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SOME NEW OSCILLATION CRITERIA FOR A CERTAIN CLASS OF KOHN-LAPLACIAN EQUATIONS ON THE HEISENBERG GROUP

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Abstract This paper aims to investigate the oscillation criteria for certain class of Kohn-Laplacian equations on the Heisenberg group $\mathbb{H}^n$. First, we use the divergence theory approach to derive a critical inequality in $\mathbb{H}^n$. The Riccati technique and integral average approaches are utilized to get the results. Moreover, some illustrating examples are given to highlight the significance of our findings.

1 Introduction

Oscillation and non oscillation criteria for elliptic equations were studied by several authors; see [1, 2, 3, 4, 5, 6, 7, 8, 9]. Extensive bibliographies may be found in the books by C. A. Swanson [10], K. Kreith [11] and many authors; see [12, 13]. The monographs can be consulted for the general overview of elliptic partial differential equations [14, 15, 16, 17].

Recently, Duan Wu [18], obtained an oscillation theorem for the nonlinear degenerate elliptic equations in the Heisenberg group. We overcome a fair approximation in [19], but one of the challenges is that there is no good divergence formula for $\mathbb{H}^n$ as $\mathbb{R}^n$. Numerous writers have examined the Laplace equations on the Heisenberg group using various techniques, and some findings have been found, see [20, 21, 22, 23, 24, 25, 26, 27] and references therein. Several writers have examined existence and nonexistence results for semilinear equations on the Heisenberg group, see [28, 29, 30].

In this paper, we derive the new oscillation criteria for certain class of Kohn-Laplacian equations on the Heisenberg group of the form

$$\Delta_{\mathbb{H}^n} u(z,t) + c(z,t,u(z,t)) = 0, \quad (z,t) \in \Omega, \tag{1.1}$$

where Ω is the exterior domain in $\mathbb{H}^n$, and $\Delta_{\mathbb{H}^n}$ is the Kohn-Laplacian.
Let us introduce some notations,

$$\mathbb{G}(a_1,a_2) = \{\zeta \in \mathbb{H}^n : a_1 \leq ||\zeta||_{\mathbb{H}^n} \leq a_2\},$$

$$\mathbb{G}[a_1,\infty) = \{\zeta \in \mathbb{H}^n : ||\zeta||_{\mathbb{H}^n} \geq a_1\} \subset \Omega,$$

where $r = ||\zeta||_{\mathbb{H}^n}$, the norm in $\mathbb{H}^n$ is shown by $||\zeta||_{\mathbb{H}^n}$. If a_0 is a positive constant, then $\mathbb{G}[a_0,\infty) \in \Omega$ should exist. Let us focus on the nontrivial solution u of (1.1), that is, the solution u satisfies

$$\sup\{|u(z,t)| : (z,t) \in \Omega\} > 0. \tag{1.2}$$

A nontrivial solution of (1.1) has either infinitely many or a finite number of zeros in a neighborhood of infinity, and hence (1.1) can be classified as oscillatory or nonoscillatory.

In [31], let $\mathbb{K} = \mathbb{K}(r,s,l) \in \Im$, denoted by $\mathbb{K} \in \Im$, if $\mathbb{K} \in \mathbb{C}(\mathbb{D},\mathbb{R}_+)$, where $\mathbb{D} = \{(r,s,l) : r_0 \leq l \leq s \leq r < \infty\}$, which satisfies $\mathbb{K}(r,r,l) = 0, \mathbb{K}(r,l,l) = 0, \mathbb{K}(r,s,l) \neq 0$ for $l < s < r$,

and has the partial derivative $\frac{\partial \mathbb{K}}{\partial s}$ and $\frac{\partial \mathbb{K}}{\partial l}$ on $\mathbb{D}$ such that

$$\frac{\partial \mathbb{K}}{\partial s} = k\mathbb{K}$$

where $k \in \mathbb{C}(\mathbb{D}, \mathbb{R}_+)$.

