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# On submanifolds of a Sasaki-Kenmotsu manifold

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**Abstract**. In this paper, the geometry of some submanifolds of a $(2n+2)$-dimensional Sasaki-Kenmotsu with the structure $(\varphi, \xi, \psi, \eta, \omega, g)$ are studied. Certain conditions for these submanifolds which are tangent to its characteristic vector fields $\xi, \psi$ (resp., are normal to at least one of $\xi, \psi$ ) to be $\varphi$-invariant, anti $\varphi$-invariant, minimal and totally geodesic, have been investigated. Concrete examples are given.

## 1 Introduction

The study of geometric structures defined via tensor fields and their behavior under lifting operations plays a central role in modern differential geometry. In particular, manifolds endowed with additional tensorial structures often exhibit rich geometric and topological properties, which have been extensively investigated in various contexts.

A fundamental class of such structures is given by $f$-structures introduced by Yano [15] on a $(2n+s)$-dimensional manifold, defined by a tensor field $f$ of type $(1,1)$ satisfying

$$f^3 + f = 0.$$

This condition induces a natural decomposition of the tangent bundle and provides a unified framework for several classical geometries. Within this setting, Blair [6] introduced the notion of $K$-manifolds as special metric $f$-manifolds, which generalize Kähler, Sasakian, and cosymplectic geometries. In particular, $S$-manifolds and $C$-manifolds arise as natural extensions of Sasakian and cosymplectic structures, respectively.

Further developments in this direction led to the introduction of trans-$S$-manifolds by Alegre et al. [2], which generalize trans-Sasakian structures [12] within the framework of metric $f$-manifolds. These constructions highlight the flexibility of $f$-structures in describing a wide variety of geometric settings.

The geometry of submanifolds in manifolds equipped with additional structures has been widely studied, especially in almost contact metric geometry (see, for example, [1, 5, 11, 13, 14]). In such ambient spaces, the interaction between the structure tensor, characteristic vector fields, and the induced geometry on submanifolds plays a crucial role. Invariant, anti-invariant, totally geodesic, and minimal submanifolds have been intensively investigated in Sasakian, Kenmotsu, and cosymplectic manifolds, where compatibility conditions between the second fundamental form and the ambient structure strongly influence the geometry of the submanifolds.

More recently, generalized contact-type structures involving multiple distinguished vector fields have attracted increasing attention. The authors in [4] introduced a four-dimensional almost bi-contact metric structure as a class of metric $f$-structures endowed with two global vector fields, and studied several associated geometric properties. This framework extends classical almost contact geometry and provides a natural setting for manifolds equipped with multiple characteristic directions.

In parallel, Beldjilali and Gezer [3] introduced Sasaki-Kenmotsu structures on even-dimensional Riemannian manifolds, characterized by a tensor field $\varphi$, two global vector fields $\xi$ and $\psi$, and their associated 1-forms $\eta$ and $\omega$. These structures generalize simultaneously Sasakian and Kenmotsu geometries in different transversal distributions and exhibit behavior distinct from trans-Sasakian and trans-$S$ structures.

Motivated by these developments, and in view of the presence of two distinguished vector fields in both almost bi-contact and Sasaki-Kenmotsu geometries, we study submanifolds of a Sasaki-Kenmotsu manifold $(\widetilde{M}^{2n+2}, \varphi, \xi, \psi, \eta, \omega, g)$. We focus on the cases where the vector fields $\xi$ and $\psi$ are tangent, partially tangent (i.e., one is tangent while the other is normal), or normal to the submanifold. In this setting, we investigate properties of $\varphi$-invariant and anti-$\varphi$-invariant submanifolds, and derive conditions under which these submanifolds are minimal, totally geodesic, or totally umbilical.

## 2 Definitions and basic formulas

Let $\widetilde{M}^n$ be an $n$-dimensional Riemannian manifold and $M^m$ an $m$-dimensional submanifold of $\widetilde{M}^n$. Let $g$ be the metric tensor field on $\widetilde{M}^n$ as well as the metric induced on $M^m$. We denote by $\widetilde{\nabla}$ the covariant differentiation in $\widetilde{M}^n$ and by $\nabla$ the covariant differentiation in $M^m$. Let $T\widetilde{M}$ (resp. $TM$) be the Lie algebra of vector fields on $\widetilde{M}^n$ (resp. on $M^m$) and $T^\perp M$ the set of all vector fields normal to $M^m$. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \widetilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{2.1}$$

for any $X, Y \in TM$, $V \in T^\perp M$, where $\nabla^\perp$ is the connection in the normal bundle, $h$ is the second fundamental form of $M^m$ and $A_V$ is the Weingarten endomorphism associated with $V$. $A_V$ and $h$ are related by

$$g(A_V X, Y) = g\big(h(X,Y), V\big). \tag{2.2}$$

The submanifold $M^m$ is said to be *totally geodesic* in $\widetilde{M}^n$ if its second fundamental form is identically zero and is said to be *minimal* if $H \equiv 0$, where $H$ is the mean curvature vector defined by $H = \frac{1}{m}\,\mathrm{trace}(h)$. From this definition it is clear that any totally geodesic submanifold is obviously a minimal submanifold.
The submanifold $M$ is known to be *totally umbilical* if

$$h(X,Y) = g(X,Y)H, \tag{2.3}$$

for any $X, Y \in TM$ [7].

