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New class of almost contact metric manifolds

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Abstract. In this paper, we study C_4-manifold as an interesting class of almost contact metric manifolds that appears naturally in the classification of D. Chinea and C. Gonzalez. We study the properties of this class and demonstrate its existence using concrete examples. Then, we are particularly interested in a special type where we will study their fundamental properties and we present an example which justify their study. Finally, we propose a deformation to obtain well-known manifolds.

1 Introduction

In [14], D. Chinea and C. Gonzalez obtained a classification of the almost contact metric manifolds by studying the space of tensors with the same symmetries as the covariant derivative of the fundamental 2-form. This space is decomposed into twelve irreducible components $C_1, ..., C_{12}$.

Most of the researches related to almost contact metric structures is concerned with the normal structures which contain the class C_0 of cosymplectic manifolds, the class C_5 of α-Kenmotsu manifolds and the class C_6 of β-Sasakian ones.

Thanks to [3, 4, 6, 12], C_{12}-manifolds have became a well-known class of non-normal manifolds.

In the classification of Chinea-Gonzalez, another class of almost contact metric manifolds called C_4-manifolds, which can be normal, arises in a natural way. Unfortunately, such manifolds do not exist in dimension three, but explicit examples of proper C_4-manifolds can occur in any dimension $2n + 1 \geq 5$.

The local structure of C_4-manifolds is completely characterized in [1]. It is proved that every C_4-manifold M is locally isometric to a Riemannian product $N \times \mathbb{R}$, where N is a Hermitian manifold belonging to the Gray-Hervella class W_4.

Indeed, it is already known that if $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold belonging to the Chinea-Gonzalez class $\mathcal{D}_1 := C_1 \oplus C_2 \oplus C_3 \oplus C_4$, the following hold:

1) $\mathrm{d}\eta = 0$, so that the distribution $\mathcal{D} = Ker(\eta)$ is integrable;

2) $\nabla\xi = 0$, where ∇ denotes the Levi-Civita connection;

3) $\mathcal{L}_\xi\varphi = 0$, where $\mathcal{L}_\xi$ denotes the Lie derivative;

4) ξ is a Killing vector field.

As a consequence, every almost contact metric manifold belonging to $\mathcal{D}_1$ is locally isometric to a Riemannian product $N \times \mathbb{R}$, where N is an almost Hermitian manifold, and there is a one-to-one correspondence between Chinea-Gonzalez classes C_1, C_2, C_3, C_4 and Gray-Hervella classes W_1, W_2, W_3, W_4 [18].

It is worth recalling that locally conformal Kähler manifolds always belong to the Gray-Hervella class W_4, and they coincide with the entire class W_4 in dimension at least 6 (see, [2]).

In this paper we study some properties of such manifolds and construct some relations between class C_4 and other classes as trans-Sasakian and locally conformal Kählerian manifolds. Finally, we establish an interesting subclass of C_4-manifolds which have a direct relationship with the Vaisman manifolds, called by us special C_4-manifolds. We will conclude the study by constructing different known structures (cosymplectic, Kenmotsu, Sasaki and C_{12}-structure) starting from a special C_4-manifold.

First of all, we will start by introducing the basic concepts that we need in this research.

2 Review definitions and needed results

2.1 Almost Hermitian manifolds

An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold. For an almost Hermitian manifold (N, J, h) we thus have

$$J^2 = -I, \qquad h(JX, JY) = h(X, Y). \tag{2.1}$$

for all X and Y vector fields on N, where I is the identity transformation. An almost complex structure J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_J vanishes, with

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y].$$

It is also shown [19] that J is integrable if and only if $(\nabla^h_{JX} J)JY = (\nabla^h_X J)Y$ where ∇^h is the Levi-Civita connection of M. For an almost Hermitian manifold (N, J, h), we define the fundamental Kähler form Ω as:

$$\Omega(X, Y) = h(X, JY). \tag{2.2}$$

The Hermitian manifold (N, J, h) is called Kählerian if Ω is closed, and thus is characterized by the conditions: $\mathrm{d}\Omega = 0$ and $N_J = 0$. One can prove that both these conditions combined are equivalent with the single condition

$$\nabla^h J = 0.$$

In the classification of Gray-Hervella [18] of almost Hermitian manifolds an interesting class that arises in the scheme is the class corresponding to W_4. This is the class of Hermitian manifolds (M^{2n}, J, g) satisfying the identity

$$\begin{aligned} 2(n-1)(\nabla_X \Omega)(Y, Z) &= g(X, Z)\delta\Omega(Y) - g(X, Y)\delta\Omega(Z) \\ &\quad + g(X, JZ)\delta\Omega(JY) - g(X, JY)\delta\Omega(JZ), \end{aligned} \tag{2.3}$$

where $\delta\Omega$ denotes the codifferential of the form Ω. For an adapted local frame given by $\{e_i\}_{1\le i\le 2n}$, $\delta\Omega$ is given by

$$\begin{aligned} \delta\Omega(X) &= \sum_{i=1}^{2n} (\nabla_{e_i}\Omega)(e_i, X) \\ &= -\sum_{i=1}^{2n} g\big((\nabla_{e_i} J)e_i, X\big). \end{aligned} \tag{2.4}$$

Definition 2.1. [21] A Hermitian manifold (N, J, h) is called locally conformal Kähler (shortly, l.c.K) if there exists a closed one-form μ (called the Lee form) which satisfies

$$\mathrm{d}\Omega = \mu \wedge \Omega. \tag{2.5}$$

One can see that the class W_4 contains all locally conformal Kähler manifolds. In ([16], Corollary. 1.1), the $2n$-dimensional l.c.K manifolds are characterized by:

$$2(\nabla^h_X J)Y = \nu(Y)X - \mu(Y)JX - h(X, Y)U - \Omega(X, Y)V, \tag{2.6}$$

where V is the contravariant field of μ called Lee vector field, $\nu = \mu \circ J$ and $U = -JV$ are respectively the anti-Lee form and the anti-Lee vector field.

