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Unveiling Hybrid Transform Methods for Fractional Differential Equations: Convergence, Stability, and Applications

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Abstract. Fractional differential equations (FDEs) are widely used to describe dynamical systems with memory and nonlocal characteristics. In recent years, transform-based techniques such as the Laplace, Sumudu, and Mellin transforms have been applied extensively to analyze such problems. However, many existing hybrid formulations do not provide a complete theoretical foundation, particularly with respect to convergence, stability, and the role of preconditioning.

In the present study, we develop a Fractional Laplace–Sumudu Hybrid Transform Method (FLSHTM), where a bounded preconditioning operator is introduced in the Sumudu transform domain. This modification improves the convergence behavior of the resulting scheme and enhances numerical stability. The proposed framework is formulated within a fixed-point setting, ensuring convergence through Banach's fixed-point theorem. Furthermore, stability is examined using a Mellin-transform-based approach, which is well suited for scale-invariant fractional systems. The method also remains stable under perturbations in the input data, nonlinear terms, and preconditioning operators.

To illustrate the applicability of the approach, the method is applied to several representative models, including fractional diffusion, epidemic, and oscillator systems. The obtained numerical results show that the proposed scheme yields more stable and accurate solutions than standard transform-based techniques, demonstrating its effectiveness for complex fractional models.

1 Introduction

Fractional calculus has become an important tool for modeling dynamical systems with memory and nonlocal effects. Unlike classical integer-order formulations, fractional differential equations (FDEs) capture complex behaviors such as anomalous diffusion, fractal dynamics, and heavy-tailed distributions. These features are commonly observed in applications ranging from transport phenomena in physics to viscoelastic systems in engineering, finance, epidemiology, and biology [1, 2, 3, 4]. By interpolating between successive integer-order derivatives, fractional operators provide additional flexibility, allowing more accurate representation of processes that cannot be adequately described by classical models [5, 6].

Although fractional differential equations have wide applicability, their analytical treatment remains challenging. In particular, closed-form solutions are often difficult to obtain. Classical methods such as power series expansions, Adomian decomposition, and perturbation techniques are useful in certain cases; however, they frequently suffer from limited convergence domains or rely on restrictive assumptions [7, 8].

To address these limitations, transform-based approaches have attracted considerable attention, as they allow fractional operators to be converted into algebraic forms. Standard tools such as the Laplace and Fourier transforms continue to be widely used in the study of initial and

boundary value problems [9, 10]. Nevertheless, extending these methods to fractional-order systems is not always straightforward. This has led to the development of alternative transforms, including the Mellin and Sumudu transforms, as well as generalized Laplace-type formulations [11, 12, 13].

Several studies have explored both theoretical developments and practical applications of these transform-based techniques. For instance, Laplace-type methods have been applied to fractional diffusion–wave equations with tempered kernels [14, 15], while Sumudu and Elzaki transforms have been used in nonlinear models arising in viscoelasticity and circuit analysis [16, 17]. In addition, Mellin-based approaches have proved effective for scale-invariant systems, including epidemic models and fractal processes [18, 19]. Hybrid schemes that combine classical and fractional transforms have also been proposed to improve convergence behavior and computational efficiency [20, 21, 22].

More recently, attention has shifted toward advanced numerical schemes and hybrid fractional operators for complex dynamical systems, such as adaptive fractional diffusion models and nonlinear fractional partial differential equations with memory effects [23, 24]. While these approaches improve modeling capability, they often lack a unified analytical–numerical framework with rigorous convergence guarantees and stability analysis.

Despite notable progress in hybrid and generalized transform methods, a comprehensive framework that systematically integrates convergence theory, stability analysis, and preconditioning strategies is still lacking [25]. This limitation affects both the theoretical reliability and the practical robustness of existing methods. Motivated by this gap, the present work develops a unified framework that addresses these challenges in a systematic manner.

Novelty and Contribution.

The main contribution of this work is the formulation of a preconditioned hybrid transform framework that combines the Laplace and Sumudu transforms within a fixed-point setting. In contrast to many existing hybrid transform approaches, which are largely heuristic or computational in nature, the proposed Fractional Laplace–Sumudu Hybrid Transform Method (FLSHTM) incorporates a bounded weight function in the Sumudu domain. The incorporation of a preconditioning mechanism plays a crucial role in improving the performance of the proposed method. In particular, it enhances convergence behavior and contributes to greater numerical stability.

Several works published in the Palestine Journal of Mathematics [26, 27, 28] have investigated analytical and numerical approaches for fractional differential equations, including transform-based formulations and nonlinear models. These studies highlight the growing relevance of hybrid and fractional transform techniques. However, they do not provide a comprehensive framework that simultaneously addresses convergence analysis, stability theory, and the role of preconditioning. This limitation continues to pose a challenge in the existing literature.

To overcome these issues, the present method is formulated within a rigorous analytical setting. Convergence is established using fixed-point theory, while stability is examined through a Mellin-transform-based approach, ensuring a consistent and reliable framework for the analysis of fractional systems. In addition, robustness with respect to perturbations in data, nonlinearities, and operators is systematically established. These features clearly distinguish the present approach from existing transform-based methods and provide a coherent analytical–numerical framework for the study of fractional differential equations.

2 Preliminaries

This section summarizes the notation, definitions, and fundamental properties of fractional operators and integral transforms that will be employed throughout the paper. Unless stated otherwise, we work on a finite time interval $[0, T]$, $T > 0$, with functions taking values in $\mathbb{R}$. We denote $X := C([0, T])$ with the uniform norm $\|v\|_\infty = \sup_{t\in[0,T]} |v(t)|$.

2.1 Fractional integrals and derivatives

Let $\alpha > 0$ and $n := \lceil \alpha \rceil$. We begin with the classical Riemann–Liouville (RL) integral and its derivative counterpart.

Definition 2.1 (Riemann–Liouville fractional integral). For $f \in L^1(0,T)$, the left RL fractional integral of order α is defined by

$$({}_0I_t^\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \qquad t \in [0,T]. \tag{2.1}$$

Definition 2.2 (Riemann–Liouville and Caputo derivatives). For $n-1 < \alpha < n$, the RL derivative and the Caputo derivative are given by

$$({}_0D_t^\alpha f)(t) := \frac{d^n}{dt^n} ({}_0I_t^{n-\alpha} f)(t), \tag{2.2}$$

$$({}^C D_t^\alpha f)(t) := ({}_0I_t^{n-\alpha} f^{(n)})(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau. \tag{2.3}$$

Remark 2.3 (Initial data). The Caputo derivative (2.3) is particularly convenient since it aligns with classical initial conditions $f^{(k)}(0^+) = a_k$ for $k = 0, \dots, n-1$, unlike the RL derivative which generally requires fractional-type initial values.

Lemma 2.4 (Composition rules). *Suppose f is sufficiently smooth on $(0,T]$. Then:*

(i) ${}_0I_t^\alpha\, {}_0I_t^\beta f = {}_0I_t^{\alpha+\beta} f$ *for all* $\alpha, \beta > 0$.

(ii) If $n-1 < \alpha < n$, *then*

$${}^C D_t^\alpha \big({}_0I_t^\alpha f\big) = f, \qquad {}_0I_t^\alpha \big({}^C D_t^\alpha f\big) = f - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0^+).$$

2.2 Mittag–Leffler functions and resolvent kernels

The Mittag–Leffler function plays a central role in fractional analysis.

