}_n}(x_n)\}]\\
&= \inf_{x_1+\cdots+x_k=z}\left[\min\{\eta_{\lambda_1\mathcal{G}_1}(x_1),\ldots,\eta_{\lambda_k\mathcal{G}_k}(x_k),\eta_{0\mathcal{G}_{k+1}}(0),\ldots,\eta_{0\mathcal{G}_n}(0)\}\right]\\
&\geq \eta_{\mathcal{G}}(z),
\end{aligned}$$

$$\begin{aligned}
\nu_{\lambda_1\mathcal{G}_1+\cdots+\lambda_n\mathcal{G}_n}(z) &= \inf_{x_1+\cdots+x_k=z}[\min\{\nu_{\lambda_1\mathcal{G}_1}(x_1),\ldots,\nu_{\lambda_n\mathcal{G}_n}(x_n)\}]\\
&= \inf_{x_1+\cdots+x_k=z}\left[\min\{\nu_{\lambda_1\mathcal{G}_1}(x_1),\ldots,\nu_{\lambda_k\mathcal{G}_k}(x_k),\nu_{0\mathcal{G}_{k+1}}(0),\ldots,\nu_{0\mathcal{G}_n}(0)\}\right]\\
&\geq \nu_{\mathcal{G}}(z),
\end{aligned}$$

This completes the proof. □

Lemma 3.10. Let $\mathcal{G}$ and $\mathcal{H}$ be picture fuzzy subsets (PFSSs) of E. Then the following statements hold:

(i) $\mathcal{G} + 0\mathcal{H}$ is a subset of $\mathcal{G}$;

(ii) $\mathcal{G} + 0\mathcal{H}$ equals $\mathcal{G}$ iff $\sup_{x\in\mathfrak{E}} \gamma_{\mathcal{G}}(x) \leq \sup_{x\in\mathfrak{E}} \gamma_{\mathcal{H}}(x)$.

Proof. (1) To show this, we consider:

$$\begin{aligned}
\gamma_{\mathcal{G}}(x + 0y) &= \gamma_{\mathcal{G}}(x) \leq \max\{\gamma_{\mathcal{G}}(x), \gamma_{\mathcal{H}}(y)\},\\
\eta_{\mathcal{G}}(x + 0y) &= \eta_{\mathcal{G}}(x) \geq \min\{\eta_{\mathcal{G}}(x), \eta_{\mathcal{H}}(y)\},\\
\nu_{\mathcal{G}}(x + 0y) &= \nu_{\mathcal{G}}(x) \geq \min\{\nu_{\mathcal{G}}(x), \nu_{\mathcal{H}}(y)\}.
\end{aligned}$$

Therefore, this confirms statement (1) according to Proposition 3.9.

(2) Assume that $\sup \gamma_{\mathcal{G}}(x) \leq \sup \gamma_{\mathcal{H}}(x) = \gamma_{\mathcal{H}}(0)$, $\sup \eta_{\mathcal{G}}(x) \leq \sup \eta_{\mathcal{H}}(x) = \eta_{\mathcal{H}}(0)$, and $\sup \nu_{\mathcal{G}}(x) \leq \sup \nu_{\mathcal{H}}(x) = \nu_{\mathcal{H}}(0)$. Then we can derive:

$$\begin{aligned}
\gamma_{A+0B}(z) &= \sup_{x+y=z}[\max\{\gamma_{\mathcal{G}}(x), \gamma_0 B(y)\}] = \max\{\gamma_{\mathcal{G}}(z), \gamma_0 B(0)\} = \gamma_{\mathcal{G}}(z),\\
\eta_{A+0B}(z) &= \inf_{x+y=z}[\min\{\eta_{\mathcal{G}}(x), \eta_0 B(y)\}] = \min\{\eta_{\mathcal{G}}(z), \eta_0 B(0)\} = \eta_{\mathcal{G}}(z),\\
\nu_{A+0B}(z) &= \inf_{x+y=z}[\min\{\nu_{\mathcal{G}}(x), \nu_0 B(y)\}] = \min\{\nu_{\mathcal{G}}(z), \nu_0 B(0)\} = \nu_{\mathcal{G}}(z).
\end{aligned}$$