Recall that the covariant derivative $\widetilde{\nabla}h$ of the second fundamental form $h$ is given by

$$\widetilde{\nabla}_X h(Y,Z) = \nabla^\perp_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.4}$$

where $\widetilde{\nabla}h$ is a normal bundle-valued tensor of type $(0,3)$. If $\widetilde{\nabla}h = 0$, then $M$ is called *parallel* [9].

A $(2n+2)$-dimensional differentiable manifold $(\widetilde{M}, g)$ endowed with an almost bi-contact structure $(\varphi, \xi, \psi, \eta, \omega, g)$, that is, $\varphi$ is a $(1,1)$-tensor field, $\xi$ and $\psi$ are characteristic vector fields corresponding to the 1-forms $\eta$ and $\omega$, respectively, which satisfies the following conditions:

$$\eta(\xi) = \omega(\psi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi + \omega \otimes \psi, \tag{2.5}$$

$$\eta(\psi) = \omega(\xi) = \varphi\xi = \varphi\psi = 0, \quad \eta \circ \varphi = \omega \circ \varphi = 0, \tag{2.6}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) - \omega(X)\omega(Y), \tag{2.7}$$

for all $X, Y \in T\widetilde{M}$, where $I$ is the identity transformation of $T\widetilde{M}$.
It follows that

$$\eta(X) = g(X,\xi), \quad \omega(X) = g(X,\psi) \quad \text{and} \quad g(X,\varphi Y) = -g(\varphi X, Y).$$

The fundamental 2-form $\phi$ of the almost bi-contact metric structure is defined by

$$\phi(X,Y) = g(X,\varphi Y).$$

The almost bi-contact structure $(\varphi,\xi,\psi,\eta,\omega)$ is *normal* if and only if [3]

$$N^{(1)}(X,Y) = [\varphi,\varphi](X,Y) + 2d\eta(X,Y)\xi + 2d\omega(X,Y)\psi = 0,$$

where $[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y]$.

**Definition 2.1.** [3] An almost bi-contact metric manifold is said to be a *Sasaki-Kenmotsu manifold* if it is normal, $\omega$ is closed and

$$\mathrm{d}\eta = \phi - 2\eta\wedge\omega. \tag{2.8}$$

Differentiating the formula (2.8) and using $\mathrm{d}\omega = 0$ on a Sasaki-Kenmotsu manifold, we obtain $\mathrm{d}\phi = 2\mathrm{d}\eta\wedge\omega$, which easily gives $\mathrm{d}\phi = 2\omega\wedge\phi$.

Let $\mathcal{D}_\eta := \{X \in T\widetilde{M} \mid \eta(X) = 0\}$ and $\mathcal{D}_\omega := \{X \in T\widetilde{M} \mid \omega(X) = 0\}$ be two distributions of rank $2n+1$ transversal to the characteristic vector fields $\xi$ and $\psi$, respectively. Here we note that $\mathrm{d}\omega = 0$ and $\mathrm{d}\phi = 2\omega\wedge\phi$ on $\mathcal{D}_\eta$; therefore, $(\varphi,\psi,\omega,g)$ is a Kenmotsu structure on $\mathcal{D}_\eta$. On the other hand, we have $\mathrm{d}\eta = \phi$ on $\mathcal{D}_\eta$ which implies that $(\varphi,\xi,\eta,g)$ is a Sasaki structure on $\mathcal{D}_\omega$. Hence the structure is called a Sasaki-Kenmotsu structure.

Moreover, a Sasaki-Kenmotsu manifold $\widetilde{M}^{2n+2}$ satisfies [3]

$$(\widetilde{\nabla}_X\varphi)Y = g(\varphi X,\varphi Y)\xi + \eta(Y)\varphi^2 X + g(\varphi X,Y)\psi - \omega(Y)\varphi X, \tag{2.9}$$

$$\widetilde{\nabla}_X\xi = \varphi X - 2\eta(X)\psi, \tag{2.10}$$

$$\widetilde{\nabla}_X\psi = -\varphi^2 X + 2\eta(X)\xi, \tag{2.11}$$

for all $X,Y \in T\widetilde{M}$.

**Remark 2.2.** We point out that we corrected the sign of the second term in equation (2.9) compared to the reference [3].

**Example 2.3.** Let $(\widetilde{M},g)$ be a Riemannian manifold where

$$\widetilde{M} = \{(x_1,x_2,y_1,y_2,z,t) \in \mathbb{R}^6 \mid t > 0\},$$

and let $(\tilde{e}_i)$ be the frame of vector fields on $M$ defined by

$$\tilde{e}_1 = \frac{1}{t}\frac{\partial}{\partial x_1}, \quad \tilde{e}_2 = \frac{1}{t}\frac{\partial}{\partial x_2}, \quad \tilde{e}_3 = \frac{1}{t}\left(\frac{\partial}{\partial y_1} + 2x_1\frac{\partial}{\partial z}\right),$$

$$\tilde{e}_4 = \frac{1}{t}\left(\frac{\partial}{\partial y_2} + 2x_2\frac{\partial}{\partial z}\right), \quad \tilde{e}_5 = \frac{1}{t^2}\frac{\partial}{\partial z}, \quad \tilde{e}_6 = t\frac{\partial}{\partial t}.$$

Let $(\theta^i)$ be the dual frame of differential 1-forms

$$\theta^1 = tdx_1, \quad \theta^2 = tdx_2, \quad \theta^3 = tdy_1, \quad \theta^4 = tdy_2,$$

$$\theta^5 = t^2\big(dz - 2(x_1dy_1 + x_2dy_2)\big) \quad \text{and} \quad \theta^6 = \frac{1}{t}dt.$$