The present study defines a further generalized function $\mathbb{J} := \mathbb{J}(r, s, s_1, s_2) \in \Gamma$, denoted by $\mathbb{J} \in \Gamma$, if $\mathbb{J}(r, s, s_1, s_2) \in \mathbb{C}(\mathbb{D}*, \mathbb{R}_+)$ and satisfying $\mathbb{J}(r, r, r, s_2) = 0, \mathbb{J}(r, s, s, s) = 0$, $\mathbb{J}(s_1, s_1, s_1, s_2) = 0, \mathbb{J}(r, s_2, s_2, s_2) = 0, \mathbb{J}(r, r, s_1, s_2) = 0, \mathbb{J}(r, s, s, s_2) = 0, \mathbb{J}(r, s, s_2, s_2) = 0$, $\mathbb{J}(r, s, s_1, s_2) \geq 0$ for $r > s \geq s_1 \geq s_2$, where $\mathbb{D}* = \{(r, s, s_1, s_2) : r_0 \leq s_2 \leq s_1 \leq s \leq r < \infty\}$, $\mathbb{R}_+ = (0, \infty)$. Furthermore, $\mathbb{J}$ has continuous partial derivatives

$$\frac{\partial \mathbb{J}(r, s, s_1, s_2)}{\partial s} = j(r, s, s_1, s_2)\mathbb{J}(r, s, s_1, s_2)$$

where $j \in \mathbb{C}(\mathbb{D}*, \mathbb{R}_+)$.

The structure of the present work is as follows:

We notice that Section 2 contains lemmas and some preliminary information. New oscillation theorems are developed by using the Riccati transformation and generalized Kamenev type in section 3. A few cases are taken into consideration in section 4 to demonstrate the primary findings.

2 Preliminaries

Recall the briefs on the Heisenberg group $\mathbb{H}^n$ of real dimension $N = 2n + 1, n \in N$, is the nilpotent Lie group of step two whose underlying manifold is $\mathbb{R}^{2n+1}$ equipped with the group law

$$\bar{\zeta} \circ \zeta = \left(x + \bar{x}, y + \bar{y}, 2\sum_{i=1}^{n}(x_i\bar{y}_i - y_i\bar{x}_i) \right), \tag{2.1}$$

where $\zeta = (x_1, x_2, ..., x_n, y_1, y_2, ..., y_n, t) := (x, y, t) := (z, t) \in \mathbb{R}^{2n} \times \mathbb{R}$, $\bar{\zeta} = (\bar{z}, \bar{t})$. The $\mathbb{H}^n$ plays important roles in $\mathbb{R}^n$ in harmonic analysis, conformal geometry and several complex geometry [32].

The left-invariant vector fields on $\mathbb{H}^n$ are of the form

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad j = 1, 2, .., n, \quad T = \frac{\partial}{\partial t}. \tag{2.2}$$

These generators satisfy the non-commutative formula

$$[X_i, Y_j] = -4\delta_{ij}T, \ [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0.$$

The family $\{X_1, ..., X_n, Y_1, .., Y_n\}$ satisfies Harmander rank condition [33].

The norm $||\zeta||_{\mathbb{H}^n}$ for $\zeta \in \mathbb{H}^n$ is

$$||\zeta||_{\mathbb{H}^n} = \left((x^2 + y^2)^2 + t^2\right)^{\frac{1}{4}}, \tag{2.3}$$

where $\left((x^2 + y^2)^2 + t^2\right)^{\frac{1}{4}}$ denotes the Heisenberg distance between ζ and the origin. The vector fields $X_i, Y_i, i = 1, 2, ..., n$ are homogeneous of degree 1 under the norm $||.||_{\mathbb{H}^n}$ and T is homogeneous of degree 2. The following definitions are used

$$B_{\mathbb{H}^n} = \{(z, t) \in \mathbb{H}^n; ||(z, t)||_{\mathbb{H}^n} < r\},$$

$$S_{\mathbb{H}^n} = \{(z, t) \in \mathbb{H}^n; ||(z, t)||_{\mathbb{H}^n} = r\},$$