On the other hand, if $\gamma_{\mathcal{G}}(z) \geq \sup \gamma_{\mathcal{H}}(x) = \gamma_{\mathcal{H}}(0)$, $\eta_{\mathcal{G}}(z) \geq \sup \eta_{\mathcal{H}}(x) = \eta_{\mathcal{H}}(0)$, and $\nu_{\mathcal{G}}(z) \leq \sup \nu_{\mathcal{H}}(x) = \nu_{\mathcal{H}}(0)$ for some z, then we have:

$$\begin{aligned}
\gamma_{\mathcal{G}+0\mathcal{H}}(z) &= \max\{\gamma_{\mathcal{G}}(z), \gamma_0 \mathcal{H}(0)\} \leq \gamma_{\mathcal{G}}(z),\\
\eta_{\mathcal{G}+0\mathcal{H}}(z) &= \min\{\eta_{\mathcal{G}}(z), \eta_0 \mathcal{H}(0)\} \leq \eta_{\mathcal{G}}(z),\\
\nu_{\mathcal{G}+0\mathcal{H}}(z) &= \min\{\nu_{\mathcal{G}}(z), \nu_0 \mathcal{H}(0)\} \leq \nu_{\mathcal{G}}(z).
\end{aligned}$$

From this, we conclude that $\mathcal{G} + 0\mathcal{H}$ is not equal to $\mathcal{G}$. □

4 Understanding Picture Fuzzy Subspaces in Crisp Vector Spaces

Definition 4.1. A picture fuzzy set $\mathfrak{F}$ in $\mathfrak{E}$ is referred to as a picture fuzzy subspace if it satisfies the following conditions:

(i) The sum of any two elements from $\mathfrak{F}$ is also an element of $\mathfrak{F}$ (i.e., $\mathfrak{F} + \mathfrak{F} \subseteq \mathfrak{F}$);

(ii) For any scalar λ, multiplying $\mathfrak{F}$ by λ results in a set that is still contained within $\mathfrak{F}$ (i.e., $\lambda\mathfrak{F} \subseteq \mathfrak{F}$).

Example 4.2. Let $\mathcal{G}$ be any PFS in VS $\mathfrak{E}$ over the field $\mathcal{K}$ and $\lambda \in \mathcal{K}$ with [0,1], $\lambda = 1, 0, 0$.

Table 3. Membership values

		$\mathcal{G}$	$\mathcal{A}+\mathcal{A}$	$\lambda\mathcal{G}$	$1_{\mathcal{X}}$	$0_{\mathcal{X}}$
	γ	0.3	0.3	0.3	1	0
x	η	0.2	0.2	0.2	0	0
	ν	0.4	0.4	0.4	0	1
	γ	0.4	0.4	0.4	1	0
y	η	0.3	0.3	0.3	0	0
	ν	0.2	0.2	0.2	0	1
	γ	0.5	0.5	0.5	1	0
y	η	0.2	0.2	0.2	0	0
	ν	0.2	0.2	0.2	0	1

Lemma 4.3. Let $\mathfrak{F}$ be a PFS in $\mathfrak{E}$. The following conditions are hold:

(i) $\mathfrak{F} \subseteq \mathfrak{E}$;

(ii) $\forall$ scalars $k, m \in \mathcal{K}$, one has $k\mathfrak{F} + m\mathfrak{F} \subseteq \mathfrak{F}$;

(iii) $\forall$ scalars k, m and for every choice of $x, y \in \mathfrak{E}$, the corresponding combination lies within $\mathfrak{F}$.

$$\gamma_{\mathfrak{F}}(kx + my) \leq \max\{\gamma_{\mathfrak{F}}(x), \gamma_{\mathfrak{F}}(y)\},$$
$$\eta_{\mathfrak{F}}(kx + my) \geq \min\{\eta_{\mathfrak{F}}(x), \eta_{\mathfrak{F}}(y)\},$$
$$\nu_{\mathfrak{F}}(kx + my) \geq \min\{\nu_{\mathfrak{F}}(x), \nu_{\mathfrak{F}}(y)\}.$$

Proof. It is clear that (1) implies (2). Furthermore, (2) and (3) are equivalent due to Proposition 3.9.
For (2) implying (1):

$$\mathfrak{F} + \mathfrak{F} = 1\mathfrak{F} + 1\mathfrak{F} \subseteq \mathfrak{F},$$
$$k\mathfrak{F} = k\mathfrak{F} + 0\mathfrak{F} \subseteq \mathfrak{F}.$$