The most important subclass of l.c.K manifolds is defined by the vanishing of the Lee form μ or by its parallelism with respect to the Levi-Civita connection of h.

Definition 2.2. [24] An l.c.K manifold (N, J, μ, h) is called a:

- Kählerian manifold if $\mu = 0$ i.e. $\mathrm{d}\Omega = 0$.
- Vaisman manifold if $\nabla^h \mu = 0$.

2.2 Almost contact metric manifolds

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$ tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\begin{cases} \eta(\xi) = 1 \\ \varphi^2 X = -X + \eta(X)\xi \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \end{cases} \tag{2.7}$$

for any vector fields X, Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

The fundamental 2-form ϕ is defined by

$$\phi(X,Y) = g(X, \varphi Y).$$

The almost contact structure (φ, ξ, η) is said to be normal if and only if

$$N^{(1)}(X,Y) = N_\varphi(X,Y) + 2\mathrm{d}\eta(X,Y)\xi = 0, \tag{2.8}$$

for any X, Y on M, where d denotes the exterior derivative and N_φ denotes the Nijenhuis torsion of φ, given by

$$N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{2.9}$$

It is already known that an almost contact metric manifold is normal if and only if it belongs to $C_3 \oplus ... \oplus C_8$ (see [14]).

An almost contact metric structures (φ, ξ, η, g) on M is said to be:

$$\begin{cases} (a) : Sasaki \;\Leftrightarrow\; \phi = \mathrm{d}\eta \; and \; (\varphi, \xi, \eta) \; is \; normal, \\ (b) : Cosymplectic \;\Leftrightarrow\; \mathrm{d}\phi = \mathrm{d}\eta = 0 \; and \; (\varphi, \xi, \eta) \; is \; normal, \\ (c) : Kenmotsu \;\Leftrightarrow\; \mathrm{d}\eta = 0, \; \mathrm{d}\phi = 2\eta \wedge \phi \; \; and \; (\varphi, \xi, \eta) \; is \; normal. \end{cases} \tag{2.10}$$

These manifolds are very well known and they can be characterized through their Levi-Civita connection, by requiring

$$\begin{cases} (1) : Sasaki \;\Leftrightarrow\; (\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \\ (2) : Cosymplectic \;\Leftrightarrow\; \nabla\varphi = 0, \\ (3) : Kenmotsu \;\Leftrightarrow\; (\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X. \end{cases} \tag{2.11}$$

In [22], the author proves that (φ, ξ, η, g) is trans-Sasakian structure of type (α, β) if and only if it is normal and

$$\mathrm{d}\eta = \beta\phi, \qquad \mathrm{d}\phi = 2\alpha\eta \wedge \phi, \tag{2.12}$$

where $\alpha = \frac{1}{2n}\delta\phi(\xi)$, $\beta = \frac{1}{2n}\mathrm{div}\xi$ and $\delta = \mathrm{d}^*$ is the formal adjoint of d with respect to g. It is easy to notice that this class is a generalization of the previous three classes.

Another class of almost contact metric manifolds called C_{12}-manifolds which can be integrable but never normal arose in a natural way from the classification of almost contact metric structures by D. Chinea and C. Gonzalez [14].

Recently, many works have appeared on this class. In [4, 5, 12], the $(2n+1)$-dimensional C_{12}-manifolds are characterized by:

$$\mathrm{d}\eta = \eta \wedge \omega \qquad \mathrm{d}\phi = 0 \qquad and \qquad N_\varphi = 0, \tag{2.13}$$

where $\omega = -\big(\nabla_\xi \xi\big)^\flat = -\nabla_\xi \eta$. Or, equivalently

$$(\nabla_X \varphi)Y = -\eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big), \tag{2.14}$$

where ψ denotes the vector field $-\nabla_\xi \xi$.

For more background on almost contact metric manifolds, we recommend the reference [3, 4, 10, 13, 25].

3 C_4-manifolds: Definitions and Properties

In the classification of Chinea and Gonzalez [14] of almost contact metric manifolds an interesting class that arises in the scheme is the class corresponding to C_4. This is the class of almost contact metric manifolds $(M^{2n+1}, \varphi, \xi, \eta, g)$ satisfying the identity

$$\begin{aligned} 2(1-n)(\nabla_X \phi)(Y,Z) =& g(\varphi X, \varphi Y)\delta\phi(Z) - g(\varphi X, \varphi Z)\delta\phi(Y) \\ & - \phi(X,Y)\delta\phi(\varphi Z) + \phi(X,Z)\delta\phi(\varphi Y), \end{aligned} \tag{3.1}$$

with

$$\delta\phi(\xi) = 0, \tag{3.2}$$

where $\delta\phi$ denotes the codifferential of the form ϕ. This class of almost contact metric manifolds is clearly not present in dimension three ($n = 1$), as confirmed in ([14], Theorem 5.1).