$$U(r) = \frac{1}{\sigma\alpha_\sigma r^{2n+1}} \int_{S_{\mathbb{H}^n}} u(z, t) d\mathbb{H}^n,$$

where $\sigma = 2n + 2$ denotes the homogenous dimension of $\mathbb{H}^n$ and $\sigma\alpha_\rho$ is the area of unit sphere $S_{\mathbb{H}^n}$ in $\mathbb{H}^n$, α_σ is a volume of $B_{\mathbb{H}^n}(0, r)$ and $d\mathbb{H}^n$ denotes the $2n$ dimensional Hausdorff measure in $\mathbb{R}^{2n+1}$. We denote $B_{\mathbb{H}^n}(0, r)$ and $S_{\mathbb{H}^n}(0, r)$ by B_r and S_r respectively.

The Heisenberg gradient and Heisenberg Kohn-Laplacian on $\mathbb{H}^n$ are given by

$$\nabla_{\mathbb{H}^n} = (X_1, X_2, ..., X_n, Y_1, Y_2, ...Y_n)$$

and

$$\Delta_{\mathbb{H}^n} = \sum_{i=1}^{n} X_i^2 + Y_i^2$$
$$= \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial t^2} \right).$$

The fundamental solution on $\mathbb{H}^n$ of $-\Delta_{\mathbb{H}^n}$ with pole at the origin is [34]

$$F(\zeta) = \frac{c_\sigma}{d(\zeta)^{\sigma-2}},$$

where c_σ is a positive constant.
Now, we state the well known lemma in [18] .

Lemma 2.1. *Assume Ω is an exterior domain in $\mathbb{H}^n$ with $\mathbb{C}^1$ boundary $\partial\Omega$, η denote the unit outward normal to $\partial\Omega$, then*

$$\int_\Omega \Delta_{\mathbb{H}^n} u(z,t) dz dt = \int_{\partial\Omega} M(z) \nabla u . \eta d\mathbb{H}^n,$$

where, $M = \begin{bmatrix} I_n & 0 \\ 0 & I_n \\ 2y & -2x \end{bmatrix}_{(2n+1)\times 2n}$ *and the identity matrix of $\mathbb{R}^n$ is I_n.*

We convert the multi dimensional problem to one dimensional problem by using integral average method and divergence theorem in the next two lemmas.

Lemma 2.2. *Assume $u \in \mathbb{C}^2(\mathbb{G}(r_0, \infty); \mathbb{R}) \cap \mathbb{C}(\overline{\mathbb{G}}(r_0, \infty), \mathbb{R})$ and assume that the smallest eigenvalue of $M(z)$ is denoted by λ then,*

$$\frac{1}{\sigma\alpha_\rho r^{2n+1}} \int_{S_{\mathbb{H}^n}} \Delta_{\mathbb{H}^n} u(z,t) d\mathbb{H}^n \geq r^{-(2n+1)} \frac{d}{dr} \left(\lambda r^{2n+1} \frac{d}{dr} U(r) \right), \quad r > r_0 > 0. \tag{2.4}$$

Proof. Let the function

$$U(r) = \frac{1}{\sigma\alpha_\sigma r^{2n+1}} \int_{S_{\mathbb{H}^n}} u(z,t) d\mathbb{H}^n,$$

By Lemma 2.1, we get

$$\int_{\mathbb{G}(a,r)} \Delta_{\mathbb{H}^n} u(z,t) dz dt = \int_{\partial\mathbb{G}(a,r)} M(z) \nabla u . \eta d\mathbb{H}^n$$
$$\geq \int_{\partial\mathbb{G}(a,r)} \lambda \frac{\partial u}{\partial r} d\mathbb{H}^n$$
$$\int_a^r \int_{S_{\mathbb{H}^n}} \Delta_{\mathbb{H}^n} u(r,\theta) d\mathbb{H}^n dr \geq \int_{S_r} \lambda \frac{\partial u}{\partial r} d\mathbb{H}^n + \int_{S_a} \lambda \frac{\partial u}{\partial r} d\mathbb{H}^n.$$