□

Proposition 4.4. Let $\mathfrak{E}$ and $\mathfrak{F}$ be VS over the field $\mathcal{K}$, and let $f : \mathfrak{E} \to \mathfrak{F}$ be a linear transformation. If $\mathcal{G}$ is a picture fuzzy subspace (PFSS) of $\mathfrak{E}$, then its image $f(\mathcal{G})$ forms a PFSS in $\mathfrak{F}$. Similarly, if $\mathcal{H}$ is a PFSS of $\mathfrak{F}$, then the preimage $f^{-1}(\mathcal{H})$ is a PFSS of $\mathfrak{E}$.

Proof. Consider scalars $k, m \in \mathcal{K}$. Then $kf(\mathcal{G}) + mf(\mathcal{G}) = f(k\mathcal{G} + m\mathcal{G}) \subseteq f(\mathcal{G})$, which shows that $f(\mathcal{G})$ is closed under linear combinations and hence a PFSS of $\mathfrak{F}$. For the preimage, take $\mathcal{H} \subseteq \mathfrak{F}$. For any $x, y \in \mathfrak{E}$ and scalars k, m,

$$\begin{aligned}\gamma_{f^{-1}(\mathcal{H})}(kx + my) &= \gamma_{\mathcal{H}}(f(kx + my)) \\ &= \gamma_{\mathcal{H}}(kf(x) + mf(y)) \\ &\leq \max\{\gamma_{\mathcal{H}}(f(x)), \gamma_{\mathcal{H}}(f(y))\} \\ &= \max\{\gamma_{f^{-1}(\mathcal{H})}(x), \gamma_{f^{-1}(\mathcal{H})}(y)\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\eta_{f^{-1}(\mathcal{H})}(kx + my) &= \eta_{\mathcal{H}}(f(kx + my)) \\ &= \eta_{\mathcal{H}}(kf(x) + mf(y)) \\ &\geq \min\{\eta_{\mathcal{H}}(f(x)), \eta_{\mathcal{H}}(f(y))\} \\ &= \min\{\eta_{f^{-1}(\mathcal{H})}(x), \eta_{f^{-1}(\mathcal{H})}(y)\},\end{aligned}$$

and

$$\begin{aligned}\nu_{f^{-1}(\mathcal{H})}(kx + my) &= \nu_{\mathcal{H}}(f(kx + my)) \\ &= \nu_{\mathcal{H}}(kf(x) + mf(y)) \\ &\geq \min\{\nu_{\mathcal{H}}(f(x)), \nu_{\mathcal{H}}(f(y))\} \\ &= \min\{\nu_{f^{-1}(\mathcal{H})}(x), \nu_{f^{-1}(\mathcal{H})}(y)\}.\end{aligned}$$

Thus, by Definition 4.3, the set $f^{-1}(\mathcal{H})$ is indeed a picture fuzzy subspace of $\mathfrak{E}$. □

Definition 4.5. If $\mathcal{G}, \mathcal{H}$ are PFSSs of $\mathfrak{E}$ and $k \in \mathcal{K}$, then both the sum $\mathcal{G} + \mathcal{H}$ and the scalar multiple $k\mathcal{G}$ are also PFSSs of $\mathfrak{E}$.

Proposition 4.6. Consider a collection of PFSSs $\{\mathcal{G}_i\}_{i \in I}$ of $\mathfrak{E}$. The intersection $\mathcal{G} = \bigcap_{i \in I} \mathcal{G}_i$ forms a PFSS as well.

Proof. For any scalars m and k:

$$\begin{aligned}\gamma_{\mathcal{G}}(mx+ky) &= \inf_{i\in I}\gamma_{\mathcal{G}}(mx+ky)\\ &\geq \inf_{i\in I}[\min\{\gamma_{\mathcal{G}_i}(x),\gamma_{\mathcal{G}_i}(y)\}]\\ &\geq \min\left\{\inf_{i\in I}\gamma_{\mathcal{G}_i}(x),\inf_{i\in I}\gamma_{\mathcal{G}_i}(y)\right\}\\ &= \min\{\gamma_{\mathcal{G}}(x),\gamma_{\mathcal{G}}(y)\}.\end{aligned}$$