For the non-zero Lie brackets of $(\tilde{e}_i)$, we have

$$[\tilde{e}_1,\tilde{e}_3] = [\tilde{e}_2,\tilde{e}_4] = 2\tilde{e}_5, \quad [\tilde{e}_1,\tilde{e}_6] = \tilde{e}_1, \quad [\tilde{e}_2,\tilde{e}_6] = \tilde{e}_2$$

$$[\tilde{e}_3,\tilde{e}_6] = \tilde{e}_3, \quad [\tilde{e}_4,\tilde{e}_6] = \tilde{e}_4, \quad [\tilde{e}_5,\tilde{e}_6] = 2\tilde{e}_5.$$

Define an almost bi-contact structure $(\varphi,\xi,\psi,\eta,\omega)$ on $M$ by assuming

$$\varphi\tilde{e}_1 = \tilde{e}_3, \quad \varphi\tilde{e}_3 = -\tilde{e}_1, \quad \varphi\tilde{e}_2 = \tilde{e}_4, \quad \varphi\tilde{e}_4 = -\tilde{e}_2 \quad \text{and} \quad \varphi\tilde{e}_5 = \varphi\tilde{e}_6 = 0,$$

$$\xi = \tilde{e}_5, \quad \psi = \tilde{e}_6, \quad \eta = \theta^5, \quad \omega = \theta^6.$$

For the normality (i.e., vanishing of the tensor $N_\varphi^{(1)}$), it can be checked that

$$N_\varphi^{(1)}(\tilde{e}_i, \tilde{e}_j) = [\varphi, \varphi](\tilde{e}_i, \tilde{e}_j) + 2\mathrm{d}\eta(\tilde{e}_i, \tilde{e}_j)\xi + 2\mathrm{d}\omega(\tilde{e}_i, \tilde{e}_j)\psi = 0,$$

for any $i, j$. Then, $(\varphi, \xi, \psi, \eta, \omega)$ is normal.

Let $g$ be a Riemannian metric on $M$ for which $(\tilde{e}_i)$ is an orthonormal frame, so that $g = \sum \theta^i \otimes \theta^i$. It is obvious that $(\varphi, \xi, \psi, \eta, \omega, g)$ is a bi-contact metric structure on $M$. For the fundamental form $\phi(X, Y) = g(X, \varphi Y)$, we have

$$\phi = -2\big(\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^4\big) = -2t^2(dx^1 \wedge dy^1 + dx^2 \wedge dy^2),$$

from which it follows that

$$\phi = \mathrm{d}\theta^5 + 2\theta^5 \wedge \theta^6 = \mathrm{d}\eta + 2\eta \wedge \omega.$$

Also, we can see that

$$\begin{aligned} \mathrm{d}\phi &= -4tdt \wedge (dx^1 \wedge dy^1 + dx^2 \wedge dy^2) \\ &= \frac{2}{t} dt \wedge \phi \\ &= 2\theta^6 \wedge \phi. \end{aligned}$$

Hence $(\varphi, \xi, \psi, \eta, \omega)$ is normal (i.e., $N_\varphi^{(1)} = 0$) and $\mathrm{d}\omega = \mathrm{d}\theta^6 = 0$ . As a result, $(\varphi, \xi, \psi, \eta, \omega, g)$ becomes a Sasaki-Kenmotsu structure on $\widetilde{M}$.

To verify the result in formula (2.9), we give the non-zero Levi-Civita connection corresponding to $g$ by:

$$\begin{aligned}
&\widetilde{\nabla}_{\tilde{e}_1}\tilde{e}_1 = \widetilde{\nabla}_{\tilde{e}_2}\tilde{e}_2 = \widetilde{\nabla}_{\tilde{e}_3}\tilde{e}_3 = \widetilde{\nabla}_{\tilde{e}_4}\tilde{e}_4 = -\psi,\\
&\widetilde{\nabla}_{\tilde{e}_1}\tilde{e}_3 = \widetilde{\nabla}_{\tilde{e}_2}\tilde{e}_4 = -\widetilde{\nabla}_{\tilde{e}_3}\tilde{e}_1 = -\widetilde{\nabla}_{\tilde{e}_4}\tilde{e}_2 = \xi,\\
&\widetilde{\nabla}_{\tilde{e}_1}\xi = \widetilde{\nabla}_{\xi}\tilde{e}_1 = -\widetilde{\nabla}_{\tilde{e}_3}\psi = -\tilde{e}_3,\\
&\widetilde{\nabla}_{\tilde{e}_2}\xi = \widetilde{\nabla}_{\xi}\tilde{e}_2 = -\widetilde{\nabla}_{\tilde{e}_4}\psi = -\tilde{e}_4,\\
&\widetilde{\nabla}_{\tilde{e}_1}\psi = \widetilde{\nabla}_{\tilde{e}_3}\xi = \widetilde{\nabla}_{\xi}\tilde{e}_3 = \tilde{e}_1,\\
&\widetilde{\nabla}_{\tilde{e}_2}\psi = \widetilde{\nabla}_{\tilde{e}_4}\xi = \widetilde{\nabla}_{\xi}\tilde{e}_4 = \tilde{e}_2,\\
&\widetilde{\nabla}_{\xi}\xi = -2\psi, \quad \widetilde{\nabla}_{\xi}\psi = 2\xi.
\end{aligned}$$

