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SOME RESULTS ON TANGENT BUNDLES ENDOWED WITH A GENERALIZED SYMMETRIC METRIC CONNECTION (GSMC) ON GOLDEN LORENTZIAN MANIFOLDS

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Abstract. In this paper, we investigate the liftings of the golden Lorentzian manifolds endowed with generalized symmetric metric connection in the tangent bundles. By using liftings, we investigate the curvature tensor and Ricci tensor of the golden Lorentzian manifolds with respect to GSMC in the tangent bundles. Moreover some properties of the curvature tensor, concircular curvature tensor, projective curvature tensor and conformal curvature tensor with the generalized symmetric metric connection in the tangent bundle T_0M are studied and we find that T_0M is ξ-concirculary and ξ-projectively flat with respect to the generalized symmetric metric ξ connection in golden Lorentzian manifolds to the tangent bundles.

1 Introduction

Numerous geometers, including Yano and Kobayashi [18], Yano and Ishihara [19], Tani [20], Pandey and Chaturvedi [22], have investigated the tangent bundle of differential geometry. Yano and Ishihara [19] introduced the vertical, complete, and horizontal lifts of tensors, as well as the connection from the manifold to its tangent bundle. Khan and others investigated tangent bundles on several manifolds and their submanifolds using various connections and partial differential equations, as shown in ([23],[24],[25],[26],[27], [28],[29],[30],[31],[32],[33],[34],[35],[37],[38]). Recently, Colney and Kumar [36] studied the properties of the soliton on Sasakian manifold on the tangent bundle. On the other hand, various geometers have studied the golden ratio for many ages. The golden structure was presented by Crasmareanu et al. [4] as a polynomial structure and Goldberg et al. [15], [16] was inspired by the golden mean. The polynomial structure was stated as $\Phi^2 = \Phi + I$, where I is the $(1, 1)$ tensor field. By investigating invariant submanifolds isometrically immersed in golden Riemannian manifolds in 2007, Hretcanu et al. [17] highlighted novel concepts in 2007. Work on an induced structure was done in the same ambient manifold by Ahmad and Qayyoom [2], leading to several additional investigations. Recent research on slant submanifolds of golden Riemannian manifolds done by Bahadir and Sirajuddin et al. [3] produced a variety of helpful findings. M. Ozkan [21] in 2014 has investigated golden structures on semi-Riemannian manifolds . For more information, see [5], [9], [11],[12] for the recent developments in golden differential geometry. Some related studies in Lorentzian para contact manifold are [13], [14].

In 1932, Hayden [10] proposed a metric connection on a Riemannian manifold with non-zero torsion. Numerous geometers like ([1], [7],[8],[39],[40],[41]) have investigated the characteristics of Riemannian manifolds with semi-symmetric (symmetric) and a non-metric connection.

Golab [8] in 1975 proposed the notion of quarter-symmetric linear connections in a differential manifold. A linear connection is said to be quartersymmetric if its torsion tensor T is of the form $T(X_0, Y_0) = u(Y_0)\Phi X_0 - u(X_0)\Phi Y_0$ for any vector fields X_0, Y_0 on a manifold, where u is a 1-form and Φ is a tensor of type $(1, 1)$, The quarter-symmetric connection is transformed into a semisymmetric connection if $\Phi = I$. Consequently, it is possible to think of the quarter-symmetric connection as an extension of the semi-symmetric connection. If a linear connection has a torsion tensor T that has the formula $T(X_0, Y_0) = \alpha[u(Y_0)X_0 - u(X_0)Y_0] + \beta[(Y_0)X_0 - u(X_0)Y_0]$, then that connection is said to have a generalized symmetric connection [27] for all vector fields X_0, Y_0, where α and β are smooth functions. Φ is a tensor of type $(1, 1)$ and u is a 1-form respectively. In addition, if the Riemannian metric g in M has the property that $\ddot{\nabla} g = 0$, then the connection is said to be a generalized symmetric metric connection otherwise it is non-metric.

The paper is organized as follows

- Section 1 deals with introduction.
- Section 2 is about preliminaries on vertical and complete lifts, tangent bundle and golden Lorentzian manifolds.
- Section 3 is devoted to the study of GSMC in golden Lorentzian manifolds to the tangent bundles.
- Section 4 deals with the studies of curvature tensor, Ricci tensor and scalar curvature of golden Lorentzian manifolds admitting GSMC in the tangent bundles.
- Sections 5, 6, 7 and 8 are related with the studies of concircular, projective and conformal curvature tensor of golden Lorentzian manifolds admitting GSMC in the tangent bundles follow by example.
- Section 9 is about conclusion/discussion

2 Preliminaries

Let M be a differentiable manifold and $T_0M = \bigcup_{p\in M} T_{0p}M$ be the tangent bundle, where $T_{0p}M$ is the tangent space at point $p \in M$ and $\pi : T_0M \to M$ is the natural bundle structure of T_0M over M. For any coordinate system (Q, x^h) in M, where (x^h) is local coordinate system in the neighborhood Q, then $(\pi^{-1}(Q), x^h, y^h)$ is coordinate system in T_0M, where (x^h, y^h) is induced coordinate system in $\pi^{-1}(Q)$ from (x^h) [19].