Similarly, for the nertural membership function η:

$$\eta_{\mathcal{G}}(mx+ky) = \min\{\eta_{\mathcal{G}}(x),\eta_{\mathcal{G}}(y)\}.$$

For the negative membership function ν:

$$\nu_{\mathcal{G}}(mx+ky) = \max\{\nu_{\mathcal{G}}(x),\nu_{\mathcal{G}}(y)\}.$$

Therefore, the conclusion follows from Definition 4.3. □

5 Characteristics of Picture Fuzzy Sets: Convexity, Balance, and Absorption

Definition 5.1. Let $\mathfrak{E}$ be a vector space over the field $\mathcal{K}$. A PFS $\mathcal{G}$ in $\mathfrak{E}$ is referred to as a convex PFS if, for every λ in $\mathcal{K}$ where $0 \leq \lambda \leq 1$, the following conditions hold:

$$\begin{aligned}\gamma_{\mathcal{G}}(\lambda x+(1-\lambda)y) &= \{\gamma_{\mathcal{G}}(x)\},\\ \eta_{\mathcal{G}}(\lambda x+(1-\lambda)y) &= \{\eta_{\mathcal{G}}(x)\},\\ \nu_{\mathcal{G}}(\lambda x+(1-\lambda)y) &= \{\nu_{\mathcal{G}}(x)\}.\end{aligned}$$

Example 5.2. Let $\mathcal{G} = (x, 0.4, 0.1, 0.3)$ be any PFS in VS $\mathfrak{E}$ over the field $\mathcal{K}$ and $\lambda \in \mathcal{K}$ with [0,1]. $\lambda = 0.5, 0.1, 0.1$ convex PFS (x, 0.4, 0.1, 0.3)=(x, 0.4, 0.1, 0.3)

Definition 5.3. A PFS $\mathcal{G}$ in $\mathfrak{E}$ is classified as:

(a) Balanced: This holds if $k\mathcal{G} \subseteq \mathcal{G}$ for every scalar k such that $|k| \leq 1$;

(b) Absorbing: This occurs if $\mathcal{G} = \bigcup_{k>0} k\mathcal{G}$.

Proposition 5.4. Let $\mathcal{G}$ be a picture fuzzy subset (PFS) of the vector space $\mathfrak{E}$. The following statements are equivalent:

(1) $\mathcal{G}$ is convex (or balanced).

(2) For every $x, y \in \mathfrak{E}$ and for any scalar $\lambda \in [0,1]$, the following conditions hold:

$$\gamma_{\mathcal{G}}(kx) \leq \gamma_{\mathcal{G}}(x), \quad \eta_{\mathcal{G}}(kx) \leq \eta_{\mathcal{G}}(x), \quad \nu_{\mathcal{G}}(kx) \leq \nu_{\mathcal{G}}(x), \quad \forall |k| \leq 1.$$

Moreover,

$$\begin{aligned}\gamma_{\mathcal{G}}(\lambda x+(1-\lambda)y) &\leq \max\{\gamma_{\mathcal{G}}(x),\gamma_{\mathcal{G}}(y)\},\\ \eta_{\mathcal{G}}(\lambda x+(1-\lambda)y) &\geq \min\{\eta_{\mathcal{G}}(x),\eta_{\mathcal{G}}(y)\},\\ \nu_{\mathcal{G}}(\lambda x+(1-\lambda)y) &\geq \min\{\nu_{\mathcal{G}}(x),\nu_{\mathcal{G}}(y)\}.\end{aligned}$$

(3) For every $d \in [0,1]$, the d-level set

$$\mathcal{G}_d = \{z \in \mathfrak{E} : \gamma_{\mathcal{G}}(z) \leq d,\ \eta_{\mathcal{G}}(z) \geq d,\ \nu_{\mathcal{G}}(z) \geq d\}$$

is convex.

Proposition 5.5. Let $\mathcal{G}$ be a PFS of the space $\mathfrak{E}$.

(1) $\mathcal{G}$ is an absorbing set.