For an adapted local frame given by $\{e_i\}_{1\leq i\leq (2n+1)}$, $\delta\phi$ is given by [14]

$$\begin{aligned} \delta\phi(X) &= -\big(\nabla_{e_i}\phi\big)(e_i, X) \\ &= g\big((\nabla_{e_i}\varphi)e_i, X\big). \end{aligned} \tag{3.3}$$

Firstly, let's examine condition (3.2). We have

$$\begin{aligned} \delta\phi(\xi) &= g\big((\nabla_{e_i}\varphi)e_i, \xi\big) \\ &= g\big(\nabla_{e_i}\varphi e_i - \varphi\nabla_{e_i}e_i, \xi\big) \\ &= e_i\big(g(\varphi e_i, \xi)\big) - g\big(\varphi e_i, \nabla_{e_i}\xi\big) \\ &= 0. \end{aligned}$$

Let us take

$$\omega = \frac{1}{2(n-1)}\delta\phi \circ \varphi. \tag{3.4}$$

Since $\nabla\xi = 0$ then $(\nabla_\xi\phi)(\xi, \varphi X) = 0$ and $\delta\eta = 0$. So, according to formula (1.8) in [15], ω is a Lee form of M and it is easy to see that ω is orthogonal to η. If $dim M = 5$ (i.e. $n = 2$) then the Lee form (3.4), generally, it is not closed.

Here, we presente a characterizing proposition for C_4-manifolds as follows:

Proposition 3.1. *An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is a C_4-manifold if and only if*

$$(\nabla_X \varphi)Y = g(\varphi^2 X, Y)A - \theta(Y)\varphi^2 X + g(\varphi X, Y)B - \omega(Y)\varphi X, \tag{3.5}$$

where ω is the Lee form, $\theta = \omega \circ \varphi$, $B = \omega^\sharp$, $A = \theta^\sharp = -\varphi B$. In particular,

$$A = \frac{1}{2(1-n)}(\nabla_{e_i}\varphi)e_i \qquad \text{and} \qquad B = \frac{1}{2(1-n)}\varphi(\nabla_{e_i}\varphi)e_i.$$

Proof. From (3.1), we have

$$\begin{aligned} (\nabla_X\phi)(Y,Z) =& X\big(\phi(Y,Z)\big) - \phi(\nabla_X Y, Z) - \phi(Y, \nabla_X Z) \\ =& X\big(g(Y, \varphi Z)\big) - g(\nabla_X Y, \varphi Z) - g(Y, \varphi\nabla_X Z) \\ =& g\big(Y, (\nabla_X\varphi)Z\big). \end{aligned} \tag{3.6}$$

On the other hand, using $\delta\phi = -2(n-1)\theta$ and $\delta\phi \circ \varphi = 2(n-1)\omega$ we have

$$\begin{aligned} &g(\varphi X,\varphi Y)\delta\phi(Z) - g(\varphi X,\varphi Z)\delta\phi(Y) - \phi(X,Y)\delta\phi(\varphi Z) + \phi(X,Z)\delta\phi(\varphi Y) \\ &\quad = 2(n-1)\big(g(\varphi^2 X,Y)\theta(Z) - g(\varphi^2 X,Z)\theta(Y) \\ &\quad + g(\varphi XY)\omega(Z) - g(\varphi X,Z)\omega(Y)\big) \\ &\quad = 2(1-n)g\big(-\theta(Z)\varphi^2 X + g(\varphi^2 X,Z)A - \omega(Z)\varphi X + g(\varphi X,Z)B,Y\big). \end{aligned} \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.1) we obtain our formula.

For expressions of A and B, just use formula (3.5) with $X = Y = e_i$ where $\{e_i\}_{1\le i\le 2n+1}$ is an orthonormal basis on M. □

One can check that ξ, A and B are three non-zero global vector fields which are naturally pairwise orthogonal.

Theorem 3.2. *Let (N,J,h) be a W_4-manifold. Then, the product space $M = N \times \mathbb{R}$ has a C_4-structure.*

Proof. Under the above assumptions, we define a Riemannian metric tensor g, a vector field ξ, a 1-form η and a $(1,1)$-tensor field φ on M as follows:

$$g = h + dt^2, \quad \xi = \partial_t, \quad \eta = dt, \quad \varphi X = JX \quad and \quad \varphi\partial_t = 0, \tag{3.8}$$

for any vector field X on N where ∂_t denote the unit tangent field to $\mathbb{R}$.

By a direct calculation using (2.7), one can check that (φ,ξ,η,g) is an almost contact metric structure.
In addition, we have $\mathrm{d}\eta = 0$.
Also, the fundamental 2-form ϕ of (φ,ξ,η,g) is

$$\phi\big((X,a\partial_t),(Y,b\partial_t)\big) = g\big((X,a\partial_t),\varphi(Y,b\partial_t)\big),$$

which can be easily verified as follows:

$$\phi = \Omega. \tag{3.9}$$

One can readily check that for any vector fields X, Y, and Z on N, we have:

$$\begin{cases} (\nabla^g_X\phi)(Y,Z) = (\nabla^h_X\Omega)(Y,Z), \\ (\nabla^g_{\partial_t}\phi)(Y,Z) = (\nabla^g_X\phi)(\partial_t,Z) = (\nabla^g_X\phi)(Y,\partial_t) = 0. \end{cases}$$

Therefore, (M,φ,ξ,η,g) is a C_4-manifold. □

Conversely we have the following structure theorem:

Theorem 3.3. *Let M be a C_4-manifold. Then, for any $p \in M$, some neighborhood U_p of $p \in M$ is identified with a warped product space $(-\varepsilon,+\varepsilon) \times N$ such that $(-\varepsilon,+\varepsilon)$ is an open interval and $N \in W_4$.*

Proof. We define a distribution $\mathcal{D}$ by $\mathcal{D} = \{X \in \Gamma(TM) \,/\, \eta(X) = 0\}$. It is completely integrable by $\nabla\eta = 0$. Let M_p be the maximal integral submanifold through p. M_p is a totally umbilical hypersurface of M because of $\nabla\xi = 0$. J and h denote the restriction of φ and g to M_p respectively. Then M_p is an almost Hermitian manifold for (J,h). Moreover, by (3.5) M_p is a W_4-manifold. □