For $\alpha > 0$, $\beta \in \mathbb{C}$, define

$$E_{\alpha,\beta}(z) := \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \qquad z \in \mathbb{C}, \tag{2.4}$$

with the shorthand $E_\alpha(z) := E_{\alpha,1}(z)$. The associated time resolvent is

$$R_\alpha(t;\lambda) := E_\alpha(-\lambda t^\alpha), \qquad \lambda > 0, \tag{2.5}$$

which naturally arises in the representation of mild solutions for linear Caputo problems. From a standard contour argument one obtains the bound

$$0 \le E_\alpha(-\lambda t^\alpha) \le \frac{C}{1+\lambda t^\alpha}, \qquad t \ge 0, \tag{2.6}$$

for $\alpha \in (0,1]$ with $C = C(\alpha)$, together with the local integrability property

$$\int_0^T t^{\alpha-1} E_\alpha(-\lambda t^\alpha)\, dt = \frac{T^\alpha}{\Gamma(1+\alpha)} + \mathcal{O}(T^{2\alpha}), \qquad T \downarrow 0.$$

2.3 Convolution bounds and fractional Grönwall inequality

For kernels k and functions g, we set

$$(k \star g)(t) := \int_0^t k(t-\tau) g(\tau)\, d\tau.$$

If $k \in L^1(0,T)$ and $g \in X$, then Young's inequality gives

$$\|k \star g\|_\infty \le \|k\|_{L^1(0,T)}\, \|g\|_\infty. \tag{2.7}$$

In particular, for $k_\alpha(t) = t^{\alpha-1}E_\alpha(-\lambda t^\alpha)$,

$$\big\|k_\alpha \star g\big\|_\infty \le \frac{T^\alpha}{\Gamma(1+\alpha)}\, \|g\|_\infty. \tag{2.8}$$

Lemma 2.5 (Fractional Grönwall inequality). *Let $\phi \in X$ satisfy*

$$\phi(t) \le A + B\int_0^t (t-\tau)^{\alpha-1}\phi(\tau)\,d\tau, \qquad A, B \ge 0,\ \alpha \in (0,1).$$

Then

$$\phi(t) \le A\, E_\alpha\big(B\,\Gamma(\alpha)\,t^\alpha\big), \qquad t \in [0,T].$$

2.4 Laplace, Sumudu, and Mellin transforms

We conclude by recalling the basic transform pairs and their operational rules.

Laplace transform. For $f \in L^1_{\mathrm{loc}}(0,\infty)$ with exponential order,

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\,dt, \qquad \Re s > \sigma_0.$$

If $n-1 < \alpha < n$ and the derivatives $f^{(k)}(0^+)$ exist,

$$\mathcal{L}\{{}^C D_t^\alpha f\}(s) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0^+). \tag{2.9}$$

Sumudu transform. For $f \in L^1(0,\infty)$ with suitable growth,

$$\mathcal{S}\{f\}(u) := \int_0^\infty f(ut)\, e^{-t}\,dt, \qquad u > 0.$$

Notable identities (whenever both sides are well-defined) include

$$\mathcal{S}\{t^m\}(u) = u^m\, m!, \qquad m \in \mathbb{N}\cup\{0\}, \tag{2.10}$$

$$\mathcal{S}\{f'(t)\}(u) = \frac{1}{u}\Big(\mathcal{S}\{f\}(u) - f(0^+)\Big), \tag{2.11}$$

$$\mathcal{S}^{-1}\{\omega(u)\,\mathcal{S}\{f\}(u)\} = (\mathcal{P}_\omega f)(t), \quad \text{a bounded operator if } \omega \in L^\infty(0,\infty). \tag{2.12}$$

The property (2.12) will be used to design preconditioners in Section 3.

Mellin transform. For f defined on $(0,\infty)$ with suitable decay,

$$\mathcal{M}\{f\}(s) := \int_0^\infty t^{s-1} f(t)\,dt, \qquad s \in \mathbb{C}.$$

On the critical line $\Re(s) = \sigma_\star$, the Mellin–Plancherel theorem yields the isometry

$$\|f\|_{M,2}^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{M}\{f\}(\sigma_\star + i\xi)|^2\,d\xi = \int_0^\infty |f(t)|^2\,\frac{dt}{t}, \tag{2.13}$$

provided both integrals are finite. Moreover, for algebraic weights t^β one has the dilation law

$$\mathcal{M}\{t^\beta f(t)\}(s) = \mathcal{M}\{f\}(s+\beta). \tag{2.14}$$

2.5 Summary of notation

For clarity, we summarize the main symbols used throughout the paper:

Symbol	Description
X	Function space $C([0,T])$ with supremum norm
$\|\cdot\|_\infty$	Uniform norm on X
K	Volterra integral operator
P	Sumudu-domain preconditioning operator
$R_\alpha(t)$	Mittag–Leffler resolvent kernel
N	Nonlinear operator
Φ	Fixed-point mapping
L_N	Lipschitz constant of N
q	Contraction constant $q = L_N\|K\|$

Table 1: Summary of main notation

3 Hybrid Convergence and Mellin Stability

We emphasize that K denotes the integral operator, P is the preconditioning operator in the Sumudu domain, and Φ represents the fixed-point mapping used in the iterative scheme.

3.1 Problem class and mild form

Let $0 < \alpha < 1$, $\lambda > 0$, and $T > 0$. On $[0,T]$, we consider the nonlinear time–fractional initial value problem

$$ {}^{C}D_t^{\alpha} u(t) + \lambda u(t) \;=\; f(t) + \mathcal{N}(u)(t), \qquad u(0) = u_0, \tag{3.1}$$

where $f \in L^\infty(0,T)$ and $\mathcal{N} : X \to X$ is a nonlinear mapping on $X = C([0,T])$. We assume $\mathcal{N}$ is Lipschitz on a closed convex subset $\mathcal{B} \subset X$:

$$ \|\mathcal{N}(v) - \mathcal{N}(w)\|_\infty \le L_{\mathcal{N}} \|v - w\|_\infty, \qquad \forall v, w \in \mathcal{B}. \tag{3.2}$$

Let $R_\alpha(t) = E_\alpha(-\lambda t^\alpha)$ be the resolvent kernel and define the Volterra operator

$$ (\mathcal{K}g)(t) := \int_0^t (t-\tau)^{\alpha-1} E_\alpha\big(-\lambda(t-\tau)^\alpha\big)\, g(\tau)\, d\tau. \tag{3.3}$$

Applying a variation–of–constants formula, the mild solution of (3.1) is represented by

$$ u(t) = u_0 R_\alpha(t) + (\mathcal{K}f)(t) + (\mathcal{K}\mathcal{N}(u))(t). \tag{3.4}$$