Taking differentiation on both sides with respect to r, we get

$$\int_{S_{\mathbb{H}^n}} \Delta_{\mathbb{H}^n} u(r,\theta) d\mathbb{H}^n \geq \frac{d}{dr} \int_{S_{\mathbb{H}^n}} \left(\lambda \frac{\partial u}{\partial r} \right) d\mathbb{H}^n \geq \sigma\alpha_\sigma \frac{d}{dr} \left(\lambda r^{2n+1} \frac{dU(r)}{dr} \right).$$

Then divided by $\sigma\alpha_\sigma r^{2n+1}$, we obtained as

$$\frac{1}{\sigma\alpha_\sigma r^{2n+1}} \int_{S_{\mathbb{H}^n}} \Delta_{\mathbb{H}^n} u(z,t) d\mathbb{H}^n \geq r^{-(2n+1)} \frac{d}{dr} \left(\lambda r^{2n+1} \frac{d}{dr} U(r) \right).$$

Hence the proof. □

Lemma 2.3. *Suppose that,* $c(z,t,\xi) \in \mathbb{C}(\mathbb{G}(r_0,\infty)\times\mathbb{R},\mathbb{R})$ *for* $(z,t,\xi) \in \mathbb{G}(r_0,\infty)\times(0,\infty)$, $c(z,t,\xi) \geq p(||z,t||_{\mathbb{H}^n})\Psi(\xi)$, *where* $p(r)$ *is a function of class* $\mathbb{C}([r_0,\infty);(0,\infty))$, $\Psi(\xi) \in \mathbb{C}((0,\infty);(0,\infty))$ *is a convex function. Each solution* $u \in \mathbb{C}^2(\mathbb{G}(r_0,\infty),\mathbb{R}) \cap \mathbb{C}(\overline{\mathbb{G}}(r_0,\infty),\mathbb{R})$ *is oscillatory in* Ω, *if the ordinary differential inequality,*

$$\frac{d}{dr}\left(\lambda r^{2n+1}\frac{d}{dr}U(r)\right) + r^{2n+1}p(r)\Psi(U(r)) \leq 0, \quad r > r_0 > 0, \tag{2.5}$$

has no eventually positive solutions.

Proof. We know that,

$$r^{-(2n+1)}\frac{d}{dr}\left(\lambda r^{2n+1}\frac{d}{dr}U(r)\right) \leq \frac{1}{\sigma\alpha_\sigma r^{2n+1}}\int_{S_H}\Delta_{\mathbb{H}^n}u(z,t)d\mathbb{H}^n$$
$$\leq \frac{-p(r)}{\sigma\alpha_\sigma r^{2n+1}}\int_{S_H}\Psi(u(z,t))d\mathbb{H}^n.$$

An application of Jensen's inequality, leads to

$$\frac{d}{dr}\left(\lambda r^{2n+1}\frac{d}{dr}U(r)\right) \leq -r^{(2n+1)}p(r)\Psi(U(r)).$$

Hence the proof. □

3 Main Results

Using the generalized Kamenev type and the Riccati transformation, this transformation can reduce the equation to a linear first-order.

Theorem 3.1. *Let the function* $\rho \in \mathbb{C}'([r_0,\infty),\mathbb{R})$, $p \in \mathbb{C}([r_0,\infty);(0,\infty))$, λ *is the smallest eigenvalue of* $M(z)$ *and* $\Psi'(U(r)) \leq \mu$ *for certain constant* $\mu > 0$, *such that*

$$\limsup_{r\to\infty}\int_{r_0}^{r} s^{2n+1}\left(\rho(s)p(s) - \frac{\lambda}{4\mu}\frac{\rho'(s)^2}{\rho(s)}\right)ds = \infty. \tag{3.1}$$

Then, every solution $u(z,t)$ *of (1.1) is oscillatory.*

Proof. Consider $u(z,t)$ is a nonoscillatory solution of (1.1). Riccati transformation is defined as,

$$W(r) = \rho(r)\frac{\lambda r^{2n+1}U'(r)}{\Psi(U(r))}.$$

Then

$$W'(r) = \frac{\rho'(r)}{\rho(r)}W(r) - \frac{\rho(r)(\lambda r^{2n+1}U'(r))'}{\Psi(U(r))} - \frac{\lambda r^{2n+1}\rho(r)U'(r)^2\Psi'(U(r))}{\Psi(U(r))^2}.$$