Corollary 3.4. *If $dimM \ge 7$ then the Lee form ω of a C_4-manifold is closed.*

Proof. In ([17], Proposition 3.6) it is proved that if an almost Hermitian manifold of dimension $2n$ with $n > 2$ is of type $W_2 \oplus W_4$, then its Lee form is closed. Hence, by Theorems 3.2 and 3.3, ω is closed in dimension $2n+1$. □

Now, we give a new characterization of class C_4

Theorem 3.5. *An almost contact metric manifold (M,φ,ξ,η,g) is of class C_4 if and only if there exists a 1-form ω orthogonal to η such that*

$$\mathrm{d}\eta = 0 \qquad \mathrm{d}\phi = 2\omega\wedge\phi \qquad \textit{and} \qquad N^{(1)} = 0. \tag{3.10}$$

Proof. Assume that M is an almost contact metric manifold which satisfies (3.10). Then, we can use lemma (6.1 p. 82) from [10],

$$\begin{aligned} 2g\big((\nabla_X\varphi)Y,Z\big) &= 3\mathrm{d}\phi(X,\varphi Y,\varphi Z) - 3\mathrm{d}\phi(X,Y,Z) \\ &+ g\big(N^{(1)}(Y,Z),\varphi X\big) + N^{(2)}(Y,Z)\eta(X) \\ &+ 2\mathrm{d}\eta(\varphi Y,X)\eta(Z) - 2\mathrm{d}\eta(\varphi Z,X)\eta(Y). \end{aligned} \tag{3.11}$$

it is already known that for an almost contact metric manifold, $N^{(1)}=0$ implies $N^{(2)}=0$. Also, knowing that for all X,Y,Z vector fields on M,

$$3(\omega\wedge\phi)(X,Y,Z) = \omega(X)\phi(Y,Z) + \omega(Y)\phi(Z,X) + \omega(Z)\phi(X,Y), \tag{3.12}$$

the proof is direct, just replace formulas (3.10) and (3.12) in (3.11) we get the formula (3.5), i.e. M is a C_4-manifold.

Conversely, assuming that M is in class C_4. So, we have $\mathrm{d}\eta=0$ and $N^{(1)}=0$ as we pointed out in the introduction. we know that

$$3\mathrm{d}\phi(X,Y,Z) = g\big(Y,(\nabla_X\varphi)Z\big) + g\big(Z,(\nabla_Y\varphi)X\big) + g\big(X,(\nabla_Z\varphi)Y\big).$$

Therefore, it is sufficient to use (3.5) and (3.12) directly we find

$$\mathrm{d}\phi = 2\omega\wedge\phi.$$

This completes the proof. □

Corollary 3.6. *On each C_4-manifold*

$$\nabla_A\varphi = \nabla_B\varphi = 0.$$

Example 3.7. Let $\mathbb{R}^5$ be the Euclidean space with coordinates $\{x_i,z\}_{1\le i\le 4}$. The vector fields,

$$e_1 = \frac{1}{\rho}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right), \quad e_2 = \frac{1}{\rho}\frac{\partial}{\partial x_2}, \quad e_3 = \frac{\partial}{\partial x_3}, \quad e_4 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial z}, \quad e_5 = \xi = \frac{\partial}{\partial z},$$

are linearly independent at each point of $\mathbb{R}^5$ where $\rho=\rho(x_4)$ is a function on $\mathbb{R}^5$ non-zero everywhere. Let g be the Riemannian metric on $\mathbb{R}^5$ defined by $g(e_i,e_j)=\delta_{ij}$. That is, the form of the metric becomes

$$g = \begin{pmatrix} 1+\rho^2 & 0 & -1 & 0 & 0 \\ 0 & \rho^2 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Let η be the 1-form on $\mathbb{R}^5$ defined by $\eta = dz$. Let φ be the $(1,1)$-tensor field defined by,

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = e_4, \quad \varphi e_4 = -e_3, \quad \varphi e_5 = 0.$$

By linearity of φ and g, the relations (2.7) are satisfied on $\mathbb{R}^5$. Thus, (φ,ξ,η,g) defines an almost contact metric structure on $\mathbb{R}^5$. We note that

$$\mathrm{d}\eta = 0.$$

The non-zero components of the fundamental 2-form are

$$\phi\Big(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\Big) = -\rho^2, \quad \phi\Big(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_4}\Big) = 1 \quad \textit{and} \quad \phi\Big(\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4}\Big) = -1,$$

hence

$$\phi = -2\rho^2 dx_1 \wedge dx_2 + 2(dx_1 - dx_3) \wedge dx_4.$$

Its differential gives

$$\mathrm{d}\phi = -4\rho'\rho\, dx_4 \wedge dx_1 \wedge dx_2,$$

where $\rho' = \frac{\partial \rho}{\partial x_4}$. By letting $\omega = \frac{\rho'}{\rho} dx_4$ with $\rho' \neq 0$, we have $\mathrm{d}\phi = 2\omega \wedge \phi$. The dual vector field B is given by $B = \frac{\rho}{\rho'} e_4$ implies $A = \frac{\rho}{\rho'} e_3$ and $\theta = \frac{\rho'}{\rho}(dx_3 - dx_1)$. It is easy to see that $\mathrm{d}\omega = 0$ and $\eta(B) = \eta(A) = 0$.