The non-zero components of $\widetilde{\nabla}_\varphi$ are as follows:

$$\begin{aligned}
&(\widetilde{\nabla}_{\tilde{e}_1}\varphi)\tilde{e}_1 = (\widetilde{\nabla}_{\tilde{e}_2}\varphi)\tilde{e}_2 = (\widetilde{\nabla}_{\tilde{e}_3}\varphi)\tilde{e}_3 = (\widetilde{\nabla}_{\tilde{e}_4}\varphi)\tilde{e}_4 = \xi,\\
&(\widetilde{\nabla}_{\tilde{e}_1}\varphi)\tilde{e}_3 = (\widetilde{\nabla}_{\tilde{e}_2}\varphi)\tilde{e}_4 = -(\widetilde{\nabla}_{\tilde{e}_3}\varphi)\tilde{e}_1 = -(\widetilde{\nabla}_{\tilde{e}_4}\varphi)\tilde{e}_2 = \psi,\\
&(\widetilde{\nabla}_{\tilde{e}_1}\varphi)\xi = (\widetilde{\nabla}_{\tilde{e}_3}\varphi)\psi = \tilde{e}_1, \quad (\widetilde{\nabla}_{\tilde{e}_2}\varphi)\xi = (\widetilde{\nabla}_{\tilde{e}_4}\varphi)\psi = \tilde{e}_2,\\
&(\widetilde{\nabla}_{\tilde{e}_1}\varphi)\psi = -(\widetilde{\nabla}_{\tilde{e}_3}\varphi)\xi = \tilde{e}_3, \quad (\widetilde{\nabla}_{\tilde{e}_2}\varphi)\psi = -(\widetilde{\nabla}_{\tilde{e}_4}\varphi)\xi = \tilde{e}_4.
\end{aligned}$$

One can easily check that for all $i \in \{1, \ldots, 6\}$, formula (2.9) holds.

For a submanifold $M$ of a Sasaki-Kenmotsu manifold $\widetilde{M}$, similar definitions in [10] can be used. First we put

$$\varphi X = TX + NX, \quad \text{for all} \quad X, TX \in TM \quad \text{and} \quad NX \in T^\perp M, \tag{2.12}$$

and

$$\varphi V = tV + nV, \quad \text{for all} \quad tV \in TM \quad \text{and} \quad V, nV \in T^\perp M. \tag{2.13}$$

**Definition 2.4.** A submanifold $M$ of a Sasaki-Kenmotsu manifold $\widetilde{M}$ is said to be:

- $\varphi$-invariant if $\varphi X \in TM$ ( i.e., $N = 0$),
- invariant if it is $\varphi$-invariant and $\xi, \psi$ are always tangent to $M$,
- anti-$\varphi$-invariant if $\varphi X \in T^\perp M$ ( i.e., $T = 0$),
- anti-invariant if it is anti-$\varphi$-invariant and $\xi, \psi$ are always tangent to $M$,

for any vector field $X$ tangent to $M$.

## 3 Submanifolds of a Sasaki-Kenmotsu manifold tangent to $\xi$ and $\psi$

**Lemma 3.1.** *Let $M$ be a submanifold of a Sasaki-Kenmotsu manifold $\widetilde{M}^{2n+2}$, tangent to $\xi$ and $\psi$. Then we obtain the following*

$$TX = 2\eta(X)\psi + \nabla_X\xi, \quad NX = h(X,\xi), \tag{3.1}$$

$$\nabla_X\psi = -\varphi^2 X + 2\eta(X)\xi, \quad h(X,\psi) = 0, \tag{3.2}$$

$$\nabla_\psi\xi = h(\psi,\xi) = h(\xi,\xi) = 0, \quad \nabla_\xi\psi = 2\xi \quad \textit{and} \quad \nabla_\xi\xi = -2\psi, \tag{3.3}$$

*for any $X, Y \in TM$.*

*Proof.* Using the Gauss formula, (2.10) and (2.12), then comparing the tangential and normal components of $\varphi X$, we obtain (3.1). Next, in view of (2.11) and (2.1) we obtain (3.2). The last part follows from (2.10), (2.1) and (3.2). □

### 3.1 Invariant submanifolds

In this subsection, we investigate some geometric conditions for an invariant submanifold $M$ of a Sasaki-Kenmotsu manifold $\widetilde{M}$.

**Proposition 3.2.** *Let $M$ be an invariant submanifold of a Sasaki-Kenmotsu manifold $(\widetilde{M}, \varphi, \xi, \psi, \eta, \omega, g)$. Then there exist two differentiable orthogonal distributions $D$ and $D^\perp$ on $M$ such that*

$$TM = D \oplus D^\perp \oplus \langle\xi\rangle \oplus \langle\psi\rangle,$$

*and*

$$\varphi(D) \subset D^\perp, \qquad \varphi(D^\perp) \subset D.$$

*Proof.* For an invariant submanifold $M$, $\xi$ and $\psi$ are tangent to $M$. Hence, we can write

$$TM = D' \oplus \langle\xi\rangle \oplus \langle\psi\rangle.$$

Let $X' \in D'$. Now $g(X', \varphi X') = 0$ and $g(\xi, \varphi X') = g(\psi, \varphi X') = 0$. So, $\varphi X'$ is orthogonal to $X'$, $\xi$ and $\psi$. Consequently, it is possible to write $D' = D \oplus D^\perp$, where $X' \in D \subset D'$ and $\varphi X' \in D^\perp \subset D'$. For $\varphi X' \in D^\perp$, we note that

$$\varphi(\varphi X') = \varphi^2 X' = -X' + \eta(X')\xi + \omega(X')\psi = -X' \in D.$$