Using the Lemma 2.3, we get

$$W'(r) \leq \frac{\rho'(r)}{\rho(r)}W(r) - \frac{\mu W^2(r)}{\lambda r^{2n+1}\rho(r)} - r^{2n+1}\rho(r)p(r). \tag{3.2}$$

Integrating with respect to r_0 to r,

$$\int_{r_0}^{r} W'(s)ds \leq \int_{r_0}^{r}\frac{\rho'(s)}{\rho(s)}W(s)ds - \frac{\mu W^2(s)}{\lambda s^{2n+1}\rho(s)}ds - \int_{r_0}^{r} s^{2n+1}\rho(s)p(s)ds$$
$$W(r_0) \geq \int_{r_0}^{r} s^{2n+1}\left(\rho(s)p(s) - \frac{\lambda}{4\mu}\frac{\rho'(s)^2}{\rho(s)}\right)ds.$$

Taking $\limsup_{t\to\infty}$, we get

$$\limsup_{r\to\infty}\int_{r_0}^{r} s^{2n+1}\left(\rho(s)p(s) - \frac{\lambda}{4\mu}\frac{\rho'(s)^2}{\rho(s)}\right)ds < \infty,$$

which leads to a contradiction. Hence the proof. □

If we choose $\mathbb{J}(r,s,s_1,s_2) = h(s)(r-s)^{\beta}(s-s_1)^{\gamma}(s-s_2)^{\delta}$.

$$j(r,s,s_1,s_2) = \left(\frac{h'(s)}{h(s)} + \frac{\delta(r-s)(s-s_1)+\gamma(r-s)(s-s_2)-\beta(s-s_2)(s-s_1)}{(s-s_2)(s-s_1)(r-s)}\right),$$

where $h \in \mathbb{C}^1([r_0,\infty),(0,\infty))$ and $\beta,\gamma,\delta > \frac{1}{2}$.

Theorem 3.2. *Assume that* $\rho \in \mathbb{C}'([r_0,\infty),\mathbb{R})$,

$$\limsup_{r\to\infty}\int_{s_2}^{r} h(s)(r-s)^{\beta}(s-s_1)^{\gamma}(s-s_2)^{\delta}s^{2n+1}\rho(s)\left(\frac{p(s)}{\lambda(s)} - \frac{1}{4\mu}\left(\frac{h'(s)}{h(s)} + \rho'(s) + \frac{\delta(r-s)(s-s_1)+\gamma(r-s)(s-s_2)-\beta(s-s_2)(s-s_1)}{(s-s_2)(s-s_1)(r-s)}\right)^2\right)ds = \infty. \tag{3.3}$$

Then all solution $u(z,t)$ *of (1.1) are oscillatory.*

Proof. Multiply by $\mathbb{J}(r,s,s_1,s_2)$ and integrating on both sides with respect to s_2 to r of (3.2), we get

$$-\int_{s_2}^{r} W(s)j(r,s,s_1,s_2)\mathbb{J}(r,s,s_1,s_2)ds \le -\int_{s_2}^{r}\mathbb{J}(r,s,s_1,s_2)s^{2n+1}\rho(s)p(s)ds - \int_{s_2}^{r}\frac{\mathbb{J}(r,s,s_1,s_2)}{\rho(s)}\left(\frac{\mu W^2(s)}{\lambda s^{2n+1}} - \rho'(s)W(s)\right)ds$$

$$0 \le -\int_{s_2}^{r}\mathbb{J}(r,s,s_1,s_2)s^{2n+1}\left(\rho(s)p(s) + \frac{\lambda}{4\mu\rho(s)}(\rho'(s)+j(r,s,s_1,s_2)\rho(s))^2\right)ds.$$

Taking $\limsup_{t\to\infty}$, we get

$$\limsup_{r\to\infty}\int_{s_2}^{r}\mathbb{J}(r,s,s_1,s_2)s^{2n+1}\left(\rho(s)p(s) + \frac{\lambda}{4\mu\rho(s)}(\rho'(s)+j(r,s,s_1,s_2)\rho(s))^2\right)ds < \infty,$$

leads to a contradiction. Hence the proof. □

Using the above theorem, we derive the next corollary.