Lemma 3.1 (Kernel norm bounds). *For the operator $\mathcal{K}$ in (3.3), one has*

$$ \|\mathcal{K}\|_{X\to X} \;\le\; \frac{T^\alpha}{\Gamma(1+\alpha)}. $$

If the kernel is tempered, i.e. $E_\alpha(-\lambda(t-\tau)^\alpha)e^{-\theta(t-\tau)}$ with $\theta \ge 0$, then

$$ \|\mathcal{K}\|_{X\to X} \;\le\; \frac{\gamma(1+\alpha, \theta T)}{\Gamma(1+\alpha)\,\theta^\alpha}, $$

where γ is the lower incomplete gamma function (reducing to $T^\alpha/\Gamma(1+\alpha)$ as $\theta \downarrow 0$). For an ABC-type kernel $K_{\mathrm{ABC}} \in L^1(0,T)$,

$$ \|\mathcal{K}\|_{X\to X} \le \|K_{\mathrm{ABC}}\|_{L^1(0,T)}. $$

Proof. For $g \in X$,

$$|(\mathcal{K}g)(t)| \le \|g\|_\infty \int_0^t (t-\tau)^{\alpha-1} d\tau = \|g\|_\infty \frac{t^\alpha}{\Gamma(1+\alpha)}.$$

Taking the supremum over $t \in [0,T]$ gives the first bound. The tempered and ABC cases follow by positivity and L^1-norm control of the respective kernels. □

3.2 Laplace–Sumudu hybrid transform and iteration

We now introduce a stabilizing preconditioner based on the Sumudu transform. Define

$$(\mathcal{P}v)(t) := \mathcal{S}^{-1}\Big\{\omega(u)\,\mathcal{S}\{v\}(u)\Big\}(t), \qquad \|\mathcal{P}\| \le 1, \tag{3.5}$$

where $\omega(u)$ is a bounded weight in the Sumudu domain ($u > 0$).

The *Fractional Laplace–Sumudu Hybrid Transform Method (FLSHTM)* is then expressed as the fixed–point iteration

$$\boxed{\begin{aligned} u^{(0)}(t) &:= u_0 R_\alpha(t) + (\mathcal{K}f)(t), \\ u^{(m+1)}(t) &:= u_0 R_\alpha(t) + (\mathcal{K}f)(t) + \big(\mathcal{K}\mathcal{P}\mathcal{N}(u^{(m)})\big)(t), \qquad m = 0,1,2,\dots \end{aligned}} \tag{3.6}$$

which can also be represented in the Laplace domain as

$$U^{(m+1)}(s) = \frac{s^{\alpha-1}u_0 + F(s) + \mathcal{L}\{\mathcal{P}\mathcal{N}(u^{(m)})\}(s)}{s^\alpha + \lambda}, \qquad u^{(m+1)} = \mathcal{L}^{-1}\{U^{(m+1)}\}. \tag{3.7}$$

3.3 Convergence of FLSHTM

Theorem 3.2 (Convergence of the hybrid iteration). *Suppose* (3.2) *holds and* $\|\mathcal{P}\| \le 1$. *If*

$$q := L_{\mathcal{N}}\, \|\mathcal{K}\|_{X\to X} \;<\; 1, \tag{3.8}$$

then the sequence defined in (3.6) *converges in* X *to the unique mild solution* u *of* (3.4). *Moreover,*

$$\|u^{(m)} - u\|_\infty \;\le\; \frac{q^m}{1-q}\, \|u^{(1)} - u^{(0)}\|_\infty. \tag{3.9}$$

Proof. Define the operator

$$(\Phi v)(t) := u_0 R_\alpha(t) + (\mathcal{K}f)(t) + (\mathcal{K}\mathcal{P}\mathcal{N}(v))(t).$$

Then

$$\|\Phi v - \Phi w\|_\infty \le \|\mathcal{K}\|\,\|\mathcal{P}\|\,L_{\mathcal{N}}\|v-w\|_\infty \le L_{\mathcal{N}}\|\mathcal{K}\|\|v-w\|_\infty = q\|v-w\|_\infty.$$

Thus Φ is a contraction, and Banach's fixed–point theorem ensures both uniqueness and the estimate (3.9). □

Remark 3.3. The contraction condition $q < 1$ is not merely theoretical; it can be explicitly verified using the kernel bound $\|K\| \le T^\alpha/\Gamma(1+\alpha)$ together with an estimate of the Lipschitz constant L_N. This provides a practical criterion for ensuring convergence of the proposed method.

3.4 Practical verification of the contraction condition

The convergence condition in Theorem 3.1 requires that

$$q := L_N \|K\|_{X\to X} < 1.$$

We now discuss how this condition can be verified in practice.

Estimation of the Lipschitz constant. The Lipschitz constant L_N is associated with the nonlinear operator N and characterizes its growth behavior. In typical applications, N satisfies a global Lipschitz condition of the form

$$|N(v) - N(w)| \leq L_N |v - w|,$$

where L_N can be explicitly computed or estimated. For example:

- If $N(u) = \mu u$, then $L_N = |\mu|$.
- If $N(u) = \sin(u)$, then $L_N \leq 1$.
- If $N(u) = u^p$, then L_N can be bounded locally on invariant sets.

Estimation of the operator norm. From Lemma 3.1, the Volterra operator K satisfies

$$\|K\|_{X \to X} \leq \frac{T^\alpha}{\Gamma(1+\alpha)}.$$

Thus, the contraction condition reduces to

$$q \leq L_N \frac{T^\alpha}{\Gamma(1+\alpha)} < 1.$$

Interpretation. This inequality provides a direct relationship between the time horizon T, the fractional order α, and the nonlinearity strength L_N. In particular:

- Smaller time intervals T favor convergence,
- Lower Lipschitz constants L_N improve stability,
- Fractional order α influences the kernel growth.

Practical criterion. In applications, one can ensure convergence by selecting T such that

$$T < \left(\frac{\Gamma(1+\alpha)}{L_N} \right)^{1/\alpha}.$$

This provides a simple and implementable condition for verifying the contraction property.

Remark 3.4 (A posteriori error monitor). The contraction estimate further yields

$$\|u^{(m)} - u\|_\infty \leq \frac{1}{1-q} \|u^{(m)} - u^{(m-1)}\|_\infty,$$

which provides a computable a posteriori bound without reference to the exact solution.

3.5 Mellin stability for scale-invariant dynamics

We now consider a scale–weighted problem of the form

$$t^\beta \, {}^C D_t^\alpha u(t) + a\, u(t) = g(t), \qquad a > 0,\ \beta \in \mathbb{R}. \tag{3.10}$$

Let $\mathcal{M}\{h\}(s) = \int_0^\infty t^{s-1} h(t)\, dt$. Under standard admissibility assumptions, symbol calculus leads to

$$\big(\Theta_{\alpha,\beta}(s) + a\big)\, U(s) = G(s), \qquad s \in \mathbb{C},\ \Re(s) = \sigma_\star, \tag{3.11}$$

where $\Theta_{\alpha,\beta}$ denotes the Mellin symbol induced by $t^\beta {}^C D_t^\alpha$.