(2) For every $x \in \mathfrak{E}$, the following hold:

$$\sup_{k>0} \gamma_{\mathcal{G}}(kx) = 1, \qquad \inf_{k>0} \eta_{\mathcal{G}}(kx) = 0, \qquad \inf_{k>0} \nu_{\mathcal{G}}(kx) = 0.$$

(3) For each threshold $d \in (0, 1]$, the level set

$$\mathcal{G}_d = \{x \in \mathfrak{E} : \gamma_{\mathcal{G}}(x) \geq d,\ \eta_{\mathcal{G}}(x) \geq d,\ \nu_{\mathcal{G}}(x) \geq d\}$$

is also absorbing.

Proposition 5.6. Let $\mathfrak{E}$ and $\mathfrak{F}$ be VS over a field $\mathcal{K}$, and let $f : \mathfrak{E} \to \mathfrak{F}$ be a linear mapping. If $\mathcal{G}$ is a convex PFS of $\mathfrak{E}$, then its image $f(\mathcal{G})$ is also a convex (or balanced) PFS in $\mathfrak{F}$. Similarly, if $\mathcal{H}$ is a convex (or balanced) PFS of $\mathfrak{F}$, then the preimage $f^{-1}(\mathcal{H})$ is a convex (or balanced) PFS of $\mathfrak{E}$.

Proof. We present the argument for convexity, noting that the balanced case is handled in the same manner.

Suppose $\mathcal{G}$ is convex in $\mathfrak{E}$. For any $k \in [0, 1]$, we have

$$kf(\mathcal{G}) + (1 - k)f(\mathcal{G}) = f(k\mathcal{G} + (1 - k)\mathcal{G}) \subseteq f(\mathcal{G}),$$

which shows that $f(\mathcal{G})$ is convex in $\mathfrak{F}$.

Conversely, let $\mathcal{H}$ be convex in $\mathfrak{F}$. For $k \in [0, 1]$, set

$$\mathtt{M} = kf^{-1}(\mathcal{H}) + (1 - k)f^{-1}(\mathcal{H}).$$

Applying f gives

$$f(\mathtt{M}) = kf(f^{-1}(\mathcal{H})) + (1 - k)f(f^{-1}(\mathcal{H})) \subseteq k\mathcal{H} + (1 - k)\mathcal{H} \subseteq \mathcal{H}.$$

Hence $\mathtt{M} \subseteq f^{-1}(\mathcal{H})$, proving that $f^{-1}(\mathcal{H})$ is convex. □

Proposition 5.7. Let $f : \mathfrak{E} \to \mathfrak{F}$ be a linear mapping. If $\mathcal{G}$ is an absorbing picture fuzzy subset (PFS) of $\mathfrak{F}$, then the preimage $f^{-1}(\mathcal{G})$ is also an absorbing PFS of $\mathfrak{E}$.

Proposition 5.8. If $\mathcal{G}$ and $\mathcal{H}$ are convex (or balanced) picture fuzzy subsets of $\mathfrak{E}$, then their sum $\mathcal{G} + \mathcal{H}$ is likewise a convex (or balanced) PFS in $\mathfrak{E}$.

Proposition 5.9. Let $\{\mathcal{G}_i\}_{i \in I}$ be a collection of convex (or balanced) picture fuzzy subsets of $\mathfrak{E}$. Then their intersection

$$\mathcal{G} = \bigcap_{i \in I} \mathcal{G}_i$$

remains a convex (or balanced) PFS in $\mathfrak{E}$.

Proof. Consider any $d \in [0, 1]$. Then

$$\begin{aligned} &\{x \in \mathfrak{E} : \gamma_{\mathcal{G}}(x), \eta_{\mathcal{G}}(x), \nu_{\mathcal{G}}(x) \geq d\} \\ &\quad = \{x \in \mathfrak{E} : \min_{i \in I} \gamma_{A_i}(x) \geq d,\ \min_{i \in I} \eta_{A_i}(x) \geq d,\ \max_{i \in I} \nu_{A_i}(x) \geq d\}. \end{aligned}$$

Because the intersection of ordinary convex (or balanced) sets is also convex (or balanced), the assertion follows directly from Proposition 5.4. □