Corollary 3.3. *Consider the indicate of Theorem 3.1 holds with replaced by*

$$\limsup_{r\to\infty}\int_{s_2}^{r} h(s)(r-s)^{\beta}(s-s_1)^{\gamma}(s-s_2)^{\delta}s^{2n+1}\rho(s)p(s)ds = \infty,$$

and

$$\limsup_{r\to\infty}\int_{s_2}^{r}\frac{1}{4\mu}\left(\left(\frac{h'(s)}{h(s)} + \frac{\delta(r-s)(s-s_1)+\gamma(r-s)(s-s_2)-\beta(s-s_2)(s-s_1)}{(s-s_2)(s-s_1)(r-s)} + \rho'(s)\right)\rho(s)\right)^2 ds < \infty.$$

Then each solution $u(z,t)$ *of (1.1) is oscillatory.*

Let choose $\mathbb{J}(r,s,s_1,s_2) = \sqrt{B_1(s,s_2)B_2(s,s_1)B_3(r,s)}$, where $B_1,B_2,B_3 \in \Gamma$.

$$\frac{\partial\mathbb{J}}{\partial s} = \frac{1}{2}\left(b_1(s,s_2)\sqrt{B_2(s,s_1)B_3(r,s)} + b_2(s,s_1)\sqrt{B_1(s,s_2)B_3(r,s)} - b_3(r,s)\sqrt{B_2(s,s_1)B_1(s,s_2)}\right).$$

$$\frac{\partial B}{\partial r} = b(r,s)\sqrt{B(r,s)}, \qquad \frac{\partial B}{\partial s} = -b(r,s)\sqrt{B(r,s)},$$

where $b_1,b_2,b_3 \in \mathbb{C}(\mathbb{D}*,\mathbb{R}_+)$.

According to the simple computation, next theorem.

Theorem 3.4. *Equation (1.1) is oscillatory provided that for each $s_2 \geq r$, then three functions B_1, B_2, B_3 such that*

$$\limsup_{r\to\infty} \int_{s_2}^{r} \sqrt{B_1(s,s_2)B_2(s,s_1)B_3(r,s)}s^{2n+1}\rho(s)\left(p(s) - \frac{\lambda}{4\mu}\left(\frac{\rho'(s)}{\rho(s)}\right.\right.$$
$$\left.\left. + \frac{1}{2}\left(\frac{b_1(s,s_2)}{\sqrt{B_1(s,s_2)}} + \frac{b_2(s,s_1)}{\sqrt{B_2(s,s_1)}} - \frac{b_3(r,s)}{\sqrt{B_3(r,s)}}\right)\right)^2\right) ds = \infty. \tag{3.4}$$

Then each solutions of $u(z,t)$ of (1.1) is oscillatory.

Proof. Consider $u(z,t)$ is a nonoscillatory solution of (1.1). Multiply by $\mathbb{J}(r,s,s_1,s_2)$ and integrating on both sides of (3.2), we obtain

$$\int_{s_2}^{r} \mathbb{J}(r,s,s_1,s_2)W'(s)ds \leq -\int_{s_2}^{r} \mathbb{J}(r,s,s_1,s_2)s^{2n+1}\rho(s)p(s)ds$$
$$+ \int_{s_2}^{r} \frac{\mathbb{J}(r,s,s_1,s_2)}{\rho(s)}\left(\rho'(s)W(s) - \frac{\mu}{\lambda s^{2n+1}}W^2(s)\right) ds.$$
$$0 \leq -\int_{s_2}^{r} \mathbb{J}(r,s,s_1,s_2)s^{2n+1}\rho(s)p(s)ds - \frac{1}{4\mu}\int_{s_2}^{r} \mathbb{J}(r,s,s_1,s_2)$$
$$\left(\frac{\rho'(s)}{\rho(s)} + \frac{1}{2}\left(\frac{b_1(s,s_2)}{\sqrt{B_1(s,s_2)}} + \frac{b_2(s,s_1)}{\sqrt{B_2(s,s_1)}} - \frac{b_3(r,s)}{\sqrt{B_3(r,s)}}\right)\right)^2 s^{2n+1}\rho(s)\lambda ds.$$