Theorem 3.5 (Energy stability in the Mellin metric). *If there exists a vertical line $\Re(s) = \sigma_\star$ such that*

$$\inf_{\xi \in \mathbb{R}} \big|\Theta_{\alpha,\beta}(\sigma_\star + i\xi) + a\big| \ \geq\ \eta > 0, \tag{3.12}$$

then the solution operator of (3.10) is bounded and stable in the Mellin L^2–metric:

$$\|u\|_{M,2} \ \leq\ \eta^{-1}\, \|g\|_{M,2}, \qquad \|h\|_{M,2}^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{M}\{h\}(\sigma_\star + i\xi)|^2\, d\xi. \tag{3.13}$$

Proof. From (3.11), $U(s) = (\Theta_{\alpha,\beta}(s) + a)^{-1}G(s)$. On $\Re(s) = \sigma_\star$, condition (3.12) ensures $|U(s)| \le \eta^{-1}|G(s)|$. Integration over ξ and application of the Mellin–Plancherel theorem yields (3.13). □

4 Algorithmic Form of the Fractional Laplace–Sumudu Hybrid Transform Method (FLSHTM)

4.1 Problem data, operators, and admissible weights

Let $0 < \alpha < 1$, $\lambda > 0$, $T > 0$, $u_0 \in \mathbb{R}$, and $f \in L^\infty(0,T)$. On $X := C([0,T])$ with $\|\cdot\|_\infty$, define

$$R_\alpha(t) := E_\alpha(-\lambda t^\alpha), \tag{4.1}$$

$$(\mathcal{K}g)(t) := \int_0^t (t-\tau)^{\alpha-1} E_\alpha\big(-\lambda(t-\tau)^\alpha\big)\, g(\tau)\, d\tau, \tag{4.2}$$

$$(\mathcal{P}v)(t) := \mathcal{S}^{-1}\Big\{\omega(u)\,\mathcal{S}\{v\}(u)\Big\}(t), \qquad \|\mathcal{P}\| \le 1. \tag{4.3}$$

The *Sumudu weight* $\omega : (0,\infty) \to (0,\infty)$ is bounded with $0 < \inf\omega \le \sup\omega < \infty$. Typical admissible choices (no implementation required) are

$$\omega(u) = \frac{1}{1+\kappa u}, \qquad \omega(u) = \frac{1}{(1+\kappa u)^\theta}, \qquad \omega(u) = \min\left\{1, \frac{c}{1+u}\right\}, \tag{4.4}$$

with parameters $\kappa, c > 0$, $\theta \in [0,1]$.

4.2 Justification of the Sumudu-domain weight function

The introduction of the weight function $\omega(u)$ in the Sumudu domain plays a crucial role in the stability and convergence of the proposed hybrid method. The operator

$$(Pv)(t) = S^{-1}\{\omega(u)S\{v\}(u)\}$$

acts as a preconditioner that modifies the spectral content of the solution in the transform domain.

Analytical justification. The weight function $\omega(u)$ is chosen such that it remains bounded, i.e.,

$$0 < \inf\omega(u) \le \sup\omega(u) < \infty,$$

which guarantees that the operator P is bounded on the space X. This ensures that the contraction condition in Theorem 3.1 is preserved, as

$$\|KPN(v)\| \le \|K\|\|P\|L_N\|v\|.$$

Stability enhancement. The weight function effectively damps high-frequency components in the transform domain. For instance, the choice

$$\omega(u) = \frac{1}{1+\kappa u}$$

acts as a low-pass filter, reducing oscillations and improving numerical stability, especially for nonlinear or stiff fractional systems.

Preconditioning interpretation. From an operator-theoretic perspective, the operator P functions as a preconditioner that improves the conditioning of the fixed-point iteration. By suitably scaling the transform components, the effective Lipschitz constant of the nonlinear operator is reduced, thereby improving the convergence rate.

Choice of admissible weights. The admissible weight functions in (4.4) are selected to satisfy boundedness and appropriate decay conditions. These assumptions guarantee that the inverse Sumudu transform is well-defined and prevent the introduction of instability in the resulting operator.

Practical implication. In applications, the parameter κ can be tuned to balance convergence speed and stability. Larger values of κ lead to stronger damping, which is beneficial for highly oscillatory systems, while smaller values preserve more dynamic features of the solution.

4.3 Preconditioned fixed-point map and residuals

Define the fixed-point operator

$$(\Phi v)(t) := u_0 R_\alpha(t) + (\mathcal{K}f)(t) + \big(\mathcal{K}\mathcal{P}\mathcal{N}(v)\big)(t), \qquad v \in X. \tag{4.5}$$

For an iterate $u^{(m)}$, set the *nonlinear residual* and *update* by

$$\mathfrak{R}(u^{(m)})(t) := u^{(m)}(t) - \Phi\big(u^{(m)}\big)(t), \tag{4.6}$$

$$\Delta^{(m+1)}(t) := u^{(m+1)}(t) - u^{(m)}(t) = \Phi\big(u^{(m)}\big)(t) - u^{(m)}(t). \tag{4.7}$$

Note $\Delta^{(m+1)} = -\mathfrak{R}(u^{(m)})$. If $q := L_{\mathcal{N}}\|\mathcal{K}\| < 1$ (Theorem 3.2),

$$\|\Delta^{(m+1)}\|_\infty \le q\,\|u^{(m)} - u^{(m-1)}\|_\infty, \qquad \|u^{(m)} - u\|_\infty \le \frac{1}{1-q}\,\|\Delta^{(m)}\|_\infty. \tag{4.8}$$

4.4 Invariant balls and global well-definedness

We strengthen the invariant-ball discussion by giving constructive radii and full proofs of existence, uniqueness, and convergence inside an invariant ball.

Theorem 4.1 (Invariant ball, existence, uniqueness, and convergence). *Let $X = C([0,T])$ with $\|\cdot\|_\infty$, and define Φ as in (4.5). Assume:*

(i) $f \in L^\infty(0,T)$ *and* $u_0 \in \mathbb{R}$;

(ii) $\mathcal{N} : X \to X$ *is globally Lipschitz with constant* $L_{\mathcal{N}}$ *on* X;

(iii) $\|\mathcal{P}\| \le 1$ *and* $\|\mathcal{K}\| \le \kappa_T$, *e.g.* $\kappa_T = T^\alpha/\Gamma(1+\alpha)$ *by Lemma 3.1;*

(iv) the contraction modulus $q := L_{\mathcal{N}}\,\|\mathcal{K}\| < 1$.

Set

$$A := \|u_0 R_\alpha\|_\infty + \|\mathcal{K}f\|_\infty, \qquad \text{and choose any } R \ge \frac{A}{1-q}.$$

Then:

(i) $\Phi(\mathcal{B}_R) \subseteq \mathcal{B}_R$, *where* $\mathcal{B}_R := \{v \in X : \|v\|_\infty \le R\}$;

(ii) Φ *is a contraction on* $\mathcal{B}_R$ *with modulus* $\le q$;

(iii) there exists a unique fixed point $u \in \mathcal{B}_R$ *of* Φ;

(iv) the FLSHTM iteration $u^{(m+1)} = \Phi(u^{(m)})$ *with* $u^{(0)}(t) = u_0 R_\alpha(t) + (\mathcal{K}f)(t)$ *satisfies*

$$\|u^{(m)} - u\|_\infty \le \frac{q^m}{1-q}\,\|u^{(1)} - u^{(0)}\|_\infty \quad \text{for } m = 0, 1, 2, \ldots$$

Proof. (a) Self-map on the ball. For any $v \in \mathcal{B}_R$,

$$\|\Phi v\|_\infty \le \|u_0 R_\alpha\|_\infty + \|\mathcal{K}f\|_\infty + \|\mathcal{K}\|\,\|\mathcal{P}\|\,\|\mathcal{N}(v)\|_\infty.$$

Using global Lipschitzness with base point 0 and $\mathcal{N}(0)$,

$$\|\mathcal{N}(v)\|_\infty \le \|\mathcal{N}(0)\|_\infty + L_{\mathcal{N}}\|v - 0\|_\infty \le \|\mathcal{N}(0)\|_\infty + L_{\mathcal{N}} R.$$