Let $\varphi X' = X'' \in D^\perp$. Hence, for $X' \in D$, $\varphi X' \in D^\perp$ and for $X'' \in D^\perp$, $\varphi X'' \in D$. Hence, the proposition follows. □

Now, using (3.1), we state the following corollary

**Corollary 3.3.** *Let $M$ be a totally geodesic submanifold of a Sasaki-Kenmotsu manifold tangent to $\xi$ and $\psi$, then $M$ is an invariant submanifold.*

Moreover, we have

**Corollary 3.4.** *Let $M$ be a totally umbilical submanifold of a Sasaki-Kenmotsu manifold tangent to at least one of $\xi$ and $\psi$, then $M$ is necessarily minimal (consequently $M$ is totally geodesic).*

*Proof.* Suppose that $M$ is totally umbilical, and by using (3.2) (or (3.3)), we get

$$H = g(\psi,\psi)H = g(\xi,\xi)H = 0.$$

□

**Proposition 3.5.** *Let $M$ be an invariant submanifold of a Sasaki-Kenmotsu manifold. Then, we obtain the following formulas*

$$h(X,\xi) = 0, \tag{3.4}$$

$$h(X,\varphi Y) = \varphi h(X,Y) = h(\varphi X, Y), \tag{3.5}$$

*for any vector fields $X, Y$ on $M$.*

*Proof.* Assume that $M$ is an invariant submanifold and by virtue of Lemma 3.1, we obtain directly the first relation. Since $\varphi X$ is tangent to $M$, for any $X \in TM$, and by using Gauss formula, we obtain

$$h(X,\varphi Y) = \widetilde{\nabla}_X(\varphi Y) - \nabla_X(\varphi Y). \tag{3.6}$$

On the other hand, we have

$$\begin{aligned} \varphi h(X,Y) &= \varphi\widetilde{\nabla}_X Y - \varphi\nabla_X Y \\ &= \widetilde{\nabla}_X(\varphi Y) - (\widetilde{\nabla}_X\varphi)Y - \nabla_X(\varphi Y) + (\nabla_X\varphi)Y \\ &= \widetilde{\nabla}_X(\varphi Y) - \nabla_X(\varphi Y), \end{aligned} \tag{3.7}$$

because of $\varphi\widetilde{\nabla}_X Y = (\widetilde{\nabla}_X\varphi)Y - \widetilde{\nabla}_X(\varphi Y)$ and $(\widetilde{\nabla}_X\varphi)Y$ is tangent to $M$.
So, from (3.6), (3.7) and the symmetry of $h$, we obtain (3.5). □

We choose in a Sasaki-Kenmotsu manifold $\widetilde{M}^{(2n+2)}$ a local orthonormal basis of vector fields $\{e_1,\dots,e_m,e_{m+1} = \varphi e_1,\dots,e_{2m} = \varphi e_m, e_{2m+1},\dots,e_n,\xi,\psi\}$, such that $\{e_1,\dots,e_{2m},\xi,\psi\}$ belong to the submanifold $M^{2m+2}$.

By Proposition 3.2 and since $\varphi X$ is tangent to $M$ for any $X \in TM$, we can state the following statement:

**Theorem 3.6.** *Any invariant submanifold $M$ of a Sasaki-Kenmotsu manifold is a Sasaki-Kenmotsu manifold. Hence, the following relations hold:*

$$(\nabla_X\varphi)Y = g(\varphi X,\varphi Y)\xi + \eta(Y)\varphi^2 X + g(\varphi X,Y)\psi - \omega(Y)\varphi X,$$

$$\nabla_X\xi = \varphi X - 2\eta(X)\psi, \quad \nabla_X\psi = -\varphi^2 X + 2\eta(X)\xi,$$

*for all vector fields $X, Y$ tangent to $M$.*

**Proposition 3.7.** *Every invariant submanifold of a $(2n+2)$-dimensional Sasaki-Kenmotsu manifold is minimal.*

*Proof.* By using (3.3), (3.2), (3.5), and the mean curvature vector definition, we get

$$\begin{aligned} H &= \frac{1}{2m+2}\Big(\sum_{i=1}^{2m+2} h(e_i,e_i)\Big) \\ &= \frac{1}{2m+2}\sum_{k=1}^{m}\big(h(e_k,e_k) + h(\varphi e_k,\varphi e_k)\big) \\ &= \frac{1}{2m+2}\sum_{k=1}^{m}\big(h(e_k,e_k) + \varphi^2 h(e_k,e_k)\big), \end{aligned}$$

where $\{e_i\}_{i=1}^{2m+2}$ be a local orthonormal basis on $M$. So, by $\varphi^2 h(e_k,e_k) = -h(e_k,e_k)$ we conclude that $H = 0$, which means $M$ is minimal. □

**Theorem 3.8.** *Let $M$ be an invariant submanifold of a Sasaki-Kenmotsu manifold. Then, $M$ is parallel if and only if $M$ is totally geodesic.*

*Proof.* Suppose that $M$ is parallel, then $\widetilde{\nabla}_X h(Y,\psi) = 0$, for any $X, Y \in TM$, it follows that

$$\nabla^{\perp}_X h(Y,\psi) - h(\nabla_X Y,\psi) - h(Y,\nabla_X\psi) = 0.$$

Using (3.2) in the above equation, we obtain

$$h(Y,\nabla_X\psi) = h(Y,-\varphi^2 X + 2\eta(X)\xi) = 0. \tag{3.8}$$

From (2.5), (3.2) and (3.4), the equation (3.8) reduces to $h(Y,X) = 0$ for any $X, Y \in TM$, which means $M$ is totally geodesic. The converse statement is trivial. □

### 3.2 Anti-invariant submanifolds

In this subsection, we study an anti-invariant submanifold $M$ of a Sasaki-Kenmotsu manifold $\widetilde{M}^{2n+2}$.