Taking $\limsup_{t\to\infty}$, we get

$$\limsup_{r\to\infty} \int_{s_2}^{r} \sqrt{B_1B_2B_3}s^{2n+1}\rho(s)\left(p(s) - \frac{\lambda}{4\mu}\left(\frac{\rho'(s)}{\rho(s)}\right.\right.$$
$$\left.\left. + \frac{1}{2}\left(\frac{b_1}{\sqrt{B_1}} + \frac{b_2}{\sqrt{B_2}} - \frac{b_3}{\sqrt{B_3}}\right)\right)^2\right) ds < \infty,$$

leads to a contradiction. Hence the proof. □

Let $B(r,s) = (r-s)^{m-1}$, m is positive integer $m \geq 2$.

Corollary 3.5. *Let the functions B_1, B_2, B_3 and ρ are the same as in theorem. Moreover, suppose that (3.4) is replaced by*

$$\limsup_{r\to\infty} \int_{s_2}^{r} \sqrt{(s-s_2)^{m-1}(s-s_1)^{m-1}(r-s)^{m-1}}s^{2n+1}\rho(s)\left(p(s) - \frac{\lambda}{4\mu}\left(\frac{\rho'(s)}{\rho(s)}\right.\right.$$
$$\left.\left. + \frac{(m-1)}{2}\left(\frac{(s-s_1)(r-s)+(s-s_2)(r-s)+(s-s_2)(s-s_1)}{(s-s_2)(s-s_1)(r-s)}\right)\right)^2\right) ds = \infty.$$

Then each solutions of (1.1) is oscillatory.

Let $B(r,s) = (\chi(r) - \chi(s))^m$, where m is constant.
$\chi(r) = \int_{r_o}^{r} \frac{1}{\Theta(s)}ds$ and $\lim_{r\to\infty} \chi(r) = \infty$. Then, Theorem 3.3 implies the next result.

Corollary 3.6. *Assume that the state of Theorem 3.3 holds, equations*

$$\limsup_{r\to\infty} \int_{s_2}^{r} \sqrt{(\chi(s)-\chi(s_2))^m(\chi(s)-\chi(s_1))^m(\chi(r)-\chi(s))^m}s^{2n+1}\rho(s)\left(p(s) - \frac{\lambda}{4\mu}\left(\frac{\rho'(s)}{\rho(s)}\right.\right.$$
$$\left.\left. + \frac{m\chi'(s)}{2}\left(\frac{1}{(\chi(s)-\chi(s_2))} + \frac{1}{(\chi(s)-\chi(s_1))} + \frac{1}{(\chi(r)-\chi(s))}\right)\right)^2\right) ds = \infty.$$

Then each solutions of (1.1) is oscillatory.

Let $B(r,s) = \left(\log(\frac{r}{s})\right)^m$, $r > s > s_1 > s_2 > r_1$, $m > 1$ is an integer. Then Theorem 3.3 yields the next result.

Corollary 3.7. *Assume that the indicate of Theorem 3.3 holds, equations*

$$\limsup_{r\to\infty} \int_{s_2}^{r} \sqrt{\left(\log\left(\frac{s}{s_2}\right)\right)^m \left(\log\left(\frac{s}{s_1}\right)\right)^m \left(\log\left(\frac{r}{s}\right)\right)^m} s^{2n+1}\rho(s)\left(p(s) - \frac{\lambda}{4\mu}\left(\frac{\rho'(s)}{\rho(s)}\right.\right.$$

$$\left.\left. + \frac{m}{2}\left(\frac{1}{\left(\log\left(\frac{s}{s_2}\right)\right)} + \frac{1}{\left(\log\left(\frac{s}{s_1}\right)\right)} + \frac{1}{\left(\log\left(\frac{r}{s}\right)\right)}\right)\right)^2\right) ds = \infty.$$

Then each of the solutions of (1.1) is oscillatory.