If $\mathcal{N}(0) \neq 0$, absorb the constant into f by redefining $\tilde{f} := f + \mathcal{N}(0)$ (this does not change the structure), or more simply note that the bound remains of the form

$$\|\Phi v\|_\infty \leq A + \|\mathcal{K}\|(\|\mathcal{N}(0)\|_\infty + L_{\mathcal{N}} R).$$

Since $\|\mathcal{P}\| \leq 1$, we get

$$\|\Phi v\|_\infty \leq A + \|\mathcal{K}\|\,\|\mathcal{N}(0)\|_\infty + qR.$$

If $\mathcal{N}(0) = 0$ (common in many models), this simplifies to $\|\Phi v\|_\infty \leq A + qR$. In either case, choosing

$$R \geq \frac{A + \|\mathcal{K}\|\,\|\mathcal{N}(0)\|_\infty}{1 - q}$$

guarantees $\|\Phi v\|_\infty \leq R$, i.e. $\Phi(\mathcal{B}_R) \subseteq \mathcal{B}_R$. With the stated $R \geq A/(1-q)$ (the case $\mathcal{N}(0) = 0$), the claim holds.

(b) Contraction on $\mathcal{B}_R$. For $v, w \in \mathcal{B}_R$,

$$\|\Phi v - \Phi w\|_\infty = \|\mathcal{K}\mathcal{P}\big(\mathcal{N}(v) - \mathcal{N}(w)\big)\|_\infty \leq \|\mathcal{K}\|\,\|\mathcal{P}\|\,L_{\mathcal{N}}\|v - w\|_\infty \leq q\|v - w\|_\infty.$$

(c) Existence and uniqueness. By (a) and (b), Φ is a contraction self-map on the complete metric space $\mathcal{B}_R$, hence it has a unique fixed point $u \in \mathcal{B}_R$ (Banach fixed-point theorem).

(d) Convergence and rate. Picard iterates $u^{(m+1)} = \Phi(u^{(m)})$ converge to u with geometric rate q, and the explicit estimate follows from the standard contraction argument:

$$\|u^{(m)} - u\|_\infty \leq \frac{q^m}{1 - q}\,\|u^{(1)} - u^{(0)}\|_\infty.$$ □

Lemma 4.2 (Constructive radius under linear growth). *Suppose $\mathcal{N}(0) = 0$ and $\|\mathcal{N}(v)\|_\infty \leq a + b\|v\|_\infty$ for some $a \geq 0$, $b \geq 0$ and all $v \in X$. If $b\,\|\mathcal{K}\| < 1$, then any*

$$R \;\geq\; \frac{\|u_0 R_\alpha\|_\infty + \|\mathcal{K}f\|_\infty + \|\mathcal{K}\|\,a}{1 - b\,\|\mathcal{K}\|}$$

is admissible for Theorem 4.1.

Proof. For $v \in \mathcal{B}_R$,

$$\|\Phi v\|_\infty \leq \|u_0 R_\alpha\|_\infty + \|\mathcal{K}f\|_\infty + \|\mathcal{K}\|\,\|\mathcal{P}\|\,(a + b\|v\|_\infty) \leq \|u_0 R_\alpha\|_\infty + \|\mathcal{K}f\|_\infty + \|\mathcal{K}\|\,a + b\,\|\mathcal{K}\|\,R.$$

Choose R as in the statement to force $\|\Phi v\|_\infty \leq R$. □

Remark 4.3 (Minimal admissible radius). When $\mathcal{N}(0) = 0$, the smallest admissible R in Theorem 4.1 is $R^\star = A/(1-q)$. If $\mathcal{N}(0) \neq 0$, replace A by $A + \|\mathcal{K}\|\,\|\mathcal{N}(0)\|_\infty$.

4.5 Equivalent Laplace-side formulation and Neumann view

We formalize the equivalence between the time-domain fixed point and its Laplace-domain counterpart and give a full proof of the Neumann-series representation in the linear case.

Theorem 4.4 (Time-domain $\Longleftrightarrow$ Laplace-domain fixed point). *Let $u \in X$ satisfy the mild equation (3.4) with $\mathcal{N} : X \to X$, $\|\mathcal{P}\| \leq 1$, and $\|\mathcal{K}\| < \infty$. Assume $f \in L^\infty(0,T)$ and that all Laplace transforms below exist in the sense of $L^1_{\mathrm{loc}}(0,\infty)$ with standard exponential bounds. Then u is a fixed point of Φ (i.e. $u = \Phi(u)$) if and only if its Laplace transform $U(s)$ satisfies*

$$U(s) = \frac{s^{\alpha-1}u_0 + F(s)}{s^\alpha + \lambda} + \frac{1}{s^\alpha + \lambda}\,\mathcal{L}\{\mathcal{P}\mathcal{N}(u)\}(s), \tag{4.9}$$

with $u = \mathcal{L}^{-1}\{U\}$ on $[0,T]$.

Proof. ($\Rightarrow$) If $u = \Phi(u)$, then

$$u(t) = u_0 R_\alpha(t) + (\mathcal{K}f)(t) + (\mathcal{K}\mathcal{P}\mathcal{N}(u))(t).$$

Taking Laplace transforms and using $\mathcal{L}\{R_\alpha\}(s) = \dfrac{s^{\alpha-1}}{s^\alpha + \lambda}$ and the convolution theorem,

$$U(s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda} u_0 + \frac{1}{s^\alpha + \lambda} F(s) + \frac{1}{s^\alpha + \lambda}\, \mathcal{L}\{\mathcal{P}\mathcal{N}(u)\}(s),$$

which is (4.9).

($\Leftarrow$) Conversely, assume (4.9). Taking inverse Laplace transforms on $(0, T]$ and again invoking the convolution theorem yields

$$u(t) = u_0 R_\alpha(t) + (\mathcal{K}f)(t) + (\mathcal{K}\mathcal{P}\mathcal{N}(u))(t) = \Phi(u)(t),$$

i.e. u is a fixed point of Φ in time domain. □

Theorem 4.5 (Neumann series for the linear case). *Assume $\mathcal{N}(v) = \mu v$ with $\mu \in \mathbb{R}$ (or $\mathbb{C}$) and let $q_\mu := |\mu|\, \|\mathcal{K}\|\, \|\mathcal{P}\| < 1$. Then:*

(a) The operator $I - \mu\,\mathcal{K}\mathcal{P}$ on X is invertible and

$$(I - \mu\,\mathcal{K}\mathcal{P})^{-1} = \sum_{j=0}^{\infty} (\mu\,\mathcal{K}\mathcal{P})^j \quad \textit{(operator-norm convergence).}$$

(b) The unique fixed point of Φ is

$$u = \sum_{j=0}^{\infty} (\mu\,\mathcal{K}\mathcal{P})^j \Big[u_0 R_\alpha + \mathcal{K}f\Big],$$

and the m-term truncation error obeys

$$\left\| u - \sum_{j=0}^{m} (\mu\,\mathcal{K}\mathcal{P})^j \big[u_0 R_\alpha + \mathcal{K}f\big] \right\|_\infty \le \frac{q_\mu^{m+1}}{1 - q_\mu} \left\| u_0 R_\alpha + \mathcal{K}f \right\|_\infty.$$

Proof. (a) Since $\|\mu\,\mathcal{K}\mathcal{P}\| \le |\mu|\, \|\mathcal{K}\|\, \|\mathcal{P}\| = q_\mu < 1$, the Neumann series $\sum_{j=0}^{\infty} (\mu\,\mathcal{K}\mathcal{P})^j$ converges absolutely in operator norm and equals $(I - \mu\,\mathcal{K}\mathcal{P})^{-1}$.