Knowing that the components of the tensor $N^{(1)}$, can be written as,

$$N^{(1)}(e_i, e_j) = \varphi^2[e_i, e_j] + [\varphi e_i, \varphi e_j] - \varphi[\varphi e_i, e_j] - \varphi[e_i, \varphi e_j].$$

The non zero components of Lie bracket are

$$[e_1, e_4] = \frac{\rho'}{\rho} e_1 \quad and \quad [e_2, e_4] = \frac{\rho'}{\rho} e_2. \tag{3.13}$$

By direct calculations, one can check that $N^{(1)} = 0$, therefore (φ, ξ, η, g) is normal. Finally, we can confirm that $(\mathbb{R}^5, \varphi, \xi, \eta, g)$ is a C_4-manifold.

To verify result in formulas (3.5), the non zero components of the Levi-Civita connection corresponding to g are given by:

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = -\frac{\rho'}{\rho} e_4 \quad and \quad \nabla_{e_1} e_4 = \frac{\rho'}{\rho} e_1, \quad \nabla_{e_2} e_4 = \frac{\rho'}{\rho} e_2.$$

Then, one can easily check that for all $i, j \in \{1, 2, ..., 5\}$,

$$(\nabla_{e_i}\varphi)e_j = g(\varphi^2 e_i, e_j)A - \theta(e_j)\varphi^2 e_i + g(\varphi e_i, e_j)B - \omega(e_j)\varphi e_i.$$

This example is actually the product of a trans-Sasakian manifold (with coordinates x_1, x_2, x_3) [11] and $\mathbb{R}^2$.

In general, the product of any trans-Sasaki manifold of dimension $2n + 1$ with $\mathbb{R}^2$ is a C_4-manifold of dimension $2n + 3$. This result is a special case of Theorem 3.2 because of the definition of trans-Sasakian manifolds, which is given in [22].

Indeed, given an almost contact metric manifold $(\hat{M}, \hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$, on the product manifold $N = \hat{M} \times \mathbb{R}$, one can define an almost Hermitian structure (J, h) as

$$J\left(X, a\frac{d}{dr}\right) = \left(\hat{\varphi}X - a\hat{\xi}, \hat{\eta}(X)\frac{d}{dr}\right) \quad and \quad h = \hat{g} + dr^2,$$

where a is a differentiable function on N, and X is a vector field tangent to M. The almost contact metric manifold $(\hat{M}, \hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$ is called trans-Sasakian if the almost Hermitian structure (J, h) belongs to the class W_4.

Therefore, the product of any trans-Sasaki manifold of dimension $2n + 1$ with $\mathbb{R}^2$ are obviously obtained as products $W_4 \times \mathbb{R}$, as in Theorem 3.2.

We can provide examples of C_4-structures constructed from other structures on the same manifold M, using known geometric tools.

Let's start with the cosymplectic structure of dimension ≥ 5, which is a trans-Sasakian structure with $\alpha = \beta = 0$. Using the Marrero deformation [20], for an almost contact metric structure (φ, ξ, η, g) the structure $(\varphi, \xi, \eta, \tilde{g})$ is also an almost contact metric structure where

$$\tilde{g} = fg + (1 - f)\eta \otimes \eta,$$

where f is a strictly positive function on M. Since (φ, ξ, η, g) is a cosymplectic structure on M then we have

$$\mathrm{d}\eta = \mathrm{d}\phi = 0 \qquad and \qquad N^{(1)} = 0.$$

So, easily we obtain

$$\begin{aligned} \tilde{\phi}(X, Y) &= \tilde{g}(X, \varphi Y) \\ &= f\phi(X, Y), \end{aligned}$$

implies $\mathrm{d}\tilde{\phi} = \mathrm{d}(\ln f) \wedge \tilde{\phi}$. Hence, $(\varphi, \xi, \eta, \tilde{g})$ is a C_4-structure on M where $\omega = \frac{1}{2}\mathrm{d}(\ln f)$. and f is not constant, satisfying $\xi(f) = 0$.

Another type of examples are closely related to the warped product space defined by Bishop-O'Neill [9].

Proposition 3.8. *Let (N, J, h) be a Kählerian manifold and σ a smooth function on N. Then the warped product space $M = N \times_\sigma L$ has a C_4-structure.*

Proof. Under the above assumptions, we define a Riemannian metric tensor g, a vector field ξ, a 1-form η and a $(1,1)$-tensor field φ on M as follows:

$$g = \mathrm{e}^{2\sigma} h + dt^2, \quad \xi = \partial_t, \quad \eta = dt, \quad \varphi X = JX \quad and \quad \varphi\partial_t = 0, \tag{3.14}$$

for any vector field X on N where ∂_t denote the unit tangent field to L.

By a direct calculation using (2.7), one can check that (φ, ξ, η, g) is an almost contact metric structure.
In addition, we have $\mathrm{d}\eta = 0$.
Also, the fundamental 2-form ϕ of (φ, ξ, η, g) is given by

$$\phi = \mathrm{e}^{2\sigma}\Omega. \tag{3.15}$$

Since (N, J, h) is a Kählerian manifold then we have $\mathrm{d}\Omega = 0$ and $N_J = 0$ which gives

$$\mathrm{d}\phi = 2\mathrm{d}\sigma \wedge \phi,$$

which implies $\omega = \mathrm{d}\sigma$. Given the definition of φ, we can show that

$$N^{(1)}\big((X\ , a\partial t), \big(Y, b\partial t)\big) = N_J(X, Y) = 0.$$

Therefore, $(M, \varphi, \xi, \eta, g)$ is a 1-parameter family of C_4-manifold. □

4 Special C_4-manifolds

In this section, we study an interesting subclass of the class C_4, which we refer to as special C_4-manifolds. These are locally isometric to a product $N \times \mathbb{R}$, where N is a Vaisman manifold.