**Proposition 3.9.** *Let $M$ be a submanifold of a Sasaki-Kenmotsu manifold. Then, $M$ is anti-invariant if and only if it satisfies*

$$\nabla_X \xi = -2\eta(X)\psi, \tag{3.9}$$

*for any vector field $X$ tangent to $M$.*

*Proof.* In view of Lemma 3.1 and the anti-invariant submanifold definition, we obtain the above relation. □

**Proposition 3.10.** *Let $M$ be an $(n+2)$-dimensional anti-invariant submanifold of a $(2n+2)$-dimensional almost bi-contact metric manifold. Then, $M$ is tangent to the global vector fields $\xi$ and $\psi$.*

*Proof.* Since $M$ is an anti-invariant submanifold, we have $\varphi T_x M = T_x^{\perp} M$, at each point $x$ of $M$. Then, $g(\xi, \varphi X) = -g(\varphi\xi, X) = 0$, and also $g(\psi, \varphi X) = -g(\varphi\psi, X) = 0$, for any vector field $X$ tangent to $M$. Thus, $M$ is tangent to $\xi$ and $\psi$. □

**Proposition 3.11.** *Let $M$ be an $(n+2)$-dimensional anti-invariant submanifold of a $(2n+2)$-dimensional Sasaki-Kenmotsu manifold. Then, $M$ cannot be totally umbilical.*

*Proof.* Assume that $M$ is an anti-invariant totally umbilical submanifold such that $\xi, \psi \in TM$. So, $h(X,Y) = g(X,Y)H$, for any vector fields $X$ and $Y$ tangent to $M$.
Taking $Y = \psi$ in the above equation and using (3.2), we get $H = 0$, which implies $h = 0$ i.e., $M$ is totally geodesic. But since $M$ is anti-invariant and from (3.1), we get $\varphi X = NX = h(X,\xi) \neq 0$, and this leads to a contradiction. □

**Theorem 3.12.** *Let $M$ be a submanifold of a Sasaki-Kenmotsu manifold tangent to $\xi$ and $\psi$. Then, $M$ is anti-invariant if and only if $\nabla T = 0$.*

*Proof.* Using (2.12), (2.13) and the Gauss-Weingarten formulas, we obtain

$$\begin{aligned}(\widetilde{\nabla}_X\varphi)Y &= \widetilde{\nabla}_X\varphi Y - \varphi\widetilde{\nabla}_X Y \\ &= \big((\nabla_X T)Y - A_{NY}X - th(X,Y)\big) \\ &\quad + \big(\underbrace{\nabla^{\perp}_X(NY) - N\nabla_X Y}_{(\nabla_X N)Y} + h(X,TY) - nh(X,Y)\big).\end{aligned}$$

Then, using (2.9) and comparing the tangential and normal components in the above formula, we have

$$\begin{cases} (\nabla_X T)Y - A_{NY}X - th(X,Y) = g(\varphi X, \varphi Y)\xi + \eta(Y)\varphi^2 X + g(\varphi X, Y)\psi - \omega(Y)TX \\ \nabla^{\perp}_X(NY) - N\nabla_X Y + h(X,TY) - nh(X,Y) = -\omega(Y)NX. \end{cases} \tag{3.10}$$

Taking $Y = \xi$ in the first equation of (3.10), using (2.6) and (3.1), we get

$$(\nabla_X T)\xi = t(NX) - \varphi^2 X. \tag{3.11}$$

$$\varphi^2 X = \varphi(TX + NX) = T^2 X + N(TX) + t(NX) + n(NX). \tag{3.12}$$

From (3.11) and (3.12), we deduce the relation

$$(\nabla_X T)\xi = -T^2 X - N(TX) - n(NX).$$

Since $(\nabla_X T)\xi$ is tangent to $M$, the last equation reduces to

$$(\nabla_X T)\xi = -T^2 X. \tag{3.13}$$

Thus, if $M$ is an anti-invariant submanifold, we can obtain directly from (3.13) that $\nabla T = 0$. Conversely, if $\nabla T = 0$, the relation (3.13) implies $T^2 = 0$; by equation (3.1), we find

$$T^2 X = T(\nabla_X \xi) = 0,$$

for any $X \in TM$. That is, $T = 0$ (i.e., $M$ is an anti-invariant submanifold). □

**Example 3.13.** Based on the above Example 2.3, let $f_1$ be an isometric immersion from $M_1$ to $\widetilde{M}$ defined by

$$f_1(x,y,z,t) = (x_1, 0, x_3, 0, x_5, x_6),$$

with $M_1 = \{(x,y,z,t) \in \mathbb{R}^4 \mid t > 0\}$, where $(x,y,z,t)$ are the standard coordinates on $\mathbb{R}^4$. The vector fields

$$e_1 = \frac{1}{t}\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{t}\left(\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z}\right), \quad e_3 = \xi_1 = \frac{1}{t^2}\frac{\partial}{\partial z}, \quad e_4 = \psi_1 = t\frac{\partial}{\partial t},$$

are linearly independent at each point of $M_1$. Let $g_1$ be the metric given by

$$g_1 = t^2dx^2 + t^2(1+4x^2t^2)dy^2 + t^4dz^2 + \frac{1}{t^2}dt^2 - 4xt^4dydz.$$