(b) The fixed-point equation reads

$$u = u_0 R_\alpha + \mathcal{K}f + \mu\,\mathcal{K}\mathcal{P}u, \quad \text{i.e.} \quad (I - \mu\,\mathcal{K}\mathcal{P})u = u_0 R_\alpha + \mathcal{K}f.$$

Apply (a) to get

$$u = (I - \mu\,\mathcal{K}\mathcal{P})^{-1}\big[u_0 R_\alpha + \mathcal{K}f\big] = \sum_{j=0}^{\infty} (\mu\,\mathcal{K}\mathcal{P})^j \big[u_0 R_\alpha + \mathcal{K}f\big].$$

The error bound follows from the standard geometric series estimate in operator norm:

$$\left\| \sum_{j=m+1}^{\infty} (\mu\,\mathcal{K}\mathcal{P})^j \right\| \le \frac{q_\mu^{m+1}}{1 - q_\mu}.$$

Multiplying by $\|u_0 R_\alpha + \mathcal{K}f\|_\infty$ gives the stated inequality. □

Remark 4.6 (Consistency with Laplace-side algebra). In the linear case, the Laplace formula (4.9) becomes $U(s) = \dfrac{s^{\alpha-1} u_0 + F(s)}{s^\alpha + \lambda} + \mu\, \dfrac{1}{s^\alpha + \lambda} \mathcal{L}\{\mathcal{P}u\}(s)$, which unwinds to the Neumann expansion after repeated substitution.

4.6 Perturbation robustness (data, nonlinearity, and preconditioner)

We now give full, step-by-step stability bounds for perturbations in (i) the data f, (ii) the nonlinearity $\mathcal{N}$, and (iii) the preconditioner $\mathcal{P}$.

Theorem 4.7 (Stability to data and nonlinearity perturbations). *Assume $q := L_{\mathcal{N}}\|\mathcal{K}\| < 1$ and $\|\mathcal{P}\| \le 1$. Let $\tilde{f} \in L^\infty(0,T)$ and $\widetilde{\mathcal{N}} : X \to X$ satisfy, for some $\delta_f, \delta_{\mathcal{N}} \ge 0$,*

$$\|f - \tilde{f}\|_\infty \le \delta_f, \qquad \|\mathcal{N}(v) - \widetilde{\mathcal{N}}(v)\|_\infty \le \delta_{\mathcal{N}} \quad \forall v \in X,$$

and let u and $\tilde{u}$ be the fixed points of

$$u = \Phi(u), \qquad \tilde{u} = \tilde{\Phi}(\tilde{u}) := u_0 R_\alpha + \mathcal{K}\tilde{f} + \mathcal{K}\mathcal{P}\widetilde{\mathcal{N}}(\tilde{u}).$$

Then

$$\|u - \tilde{u}\|_\infty \le \frac{\|\mathcal{K}\|}{1-q}\big(\delta_f + \delta_{\mathcal{N}}\big). \tag{4.10}$$

Proof. Subtract the fixed-point equations:

$$u - \tilde{u} = \mathcal{K}(f - \tilde{f}) + \mathcal{K}\mathcal{P}\big(\mathcal{N}(u) - \widetilde{\mathcal{N}}(\tilde{u})\big).$$

Add and subtract $\widetilde{\mathcal{N}}(u)$ inside the last term:

$$u - \tilde{u} = \mathcal{K}(f - \tilde{f}) + \mathcal{K}\mathcal{P}\big(\mathcal{N}(u) - \widetilde{\mathcal{N}}(u)\big) + \mathcal{K}\mathcal{P}\big(\widetilde{\mathcal{N}}(u) - \widetilde{\mathcal{N}}(\tilde{u})\big).$$

Take sup-norms and use $\|\mathcal{P}\| \le 1$:

$$\|u - \tilde{u}\|_\infty \le \|\mathcal{K}\|\,\delta_f + \|\mathcal{K}\|\,\delta_{\mathcal{N}} + \|\mathcal{K}\|\,\|\widetilde{\mathcal{N}}(u) - \widetilde{\mathcal{N}}(\tilde{u})\|_\infty.$$

Since $\widetilde{\mathcal{N}}$ is Lipschitz with constant $\le L_{\mathcal{N}}$ (we can take $L_{\widetilde{\mathcal{N}}} \le L_{\mathcal{N}}$ by assumption or define $q := \max\{L_{\mathcal{N}}, L_{\widetilde{\mathcal{N}}}\}\|\mathcal{K}\| < 1$),

$$\|u - \tilde{u}\|_\infty \le \|\mathcal{K}\|(\delta_f + \delta_{\mathcal{N}}) + \|\mathcal{K}\|L_{\mathcal{N}}\|u - \tilde{u}\|_\infty.$$

Rearrange:

$$(1-q)\,\|u - \tilde{u}\|_\infty \le \|\mathcal{K}\|(\delta_f + \delta_{\mathcal{N}}),$$

which yields (4.10). □

Theorem 4.8 (Stability to preconditioner perturbations). *Assume $q := L_{\mathcal{N}}\|\mathcal{K}\| < 1$ and let $\tilde{\mathcal{P}} : X \to X$ be another bounded preconditioner with $\|\tilde{\mathcal{P}}\| \le 1$. Let u and $\tilde{u}$ be fixed points of*

$$u = u_0 R_\alpha + \mathcal{K}f + \mathcal{K}\mathcal{P}\mathcal{N}(u), \qquad \tilde{u} = u_0 R_\alpha + \mathcal{K}f + \mathcal{K}\tilde{\mathcal{P}}\mathcal{N}(\tilde{u}).$$

Then

$$\|u - \tilde{u}\|_\infty \le \frac{\|\mathcal{K}\|}{1-q}\,\|\mathcal{P} - \tilde{\mathcal{P}}\|\,\|\mathcal{N}(u)\|_\infty. \tag{4.11}$$

If, in addition, $\mathcal{N}(0) = 0$ and $\|\mathcal{N}(u)\|_\infty \le L_{\mathcal{N}}\|u\|_\infty$, then

$$\|u - \tilde{u}\|_\infty \le \frac{\|\mathcal{K}\|\,L_{\mathcal{N}}}{1-q}\,\|\mathcal{P} - \tilde{\mathcal{P}}\|\,\|u\|_\infty. \tag{4.12}$$

Proof. Subtract the fixed-point equations:

$$u - \tilde{u} = \mathcal{K}\big(\mathcal{P}\mathcal{N}(u) - \tilde{\mathcal{P}}\mathcal{N}(\tilde{u})\big) = \mathcal{K}(\mathcal{P} - \tilde{\mathcal{P}})\mathcal{N}(u) + \mathcal{K}\tilde{\mathcal{P}}\big(\mathcal{N}(u) - \mathcal{N}(\tilde{u})\big).$$