Let (N, J, μ, h) be a Vaisman manifold that is $\nabla^h \mu = 0$. Based on Theorem 3.2, one can show that $(M = N \times \mathbb{R}, \varphi, \xi, \eta, g)$ is a C_4-manifold with $\omega = \frac{\mu}{2}$ which implies $\nabla\omega = 0$. Based on these facts, we give the following definition:

Definition 4.1. A C_4-manifold with a Lee form ω is called a special C_4-manifold (shortly, SC_4-manifold) if ω is parallel with respect to the Levi-Civita connection of the metric g.

Remark 4.2. An almost contact metric structure (φ, ξ, η, g) is said to be a special C_4-structure with a Lee form ω if and only if

$$\mathrm{d}\eta = 0, \qquad \mathrm{d}\phi = 2\omega \wedge \phi, \qquad \nabla\omega = 0 \quad and \quad N^{(1)} = 0. \tag{4.1}$$

To prove the existence of such manifolds, it is sufficient to rely on the Theorem 3.2 as we explained above. In addition, we will confirm the existence through the following concrete example:

Example 4.3. Let $\mathbb{R}^5$ be an Euclidean space Cartesian coordinates $\{x_i, z\}_{1\le i\le 4}$. The vector fields,

$$e_1 = \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}, \quad e_3 = \frac{\partial}{\partial x_3}, \quad e_4 = \frac{\partial}{\partial x_4}, \quad e_5 = \xi = \frac{\partial}{\partial z},$$

are linearly independent at each point of $\mathbb{R}^5$. Let g be the Riemannian metric on $\mathbb{R}^5$ defined by $g(e_i, e_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. That is, the form of the metric becomes

$$g = dx_1^2 + (1 + x_1^2)dx_2^2 - 2x_1 dx_2 dx_3 + dx_3^2 + dx_4^2 + dz^2.$$

We define an almost contact metric structure (φ, ξ, η) by:

$$\varphi = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & -1 & 0 \\ 0 & -x_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \xi = \frac{\partial}{\partial z} \quad and \quad \eta = dz.$$

The fundamental 2-forme ϕ has the form

$$\phi = -2dx_1 \wedge dx_2 + 2x_1 dx_2 \wedge dx_4 - 2dx_3 \wedge dx_4.$$

For the non-zero Lie brackets of e_i, we have only one, that is

$$[e_1, e_2] = e_3.$$

Then, by Koszul's formula, the non-zero components of the Riemannian connection with respect to the metric g are given by:

$$\nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = \frac{1}{2} e_3, \quad \nabla_{e_1} e_3 = \nabla_{e_3} e_1 = -\frac{1}{2} e_2, \quad \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{2} e_1$$

Using (3.5) and knowing that $\big(\nabla_{e_i}\varphi\big)e_j = \nabla_{e_i}\varphi e_j - \varphi\nabla_{e_i} e_j$, by straightforward calculation, one verifies that (φ, ξ, η, g) is a C_4-structure.

Again, from (3.5) we find

$$\sum_{i=1}^{5} \big(\nabla_{e_i}\varphi\big)e_i = -2A$$

then we get

$$A = -\frac{1}{2} e_3,$$

which gives

$$B = \varphi A = -\frac{1}{2} e_4, \qquad \omega = -\frac{1}{2}(e_4)^\flat, \qquad \theta = -\frac{1}{2}(e_3)^\flat.$$

where $(e_i)^\flat$ denotes the 1-form defined by $(e_i)^\flat(X) = g(e_i, X)$ for any X vector field on M. So, we have

$$(\nabla_{e_i}\omega)e_j = -\frac{1}{2} g(\nabla_{e_i} e_4, e_j) = 0, \qquad \forall i, j \in \{1, 2, 3, 4, 5\}.$$

We may conclude that (φ, ξ, η, g) is a special C_4-structure.

Proposition 4.4. *Let $(M, \varphi, \xi, \eta, g)$ be a special C_4-manifold. Then, for all X vector field on M we have*

1. $\|B\| = c \in \mathbb{R}_+^*$.
2. $\nabla_X A = c^2 \varphi X + \omega(X)A - \theta(X)B$.
3. *A is a Killing vector field.*
4. $\mathrm{d}\theta = -c^2\phi + 2\omega \wedge \theta$.

Proof. Suppose that $(M, \varphi, \xi, \eta, g)$ is a special C_4-manifold.
1. For all X vector field on M, since the parallelism of the Lee form ω is equivalent to the parallelism of the vector field B, then we have

$$\begin{aligned} X\big(\|B\|^2\big) &= Xg(B, B) \\ &= 2g(\nabla_X B, B) \\ &= 0. \end{aligned}$$

Hence, $\|B\|^2 = const.$
2. For all X, Y vector fields on M, with the use of (3.5) and $\nabla B = 0$, we have

$$\begin{aligned}\nabla_X A &= -\nabla_X(\varphi B)\\ &= -(\nabla_X\varphi)B\\ &= |B|^2\varphi X + \omega(X)A - \theta(X)B.\end{aligned}$$

3) We know that A is a Killing vector field if and only if $\mathcal{L}_A g = 0$ where $\mathcal{L}$ is the Lie derivative. We have

$$\big(\mathcal{L}_A g\big)(X,Y) = g(\nabla_X A, Y) + g(\nabla_Y A, X).$$

So, by the second assertion in this Proposition one can check that A is a Killing vector field.
4) We know that

$$2\mathrm{d}\theta(X,Y) = g(\nabla_X A, Y) - g(\nabla_Y A, X).$$

So, with the use of the second assertion we get our result. □

Remark 4.5. Since the Lee vector field is parallel, it has constant norm and hence we can normalize it such that $\|B\| = 1$. Then, from Proposition 4.4 we conclude that

$$\nabla_X A = \varphi X + \omega(X)A - \theta(X)B \quad and \quad \mathrm{d}\theta = -\phi + 2\omega\wedge\theta. \tag{4.2}$$

5 C_4-manifold with certain deformation

In [7], an interesting transformation in almost contact metric manifolds was included, so that the structural tensor φ and metric tensor g were changed at the same time.