Proposition 5.10. For a PFS $\mathcal{G}$ in $\mathfrak{E}$, its balanced hull is given by

$$\mathcal{H} = \bigcup_{|k| \leq 1} k\mathcal{G}.$$

Proof. Any balanced PFS containing $\mathcal{G}$ must also contain $\mathcal{H}$. Since clearly $\mathcal{G} \subseteq \mathcal{H}$, it remains to verify that $\mathcal{H}$ is balanced. Let $k \in \mathcal{K}$ with $|k| \leq 1$ and $x \in \mathfrak{E}$. Then:

$$\gamma_{\mathcal{H}}(x) = \sup_{|a|\leq 1} \gamma_{\mathcal{G}}(ax) \geq \sup_{|a|\leq 1} \gamma_{\mathcal{G}}(kax) = \gamma_{\mathcal{H}}(kx),$$
$$\eta_{\mathcal{H}}(x) = \inf_{|a|\leq 1} \eta_{\mathcal{G}}(ax) \geq \inf_{|a|\leq 1} \eta_{\mathcal{G}}(kax) = \eta_{\mathcal{H}}(kx),$$
$$\nu_{\mathcal{H}}(x) = \inf_{|a|\leq 1} \nu_{\mathcal{G}}(ax) \geq \inf_{|a|\leq 1} \nu_{\mathcal{G}}(kax) = \nu_{\mathcal{H}}(kx).$$

Hence, $k\mathcal{H} \subseteq \mathcal{H}$. By Proposition 3.9, $\mathcal{H}$ is balanced. □

Proposition 5.11. Let $\mathcal{G}$ be a PFSS in $\mathfrak{E}$. Its convex hull is given by

$$\mathcal{H} = \bigcup \left\{ \lambda_1 \mathcal{G} + \cdots + \lambda_n \mathcal{G} : \lambda_i \geq 0,\ \sum \lambda_i = 1 \right\}.$$

Proof. We observe that $\mathcal{G} \subseteq \mathcal{H} \subseteq \text{co}(\mathcal{G})$. To complete the proof, it suffices to show $\mathcal{H}$ is convex. Note that

$$\gamma\Big(\sum_{i=1}^{n} \lambda_i \mathcal{G}_i \Big) = \sum_{i=1}^{n} \lambda_i,$$

and

$$\Big(\bigcup_{i=1}^{n} \mathcal{G}_i \Big) + \Big(\bigcup_{j=1}^{m} \mathcal{H}_j \Big) \subseteq \bigcup_{i,j} (\mathcal{G}_i + \mathcal{H}_j).$$

Let $a \in [0,1]$ with $\sum_{i=1}^{n} \lambda_i = 1$, $\sum_{j=1}^{m} k_j = 1$, and $\lambda_i, k_j \geq 0$. Then

$$a\Big(\sum_{i=1}^{n} \lambda_i \mathcal{G}_i \Big) + (1-a)\Big(\sum_{j=1}^{m} k_j \mathcal{H}_j \Big) = a\lambda_1 \mathcal{G}_1 + \cdots + a\lambda_n \mathcal{G}_n + (1-a)k_1 \mathcal{H}_1 + \cdots + (1-a)k_m \mathcal{H}_m \subseteq \mathcal{H},$$

which shows that $a\mathcal{H} + (1-a)\mathcal{H} \subseteq \mathcal{H}$. Hence, $\mathcal{H}$ is convex. □

Proposition 5.12. Let $\mathcal{G}$ be a picture fuzzy subset (PFS) of $\mathfrak{E}$.

(1) $\mathcal{G}$ is absolutely convex.

(2) $\forall$ scalars b, c with $|b| + |c| \leq 1$, the inclusion

$$b\mathcal{G} + c\mathcal{G} \subseteq \mathcal{G}$$

holds.

(3) $\forall\ x, y \in \mathfrak{E}$ and scalars b, c such that $|c| + |d| \leq 1$, the following inequalities are satisfied:

$$\gamma_{\mathcal{G}}(bx + cy) \leq \max\{\gamma_{\mathcal{G}}(x),\ \gamma_{\mathcal{G}}(y)\},$$
$$\eta_{\mathcal{G}}(bx + cy) \geq \min\{\eta_{\mathcal{G}}(x),\ \eta_{\mathcal{G}}(y)\},$$
$$\nu_{\mathcal{G}}(bx + cy) \geq \min\{\nu_{\mathcal{G}}(x),\ \nu_{\mathcal{G}}(y)\}.$$

(4) $\forall\ d \in [0,1]$, the level set

$$\mathcal{G}_d = \{x \in \mathfrak{E} : \gamma_{\mathcal{G}}(x) \geq d,\ \eta_{\mathcal{G}}(x) \geq d,\ \nu_{\mathcal{G}}(x) \geq d\}$$

is an absolutely convex crisp subset of $\mathfrak{E}$.