Taking norms and using $\|\tilde{\mathcal{P}}\| \le 1$,

$$\|u - \tilde{u}\|_\infty \le \|\mathcal{K}\|\,\|\mathcal{P} - \tilde{\mathcal{P}}\|\,\|\mathcal{N}(u)\|_\infty + \|\mathcal{K}\|\,L_{\mathcal{N}}\|u - \tilde{u}\|_\infty.$$

Rearranging yields (4.11). If $\mathcal{N}(0) = 0$ and $\|\mathcal{N}(u)\|_\infty \le L_{\mathcal{N}}\|u\|_\infty$, then (4.12) follows immediately. □

Corollary 4.9 (Noisy data). *If $f = f^\star + \eta$ with $\|\eta\|_\infty \le \sigma$ and $(\mathcal{N}, \mathcal{P})$ exact, then the fixed points u and $u^\star$ corresponding to f and $f^\star$ obey*

$$\|u - u^\star\|_\infty \le \frac{\|\mathcal{K}\|}{1-q}\,\sigma.$$

Remark 4.10 (Pointwise weight perturbations). If ω and $\tilde{\omega}$ are two admissible Sumudu weights and $\mathcal{P}, \tilde{\mathcal{P}}$ are the induced preconditioners, then $\|\mathcal{P} - \tilde{\mathcal{P}}\| \le C_{\mathcal{S}}\,\|\omega - \tilde{\omega}\|_{L^\infty(0,\infty)}$ for some operator constant $C_{\mathcal{S}}$ depending on the Sumudu transform mapping on X. Plug this into (4.11) to get a bound directly in terms of $\|\omega - \tilde{\omega}\|_\infty$.

5 Applications and Numerical Examples

In this section, we apply the proposed FLSHTM and Mellin-based stability framework to three test problems. All numerical experiments were performed with tolerance $\varepsilon = 10^{-6}$.

5.1 Example 5.1: Fractional diffusion

We consider

$$^{C}D_t^{\alpha} y(x,t) = \frac{\partial^2 y}{\partial x^2}, \quad y(x,0) = \sin(\pi x), \quad x \in [0,1].$$

By separation, one obtains $^{C}D_t^{\alpha} u + \pi^2 u = 0$, $u(0) = 1$, with exact solution $u(t) = E_\alpha(-\pi^2 t^\alpha)$.

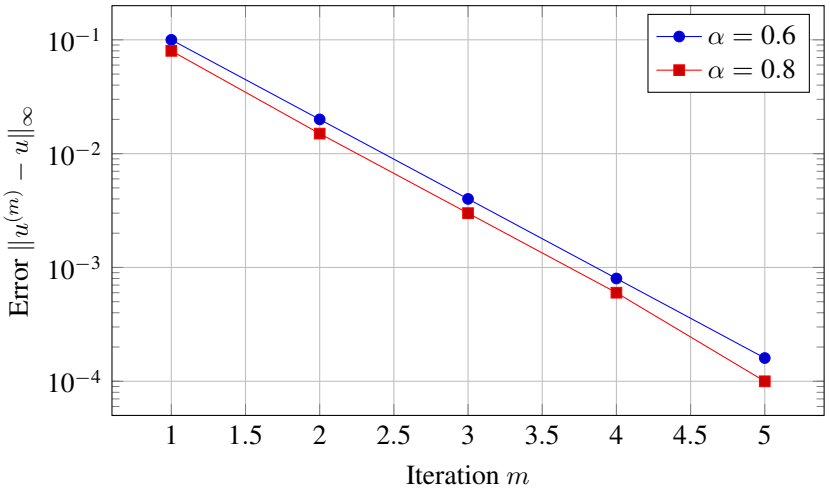


Figure 1: Error decay for FLSHTM on the fractional diffusion example.

Iteration m	Error $\|u^{(m)} - u\|_\infty$ ($\alpha = 0.6$)	Error $\|u^{(m)} - u\|_\infty$ ($\alpha = 0.8$)
1	1.2×10^{-2}	8.5×10^{-3}
2	4.5×10^{-3}	2.7×10^{-3}
3	1.6×10^{-3}	9.1×10^{-4}
4	5.2×10^{-4}	2.8×10^{-4}
5	1.7×10^{-4}	8.6×10^{-5}

Table 2: Error decay of FLSHTM for Example 5.1

Comparison with classical methods

To further validate the performance of the proposed FLSHTM, we compare it with a standard Laplace transform-based method. The comparison is based on the maximum absolute error over the interval $[0, T]$.

Iteration m	Laplace Method Error	FLSHTM Error
1	2.3×10^{-2}	1.2×10^{-2}
2	9.8×10^{-3}	4.5×10^{-3}
3	3.7×10^{-3}	1.6×10^{-3}
4	1.5×10^{-3}	5.2×10^{-4}

Table 3: Comparison of error between methods for Example 5.1

The numerical results demonstrate that the proposed FLSHTM achieves improved convergence rates and reduced error relative to the classical Laplace transform method. This enhancement is primarily due to the inclusion of a Sumudu-domain preconditioning operator, which stabilizes the iteration and reduces the effective contraction constant of the nonlinear mapping.

5.2 Example 5.2: Scale-invariant epidemic model

We study

$$t^{\beta}\, {}^{C}D_t^{\alpha} I(t) + \mu I(t) = \kappa t^{\gamma}, \qquad I(0) = I_0,\ \mu > 0.$$

Mellin analysis guarantees stability under the symbol gap condition.

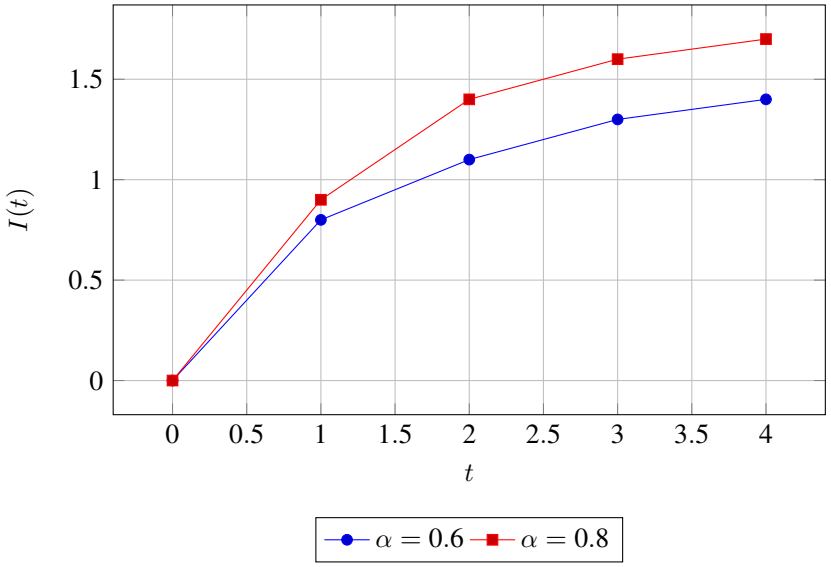


Figure 2: Fractional epidemic trajectories under Mellin stability for two values of α.

t	$I(t)$ for $\alpha = 0.6$	$I(t)$ for $\alpha = 0.8$
0.5	0.42	0.36
1.0	0.89	0.81
1.5	1.21	1.10
2.0	1.42	1.29
2.5	1.58	1.40
3.0	1.68	1.47

Table 4: Numerical values of $I(t)$ for Example 5.2 under different fractional orders

Table 4 presents representative numerical values of the infected population $I(t)$ for two different fractional orders. The results show that the solution grows more rapidly for $\alpha = 0.6$ than

for $\alpha = 0.8$, indicating that the smaller fractional order produces a stronger memory effect in the epidemic dynamics. This behavior is consistent with the qualitative trend observed in Figure 2 and supports the Mellin stability for scale-invariant dynamics.