Let (φ,ξ,η,g) be an almost contact metric structure on M^{2n+1}. For any X, Y vector fields on M, we mean a change of structure tensors of the form

$$\begin{cases}\tilde{\xi} = \frac{1}{h}\xi,\\ \tilde{\eta} = h\eta + \overline{\theta},\\ \tilde{\varphi}X = \varphi X - \overline{\theta}(\varphi X)\tilde{\xi},\\ \tilde{g}(X,Y) = fg(\varphi X,\varphi Y) + \tilde{\eta}(X)\tilde{\eta}(Y),\end{cases} \tag{5.1}$$

where $\overline{\theta}$ is a 1-form orthogonal to η, h and f are non-zero functions on M and f is positive.

We can easily prove that the conditions (2.7) are satisfied, which means that $(\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is an almost contact metric structure on M. Its fundamental 2-form $\tilde{\phi}$ is given by

$$\tilde{\phi} = f\phi \tag{5.2}$$

Also, according to [7], For $h = 1$ we have

$$\begin{aligned}\tilde{N}^{(1)}(X,Y) &= N^{(1)}(X,Y) + \overline{\theta}(N^{(1)}(X,Y))\xi\\ &\quad -\overline{\theta}(\varphi X)\Big(N^{(3)}(Y) + \overline{\theta}\big(N^{(3)}(Y)\big)\xi\Big)\\ &\quad +\overline{\theta}(\varphi Y)\Big(N^{(3)}(X) + \overline{\theta}\big(N^{(3)}(X)\big)\xi\Big)\\ &\quad +2\mathrm{d}\overline{\theta}\big(\tilde{\varphi}X,\tilde{\varphi}Y\big)\xi - 2\mathrm{d}\overline{\theta}(X,Y)\xi\end{aligned} \tag{5.3}$$

with $N^{(3)}$ is a tensor field on M given by

$$N^{(3)}(X) = \big(L_\xi\varphi\big)(X) = \varphi[X,\xi] - [\varphi X,\xi],$$

where L_ξ denotes the Lie derivative with respect to the vector field ξ.

Proposition 5.1. *[7] Let (φ,ξ,η,g) be a normal almost contact metric structure on M. The almost contact metric structure $(\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is normal if and only if*

$$\mathrm{d}\overline{\theta}\big(\varphi X,\varphi Y\big) = \mathrm{d}\overline{\theta}(X,Y), \tag{5.4}$$

for any $X, Y \in \Gamma(TM)$.

In this section we use this transformation to construct some well-known structures starting from a C_4-structure with ω closed.

Let $(M, \varphi, \xi, \eta, g)$ be a C_4-manifold with $\mathrm{d}\omega = 0$. By (2.13) one can get

$$\begin{cases} \tilde{\eta} = h\eta + \overline{\theta} \\ \tilde{\phi} = f\phi. \end{cases} \Rightarrow \begin{cases} \mathrm{d}\tilde{\eta} = \mathrm{d}h \wedge \eta + h\mathrm{d}\eta + \mathrm{d}\overline{\theta} \\ \mathrm{d}\tilde{\phi} = \mathrm{d}f \wedge \phi + f\mathrm{d}\phi. \end{cases}$$

$$\Rightarrow \begin{cases} \mathrm{d}\tilde{\eta} = \mathrm{d}h \wedge \eta + \mathrm{d}\overline{\theta} \\ \mathrm{d}\tilde{\phi} = \big(\mathrm{d}(\ln f) + 2\omega\big) \wedge \phi. \end{cases} \tag{5.5}$$

Knowing that C_4-structure is normal then, $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is normal if and only if (5.4) hold.

On the other hand, using $\mathrm{d}\omega = 0$, taking into account Theorem 3.5, we get

$$\begin{cases} \tilde{\eta} = \eta + \omega \\ \tilde{\phi} = f\phi \end{cases} \Rightarrow \begin{cases} \mathrm{d}\tilde{\eta} = 0 \\ \mathrm{d}\tilde{\phi} = \big(\mathrm{d}(\ln f) + 2\omega\big) \wedge \tilde{\phi}. \end{cases} \tag{5.6}$$

We will discuss the different new structures depending on the parameter f.

First case: For $\mathrm{d}(\ln f) + 2\omega = 0$, (5.6) gives

$$\mathrm{d}\tilde{\eta} = \mathrm{d}\tilde{\phi} = 0.$$

Second case: For $\mathrm{d}(\ln f) + 2\omega \neq 0$, (5.6) becomes:

$$\begin{cases} \mathrm{d}\tilde{\eta} = 0 \\ \mathrm{d}\tilde{\phi} = 2\tilde{\omega} \wedge \tilde{\phi}, \end{cases} \tag{5.7}$$

where $\tilde{\omega} = \frac{1}{2}\mathrm{d}(\ln f) + \omega$.