Let $\eta_1$ and $\omega_1$ be the 1-forms defined by $\eta_1(X) = g_1(X, e_3)$ and $\omega_1(X) = g_1(X, e_4)$, for any vector field $X$ tangent to $M$. Let $\varphi_1$ be the $(1,1)$-tensor field defined by

$$\varphi_1 e_1 = e_2, \quad \varphi_1 e_2 = -e_1 \quad \text{and} \quad \varphi_1 e_3 = \varphi_1 e_4 = 0.$$

The linearity of $\varphi_1$ and $g_1$ yields that

$$\eta_1(\xi_1) = \omega(\psi_1) = 1, \quad \eta_1(\psi_1) = \omega(\xi_1) = 0, \quad \varphi_1^2 e_i = -e_i + \eta_1(e_i)\xi_1 + \omega_1(e_i)\psi_1,$$

$$g_1(\varphi_1 e_i, \varphi_1 e_j) = g_1(e_i, e_j) - \eta_1(e_i)\eta_1(e_j) - \omega_1(e_i)\omega_1(e_j) \quad \text{for all} \quad 1 \le i, j \le 3.$$

Thus, $(\varphi_1, \xi_1, \eta_1, g_1)$ defines an almost bi-contact metric structure on $M_1$. The non-zero component of the fundamental 2-form is

$$\phi_1\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big) = -t^2,$$

and we have $\phi_1 = -2t^2 dx \wedge dy$. We note that

$$\mathrm{d}\eta_1 = \mathrm{d}\big(t^2(dz - 2xdy)\big) = \phi_1 - 2\eta_1 \wedge \omega_1,$$

and $\omega_1 = \frac{1}{t}dt$ which implies $\mathrm{d}\omega_1 = 0$. The non-zero components of Lie bracket are

$$[e_1, e_2] = 2e_3, \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2.$$

By direct calculations, one can check that $N^{(1)} = 0$. Therefore, in view of Definition 2.1, we conclude that $(\varphi_1, \xi_1, \eta_1, g_1)$ is a Sasaki-Kenmotsu structure on the submanifold $M_1$ of $\widetilde{M}$ tangent to $\xi$ and $\psi$.

Using Koszul's formula, the non-zero components of the Levi-Civita connection $\nabla^1$ corresponding to $g$ are given by:

$$\begin{aligned}
&\nabla^1_{e_1} e_1 = \nabla^1_{e_2} e_2 = \nabla^1_{e_4} e_4 = -\psi,\\
&\nabla^1_{e_1} e_2 = -\nabla^1_{e_2} e_1 = \xi,\\
&\nabla^1_{e_1}\xi = \nabla^1_{\xi} e_1 = -\nabla^1_{e_2}\psi = -e_2,\\
&\nabla^1_{e_1}\psi = \nabla_{e_2}\xi = \nabla^1_{\xi} e_2 = e_1,\\
&\nabla^1_{\xi}\xi = -2\psi, \quad \nabla_{\xi}\psi = 2\xi.
\end{aligned}$$

Let us take $D_1 = \langle e_1 \rangle$, $D_1^{\perp} = \langle e_2 \rangle$. Then clearly we see that

$$TM = D_1 \oplus D_1^{\perp} \oplus \langle \xi \rangle \oplus \langle \psi \rangle.$$

For any $X \in D_1$ and $Y \in D_1^{\perp}$, we can write $X = \lambda_1 e_1$ and $Y = \lambda_2 e_2$, where $\lambda_1$ and $\lambda_2$ are two scalars. Now, $\varphi_1 X = \varphi_1(\lambda_1 e_1) = \lambda_1 \varphi_1 e_1 = \lambda_1 e_2 \in D_1^{\perp} \subset TM_1$. Similarly, $\varphi_1 Y = -\lambda_2 e_1 \in D_1 \subset TM_1$. Hence, the submanifold is invariant.

Now from the values of $\nabla^1_{e_i} e_j$ and $\widetilde{\nabla}_{\tilde{e}_i}\tilde{e}_j$, we see that $h(e_i, e_j) = 0$ for all $i, j \in \{1,2,3,4\}$. This means that the submanifold $M_1$ is totally geodesic. Thus, results (3.3) and (3.4) are verified.

## 4 Submanifolds of a Sasaki-Kenmotsu normal to $\xi$ (resp. to $\psi$ )

### 4.1 Submanifold normal to $\xi$

**Theorem 4.1.** *Let $M$ be a submanifold of a Sasaki-Kenmotsu manifold normal to $\xi$. Then, $M$ is anti-$\varphi$-invariant.*

*Proof.* From (2.12) and (2.8), we have

$$g(X,TY) = g(X,\varphi Y) = \phi(X,Y) = d\eta(X,Y) + 2(\eta\wedge\omega)(X,Y),$$

for any $X,Y \in TM$. Since $\xi$ is a normal vector field on $M$, the above relation becomes $g(X,TY)=0$, which implies $M$ is anti-$\varphi$-invariant. □

**Lemma 4.2.** *Let $M$ be a submanifold of a Sasaki-Kenmotsu manifold normal to $\xi$ and tangent to $\psi$, we have*

$$TX = -A_\xi X \quad \textit{and} \quad NX = \nabla^\perp_X \xi. \tag{4.1}$$

*Moreover, if $\psi \in TM$, then we get*

$$\nabla_X \psi = X - \omega(X)\psi \quad \textit{and} \quad h(X,\psi) = 0, \tag{4.2}$$