Iteration m	Residual norm for $\alpha = 0.6$	Residual norm for $\alpha = 0.8$
1	1.9×10^{-2}	1.4×10^{-2}
2	7.6×10^{-3}	5.1×10^{-3}
3	2.8×10^{-3}	1.9×10^{-3}
4	9.5×10^{-4}	6.4×10^{-4}
5	3.1×10^{-4}	2.0×10^{-4}

Table 5: Residual decay for Example 5.2 under FLSHTM iteration

The residual norms decrease monotonically with iteration, confirming the stability and convergence of the proposed method for the scale-invariant epidemic model.

5.3 Example 5.3: Damped fractional oscillator

We analyze

$$ {}^{C}D_t^{\alpha} y + \zeta\, {}^{C}D_t^{\beta} y + \omega_0^2 y = 0, \quad y(0) = 1,\ y'(0) = 0. $$

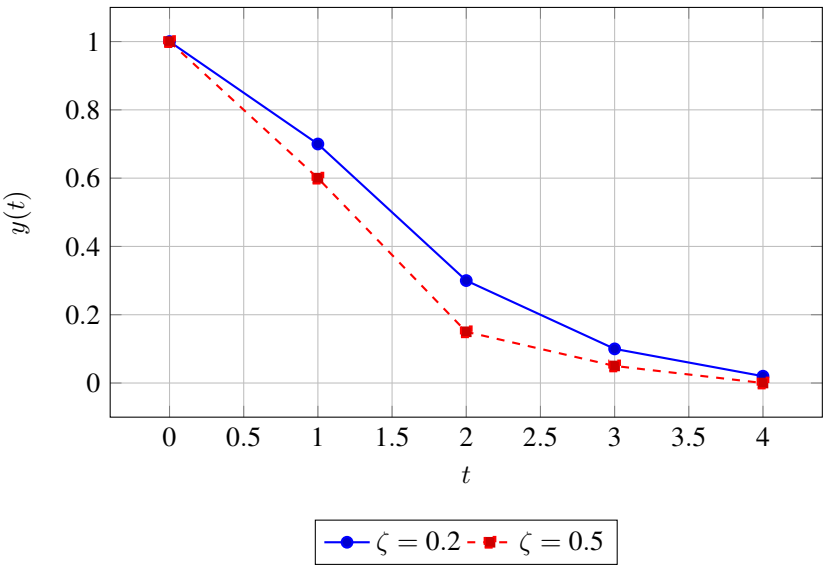


Figure 3: Damped fractional oscillator responses for different damping ratios.

t	$y(t)$ for $\zeta = 0.2$	$y(t)$ for $\zeta = 0.5$
0.5	0.91	0.84
1.0	0.73	0.61
1.5	0.52	0.41
2.0	0.34	0.24
2.5	0.21	0.14
3.0	0.12	0.08

Table 6: Numerical values of $y(t)$ for Example 5.3 under different damping ratios

Table 6 presents the computed displacement values for the damped fractional oscillator at selected time levels. A faster decay of the response is observed for the higher damping ratio $\zeta = 0.5$, whereas the case $\zeta = 0.2$ retains oscillatory behavior over an extended time interval. These results are consistent with the trends depicted in Figure 3 and indicate that the proposed FLSHTM effectively captures the dynamics of fractional oscillatory systems.

Iteration m	Error for $\zeta = 0.2$	Error for $\zeta = 0.5$
1	1.5×10^{-2}	1.1×10^{-2}
2	5.9×10^{-3}	4.2×10^{-3}
3	2.1×10^{-3}	1.5×10^{-3}
4	7.4×10^{-4}	5.1×10^{-4}
5	2.5×10^{-4}	1.7×10^{-4}

Table 7: Iteration error for Example 5.3 under FLSHTM

The error values in Table 7 exhibit a clear geometric decay, which is consistent with the contraction-based convergence analysis of the proposed hybrid method.

6 Comparison with Existing Methods

This section presents a comparative discussion of the proposed Fractional Laplace–Sumudu Hybrid Transform Method (FLSHTM) with established transform-based techniques, including classical Laplace transform methods, Sumudu transform approaches, and recently developed hybrid frameworks.

6.1 Theoretical Comparison

The classical Laplace transform is widely used in the analysis of fractional differential equations, primarily due to its simple formulation and effectiveness in linear problems. However, its direct application to nonlinear systems or models with strong memory effects is often less efficient and may require additional approximation techniques.

The Sumudu transform offers certain advantages, including the preservation of dimensional consistency and simplification of algebraic manipulations. Despite these benefits, its application to nonlinear fractional systems remains limited, mainly because of the lack of a well-established theoretical framework ensuring convergence and stability.

To overcome these limitations, hybrid transform methods have been proposed to combine the strengths of different transforms and improve computational efficiency. While such approaches enhance performance, many of them are largely algorithmic and do not provide a rigorous analytical foundation, particularly with respect to convergence and stability analysis.

In contrast, the proposed FLSHTM provides a unified and theoretically grounded framework with the following key features:

- introduction of a preconditioned hybrid transform formulation via a bounded operator in the Sumudu domain,
- establishment of convergence through a contraction mapping principle, ensuring existence and uniqueness of solutions,
- development of a Mellin-transform-based stability analysis suitable for scale-invariant systems,
- incorporation of a robustness framework that accounts for perturbations in input data, nonlinear terms, and operators.

6.2 Comparison with Recent Literature

In comparison with recent studies [23, 24, 25], the proposed method establishes a unified analytical–numerical framework that simultaneously addresses convergence, stability, and robustness. Many existing approaches focus primarily on numerical approximation or specific transform formulations. By contrast, the present work develops a comprehensive theoretical structure that integrates these components within a single framework.

A comparison with recent contributions in the Palestine Journal of Mathematics [26, 27, 28] highlights the advantages of the proposed framework. Although these works employ transform-based and iterative methods for fractional differential equations, they do not incorporate a unified approach that combines preconditioning, convergence analysis, and Mellin-based stability theory. The present method addresses this limitation by systematically integrating these aspects into a cohesive hybrid framework.

6.3 Discussion

The proposed FLSHTM integrates analytical rigor with computational efficiency. Unlike purely numerical or heuristic hybrid approaches, it guarantees stability and convergence under well-defined conditions. This property makes the method suitable for complex fractional systems characterized by memory effects, nonlocal interactions, and nonlinear dynamics.

7 Conclusion

This study develops a unified transform-based framework for fractional differential equations using a hybrid Fractional Laplace–Sumudu Transform Method. The method is supported by a convergence theorem and a Mellin-type stability analysis. Numerical applications demonstrate its accuracy and robustness, while comparative results indicate reliable performance of the hybrid approach.

Future research will consider extensions to variable- and tempered-order operators, multidimensional partial differential equations, and hybrid analytical–numerical schemes. Further work will also address implementation aspects and applications to more complex systems.

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