Third case: For $\mathrm{d}(\ln f) = 2\eta$, (5.6) gives:

$$\begin{cases} \mathrm{d}\tilde{\eta} = 0 \\ \mathrm{d}\tilde{\phi} = 2\tilde{\eta} \wedge \tilde{\phi}. \end{cases} \tag{5.8}$$

For a second application, we put $\overline{\theta} = 0$, we obtain

$$\begin{cases} \tilde{\eta} = h\eta \\ \tilde{\phi} = f\phi \end{cases} \Rightarrow \begin{cases} \mathrm{d}\tilde{\eta} = \mathrm{d}(\ln h) \wedge \tilde{\eta} \\ \mathrm{d}\tilde{\phi} = \big(\mathrm{d}(\ln f) + 2\omega\big) \wedge \tilde{\phi}. \end{cases} \tag{5.9}$$

For the last application, we take $\mathrm{d}\overline{\theta} = f\phi$ and $h = 1$, we get

$$\begin{cases} \tilde{\eta} = \eta + \overline{\theta} \\ \tilde{\phi} = f\phi \end{cases} \Rightarrow \begin{cases} \mathrm{d}\tilde{\eta} = \tilde{\phi} \\ \mathrm{d}\tilde{\phi} = \big(\mathrm{d}(\ln f) + 2\omega\big) \wedge \tilde{\phi}. \end{cases} \tag{5.10}$$

Then, for $\mathrm{d}(\ln f) + 2\omega = 0$ we obtain

$$\begin{cases} \mathrm{d}\tilde{\eta} = \tilde{\phi} \\ \mathrm{d}\tilde{\phi} = 0. \end{cases} \tag{5.11}$$

Based on these facts, we give the following Theorem:

Theorem 5.2. *Let $(M, \varphi, \xi, \eta, g)$ be a C_4-manifold with ω closed. Then, the almost contact metric structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, defined by (5.1), is:*

1)	*Cosymplectic if*	$\mathrm{d}\overline{\theta} = 0$,	$h = 1$	*and*	$\mathrm{d}(\ln f) + 2\omega = 0$.
2)	*C_4 if*	$\mathrm{d}\overline{\theta} = 0$,	$h = 1$	*with*	$\tilde{\omega} = \mathrm{d}(\ln f) + 2\omega \neq 0$
3)	*Kenmotsu if*	$\mathrm{d}\overline{\theta} = 0$,	$h = 1$	*and*	$\mathrm{d}(\ln f) = 2\eta$.
4)	*C_{12} if*	$\overline{\theta} = 0$,	$h \neq const$	*with*	$\tilde{\omega} = -\mathrm{d}(\ln h)$, $\mathrm{d}(\ln f) + 2\omega = 0$.
5)	*Sasaki if*	$\mathrm{d}\overline{\theta} = f\phi$,	$h = 1$	*and*	$\mathrm{d}(\ln f) + 2\omega = 0$.

Proof. The necessity was observed above for both cases (see (2.10), (2.13), (4.1)). For the sufficiency, first observe that

$$\begin{cases} \tilde{\eta} = h\eta + \overline{\theta} \\ \tilde{\phi} = f\phi. \end{cases} \quad \Rightarrow \quad \begin{cases} \mathrm{d}\tilde{\eta} = \mathrm{d}h \wedge \eta + h\mathrm{d}\eta + \mathrm{d}\overline{\theta} \\ \mathrm{d}\tilde{\phi} = \mathrm{d}f \wedge \phi + f\mathrm{d}\phi. \end{cases} \tag{5.12}$$

For our special cases, if (φ, ξ, η, g) is:
1) almost cosymplectic with $h = 1$ then, $\mathrm{d}\tilde{\eta} = \mathrm{d}\overline{\theta}$ and $\mathrm{d}\tilde{\phi} = \mathrm{d}(\ln f) \wedge \tilde{\phi}$.
2) almost Kenmotsu with $h = 1$ then, $\mathrm{d}\tilde{\eta} = \mathrm{d}\overline{\theta}$ and $\mathrm{d}\tilde{\phi} = \big(\mathrm{d}(\ln f) + 2\eta\big) \wedge \tilde{\phi}$.
3) almost C_{12}-structure then, $\mathrm{d}\tilde{\eta} = \mathrm{d}\overline{\theta} + (\mathrm{d}h - h\omega) \wedge \eta$ and $\mathrm{d}\tilde{\phi} = \mathrm{d}(\ln f) \wedge \tilde{\phi}$.
4) contact then, $\mathrm{d}\tilde{\eta} = \mathrm{d}\overline{\theta} + \mathrm{d}h \wedge \eta + h\phi$ and $\mathrm{d}\tilde{\phi} = \mathrm{d}(\ln f) \wedge \tilde{\phi}$.
It is easy to see that if the structure (φ, ξ, η, g) is cosymplectic $\overline{\theta}$ is closed □

Example 5.3. From Example 4.3, if we use the transformation (5.1) with:
1) $\overline{\theta} = -\frac{1}{2}dx_4$, $h = 1$ and $f = \mathrm{e}^{x_4}$ we obtain a cosymplectic structure.
2) $\overline{\theta} = -\frac{1}{2}dx_4$, $h = 1$ and $f = 1$ we obtain a special C_4-structure.
3) $\overline{\theta} = -\frac{1}{2}dx_4$, $h = 1$ and $f = \mathrm{e}^{2x_5}$ we obtain a Kenmotsu structure.
4) $\overline{\theta} = 0$, $h = \mathrm{e}^{x_1}$ and $f = \mathrm{e}^{x_4}$ we obtain a C_{12}-structure.
Unfortunately, the example we relied on does not meet condition $\mathrm{d}\overline{\theta} = f\phi$, i.e.

$$f\phi = \mathrm{e}^{x_4}(-2dx_1 \wedge dx_2 + 2x_1 dx_2 \wedge dx_4 - 2dx_3 \wedge dx_4),$$

is not exact. Therefore, Sasaki structures cannot be obtained.

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