*for any $X \in TM$.*

*Proof.* Assuming that $\xi \in T^\perp M$ and $\psi \in TM$, from the Gauss-Weingarten formulas and (2.10) (resp., (2.11) and (2.5)), we get

$$\begin{aligned} \widetilde{\nabla}_X \xi &= -A_\xi X + \nabla^\perp_X \xi \\ &= \varphi X \end{aligned}$$

and

$$\begin{aligned} \widetilde{\nabla}_X \psi &= \nabla_X \psi + h(X,\psi) \\ &= X - \omega(X)\psi, \end{aligned}$$

for any $X \in TM$. Comparing the tangent and normal parts of the two above relations yields (4.1) and (4.2) respectively. □

**Corollary 4.3.** *Let $M$ be a totally umbilical submanifold of a Sasaki-Kenmotsu manifold, such that $\xi \in T^\perp M$ and $\psi \in TM$. Then, $M$ is minimal (consequently it is totally geodesic).*

*Proof.* Setting $X = \psi$ in (4.2), we get

$$H = g(\psi,\psi)H = h(\psi,\psi) = 0.$$

□

**Theorem 4.4.** *Any submanifold of a Sasaki-Kenmotsu manifold normal to $\xi$ and tangent to $\psi$ is totally geodesic.*

*Proof.* If $M$ is normal to $\xi$, then in view of Theorem 4.1 $M$ is an anti-$\varphi$-invariant submanifold (i.e., $T=0$) and by using (4.1), we get

$$g(TX,Y) = -g(A_\xi X, Y) = 0.$$

That implies $h(X,Y)=0$ for any $X,Y \in TM$, which means $M$ is totally geodesic. □

**Example 4.5.** Again, based on the above Example 2.3, let $f_2$ be an isometric immersion from $M_2$ to $\widetilde{M}$ defined by

$$f_2(x,y,t) = (x_1, x_2, 0, 0, 0, x_6),$$

with $M_2 = \{(x, y, t) \in \mathbb{R}^3 \mid t > 0\}$, where $(x, y, t)$ are the standard coordinates on $\mathbb{R}^3$. The vector fields

$$e_1 = \frac{1}{t}\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{t}\frac{\partial}{\partial y}, \quad e_3 = t\frac{\partial}{\partial t},$$

are linearly independent at each point of $M_2$. Let $g_2$ be the metric given by

$$g_2 = t^2(dx^2 + dy^2 + \frac{1}{t^4}dt^2).$$

The non-zero components of Lie bracket are

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

Using Koszul's formula, the non-zero components of the Levi-Civita connection $\nabla^2$ corresponding to $g$ are given by:

$$\nabla^2_{e_1} e_1 = \nabla^2_{e_2} e_2 = -e_3, \quad \nabla^2_{e_1} e_3 = e_1, \quad \nabla^2_{e_2} e_3 = e_2.$$

Let $U = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ and $V = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3$, be two vector fields on $M^2$ where $\lambda_i$ and $\mu_i$ are scalars, $i \in \{1, 2, 3\}$. Then,

$$\begin{aligned} h(U, V) \quad = \quad & \lambda_1\mu_1 h(e_1, e_1) + \lambda_1\mu_2 h(e_1, e_2) + \lambda_1\mu_3 h(e_1, e_3) \\ & + \lambda_2\mu_1 h(e_2, e_1) + \lambda_2\mu_2 h(e_2, e_2) + \lambda_2\mu_3 h(e_2, e_3) \\ & + \lambda_3\mu_1 h(e_3, e_1) + \lambda_3\mu_2 h(e_3, e_2) + \lambda_3\mu_3 h(e_3, e_3). \end{aligned}$$

From the values of $\nabla^2_{e_i} e_j$ and $\widetilde{\nabla}_{\tilde{e}_i}\tilde{e}_j$ calculated before and from the relation $h(e_i, e_j) = \widetilde{\nabla}_{\tilde{e}_i}\tilde{e}_j - \nabla^2_{e_i} e_j$, we see that $h(U, V) = 0$, for all $U, V$ vector fields on $M^2$. Hence, the submanifold $M^2$ is totally geodesic. Thus, Theorem (4.4) is verified.

### 4.2 Submanifold normal to $\psi$

Now, suppose that $\psi \in T^\perp M$ and $\xi \in TM$, by using a similar proof of Lemma 4.2, we can prove:

**Lemma 4.6.** *Let $M$ be a submanifold of a Sasaki-Kenmotsu manifold normal to $\psi$ and tangent to $\xi$, we have*

$$TX = \nabla_X \xi \quad \textit{and} \quad NX = 2\eta(X)\psi + h(X, \xi), \tag{4.3}$$

*and*

$$A_\psi X = -X - \eta(X)\xi \quad \textit{and} \quad \nabla^\perp_X \psi = 0, \tag{4.4}$$

*for any $X \in TM$.*

**Proposition 4.7.** *Any submanifold of a Sasaki-Kenmotsu manifold normal to $\psi$ and tangent to $\xi$ cannot be totally geodesic.*

*Proof.* From (4.3), we put $X = \xi$, we obtain $h(\xi, \xi) = -2\psi \neq 0$. □

By using (4.3) of Lemma 4.6, we can easily prove:

**Proposition 4.8.** *Let $M$ be a submanifold of a Sasaki-Kenmotsu manifold normal to $\psi$ and tangent to $\xi$. Then $\xi$ is parallel i.e., $\nabla\xi = 0$, if and only if $M$ is anti-$\varphi$-invariant.*

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