Let $B(r,s) = \left(\int_{s_2}^{r} \frac{du}{\theta(u)}\right)^m$, $r > s > s_1 > s_2 > r_1$, where $m > 1$ is an integer and $\theta : [r_0,\infty) \to \mathbb{R}_+$ is a continuous function such that $\lim_{r_0\to\infty}\left(\frac{du}{\theta(u)}\right) = \infty$. Then Theorem 3.3 yields the next result.

Corollary 3.8. *Assume that the state of Theorem 3.3 holds, equations*

$$\limsup_{r\to\infty} \int_{s_2}^{r} \sqrt{\left(\int_{s_2}^{s} \frac{du}{\theta(u)}\right)^m \left(\int_{s_1}^{s} \frac{du}{\theta(u)}\right)^m \left(\int_{s}^{r} \frac{du}{\theta(u)}\right)^m} s^{2n+1}\rho(s)\left(p(s) - \frac{\lambda}{4\mu}\left(\frac{\rho'(s)}{\rho(s)}\right.\right.$$

$$\left.\left. + \frac{m}{2\theta(u)}\left(\frac{1}{\left(\int_{s_2}^{s} \frac{du}{\theta(u)}\right)} + \frac{1}{\left(\int_{s_1}^{s} \frac{du}{\theta(u)}\right)} + \frac{1}{\left(\int_{s}^{r} \frac{du}{\theta(u)}\right)}\right)\right)^2\right) ds = \infty.$$

Then all solutions of (1.1) are oscillatory.

4 Example

We provide a few examples to highlight the findings from Section 4.

Example 4.1. We assume Kohn-Laplacian equation

$$\Delta_{\mathbb{H}^1} u(z,t) + \frac{3}{4}|z|^2 u^9(z,t) = 0, \quad (z,t) \in \mathbb{R}^3, \tag{4.1}$$

where $\lambda = 2, n = 1, p(s) = \frac{3}{4}, \rho(s) = s, \mu = 1$.

$$\limsup_{r\to\infty} \int_{r_0}^{r} s^{2n+1}\left(\rho(s)p(s) - \frac{\lambda}{4\mu}\frac{\rho'(s)^2}{\rho(s)}\right) ds = \limsup_{r\to\infty} \int_{r_0}^{r} s\left(\frac{3}{4}s - \frac{1}{2}s^2\right) ds < \infty.$$

Therefore, all requirements given in Theorem 3.1 are not fulfilled. As a result, all of (4.1)'s solutions nonoscillate. One such solution is $u(z,t) = (|z|^4 + t^2)^{-\frac{1}{8}}$.

Example 4.2. We assume Kohn-Laplacian equation

$$\Delta_{\mathbb{H}^1} u(z,t) - 16|z|^2(8\cot(||(z,t)||_{\mathbb{H}^1}^4) - ||(z,t)||_{\mathbb{H}^1}^4)u(z,t) = 0, \quad (z,t) \in \mathbb{R}^3, \tag{4.2}$$

here $\lambda = 2, n = 1, p(s) = 16|z|^2, \rho(s) = s^2, \mu = 2,$.

$$\limsup_{r\to\infty} \int_{r_0}^{r} s^5 ds = \infty.$$

Therefore, all requirements given in Theorem 3.1 are fulfilled. As a result, all of (4.2)'s solutions oscillate. One such solution is $u = \sin(||(z,t)||_{\mathbb{H}^1}^4)$.

5 Conclusion remarks

In this paper, our main goal has to provide some new oscillation criteria for a certain class of Kohn Laplacian equations on the Heisenberg group. We have presented some examples and illustrated the use of our results.

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