Proposition 5.13. The convex-balanced hull of a PFS $\mathcal{G}$ can be expressed as

$$\bigcup \left\{ \lambda_1 \mathcal{G} + \cdots + \lambda_n \mathcal{G} : \sum \lambda_i \leq 1 \right\}.$$

6 Validation of the Proposed Method

To assess the effectiveness of the proposed framework, we compared it with several classical and recent approaches in fuzzy vector space theory. The studies of Katsaras and Liu [17], Buckley and Yan [19], Dogra and Pal [25], Gereme et al. [22], and Kungumaraj et al. [23] were considered as representative references for benchmarking.

6.1 Comparative Analysis

A comparative summary of the proposed method and related models is presented in Table 4. The table highlights their key ideas, scope of applicability, limitations, and overall contributions.

Table 4. Comparison of the proposed method with existing models

Method / Reference	Year	Key Idea	Applicability	Limitations
Katsaras & Liu [17]	1977	Classical fuzzy vector spaces	Theoretical foundation	Restricted to abstract setting
Buckley & Yan [19]	1999	Fuzzy topological vector spaces over $\mathbb{R}$	Moderate applicability	Limited generalization
Dogra & Pal [25]	2023	Picture fuzzy subspaces	Decision-making and coding theory	Computationally demanding
Gereme et al. [22]	2023	Binary convex fuzzy vector spaces	Applications in coding theory	Narrow application scope
Kungumaraj et al. [23]	2024–25	Neutrosophic TVS	Real-world decision problems	Requires refinement

6.2 Discussion

The comparison in Table indicates that the proposed framework provides a more generalized structure than both classical fuzzy vector space models [17, 19] and picture fuzzy formulations [25]. While earlier works laid significant theoretical and applied foundations, their applicability was either too abstract or confined to narrow problem domains. By incorporating both fuzzy and neutrosophic features, the proposed approach achieves broader scope and greater resilience. When applied to *picture fuzzy vector space*, the method demonstrated superior accuracy and adaptability compared with existing techniques.

6.3 Advantages

- Integrates fuzzy and neutrosophic structures within a single framework.
- Provides stronger generalization and robustness than previous models.
- Suitable for diverse applications, including decision-making, coding theory, and data analysis.

6.4 Limitations

Despite its merits, the proposed approach has certain limitations. The computational burden is relatively high compared with earlier fuzzy models [17, 19]. Moreover, its performance depends on appropriate parameter choices, which may limit scalability to very large datasets. Current validation has been performed on *picture fuzzy vector space*, and additional studies across broader contexts are needed to establish wider applicability. These limitations suggest future work in optimizing computational efficiency and extending empirical evaluations.

7 Conclusion

- Presented the concept of **picture fuzzy vector spaces** as an advancement over fuzzy and intuitionistic fuzzy vector spaces, integrating topological structures for more robust handling of uncertainty and vagueness.

- Defined and examined **key structural properties** absolute convexity, balance, and absorbency within this framework, and introduced the notion of a **convex balanced hull**.
- Formulated **propositions** illustrating the interdependence of these properties, thereby strengthening the theoretical underpinnings of picture fuzzy vector spaces.
- Highlighted the **practical applicability** of picture fuzzy topological vector spaces in diverse domains, including **neural networks, image processing, machine learning, and AI**.
- Established a **solid mathematical foundation** that bridges theory and practice, enabling more effective solutions for problems involving imprecision, vagueness, and partial truths.

8 Future Work

Picture fuzzy topological spaces(PFTS), as well as product and quotient spaces, future research can focus on:

- **Refinement of Definitions** Further formalize the definitions of PFTS to improve generality, precision, and theoretical consistency.
- **Picture Fuzzy Product Spaces** Develop a rigorous formulation of product spaces in the picture fuzzy setting to facilitate multi-dimensional and combined structure analysis.
- **Picture Fuzzy Quotient Spaces** Construct a clear framework for quotient spaces, allowing systematic study of equivalence relations and factor structures in picture fuzzy topological vector spaces.
- **Interlinked Properties** Explore how product and quotient space constructions influence fundamental attributes such as convexity, balance, and absorbency.

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