2.1 Vertical and complete lifts

Let f_0 be a function, X_0 a vector field, ω_0 a 1-form, a tensor field F_0 of type $(1, 1)$ and ∇_0 an affine connection in M. The vertical and complete lifts of a function f_0, a vector field X_0, a 1-form ω_0, a tensor field F_0 of type $(1, 1)$ and ∇_0 an affine connection are given by $f_0^v, X_0^v, \omega_0^v, F_0^v, \nabla_0^v$ and $f_0^c, X_0^c, \omega_0^c, F_0^c, \nabla_0^c$ respectively. The following formulas of complete and vertical lifts are defined by [19]

$$(f_0X_0)^v = f_0^v X_0^v, \quad (f_0X_0)^c = f_0^c X_0^v + f_0^v X_0^c, \tag{2.1}$$

$$X_0^v f_0^v = 0, \quad X_0^v f_0^c = X_0^c f_0^v = (X_0 f_0)^v, \quad X_0^c f_0^c = (X_0 f_0)^c, \tag{2.2}$$

$$\omega_0(f_0^v) = 0, \quad \omega_0^v(X_0^c) = \omega_0^c(X_0^v) = \omega_0(X_0)^v, \quad \omega_0^c(X_0^c) = \omega_0(X_0)^c, \tag{2.3}$$

$$F_0^v X_0^c = (F_0X_0)^v, \quad F_0^c X_0^c = (F_0X_0)^c, \tag{2.4}$$

$$[X_0, Y_0]^v = [X_0^c, Y_0^v] = [X_0^v, Y_0^c], \quad [X_0, Y_0]^c = [X_0^c, Y_0^c], \tag{2.5}$$

$$\nabla^c{}_{X_0^c} Y_0^c = (\nabla_{X_0} Y_0)^c, \quad \nabla^c{}_{X_0^c} Y_0^v = (\nabla_{X_0} Y_0)^v. \tag{2.6}$$

Suppose T_0M be the tangent bundle and $X_0 = X_0^i \frac{\partial}{\partial x^i}$ be a local vector field on M, then its vertical and complete lifts in the term of partial differential equations are

$$X_0^v = X_0^i \frac{\partial}{\partial y^i}, \tag{2.7}$$

$$X_0^c = X_0^i \frac{\partial}{\partial x^i} + \frac{\partial X_0^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}. \tag{2.8}$$

2.2 Golden Lorentzian manifolds

In order to define golden Lorentzian manifold, we need to define first define golden Riemannian manifold. Let M be an n-dimensional Riemannian manifold with metric g and let G be a tensor field of type $(1,1)$ on M satisfying the polynomial equation [6]

$$K(G) = G^n + a_n G^{n-1} + \cdots + a_2 G + a_1 I = 0, \tag{2.9}$$

where I denotes the identity transformation, and for every point $p \in M$ the linear transformations $I_p, G_p, G_p^2, \ldots, G_p^{n-1}$ are linearly independent. Here G^n means the composition of G with itself n times. In addition M is said to be equipped with a golden structure if the relation

$$\Phi^2 = \Phi + I, \tag{2.10}$$

is satisfied. Where I is the identify transformation, Φ is the $(1,1)$ tensor field. When the golden structure Φ is equipped with the structure (M, g), we say that it is a golden Riemannian manifold if the following conditions are satisfied for Φ-compatible metric.

$$g(\Phi X_0, Y_0) = g(X_0, \Phi Y_0), \tag{2.11}$$

for all $X_0, Y_0 \in \chi(T_0M)$ and

$$g(\Phi X_0, \Phi Y_0) = g(\Phi^2 X_0, Y_0) = g(\Phi X_0, Y_0) + g(X_0, Y_0). \tag{2.12}$$

The golden Lorentzian manifold is defined by [6]

Definition 2.1. An n-dimensional semi-Riemannian manifold (M, g) with metric g having signature $(-,+,+,...,+(n-1 \ \ times))$. Then M is called Lorentzian manifold if it is equipped with a golden structure Φ and g is Φ-compatible. [6]

The example of golden Lorentzian manifold is given as

Example 2.2. Let $\mathbb{R}^3$ represent the semi-Euclidean space and we consider the structure of g as $(-,+,+)$. If Φ stands for a $(1,1)$ tensor field, then it can be showed that if [6]

$$\Phi(S_1, S_2, S_3) = \frac{1}{2}(S_1 + \sqrt{5}S_2, S_2 + \sqrt{5}S_1, 2\psi S_3), \tag{2.13}$$

for any vector field $(S_1, S_2, S_3) \in \mathbb{R}^3$, where $\psi = \frac{1+\sqrt{5}}{2}$ is the golden mean, then

$$\Phi^2 = \Phi + I, \tag{2.14}$$

and hence Φ is a golden structure on $\mathbb{R}^3$, moreover, g may also be verified to be Φ-compatible. Thus $(\mathbb{R}^3, g, \Phi)$ becomes a golden Lorentzian manifold.

3 GSMC in a golden Lorentzian manifolds to the tangent bundle

In a golden Lorentzian manifold the GSMC of (α, β) type is given by [6]

$$\ddot{\nabla}_{X_0} Y_0 = \nabla_{X_0} Y_0 + \alpha K(X_0, Y_0) + \beta K(\Phi X_0, Y_0), \tag{3.1}$$

for all $X_0, Y_0 \in \chi(T_0M)$. where $K(X_0, Y_0) = -g(X_0, Y_0)U + u(Y_0)X_0$ and α, β represent smooth functions on the manifold, ξ is a unitary (spacelike or timelike) vector field on M, and u is a 1-form defined by

$$u(X_0) = g(\xi, X_0). \tag{3.2}$$

The torsion tensor $\ddot{T}$ with respect to $\ddot{\nabla}$ is given by

$$\ddot{T}(X_0, Y_0) = -\alpha\Big(u(X_0)Y_0 - u(Y_0)X_0\Big) - \beta\Big(u(X_0)\Phi Y_0 - u(Y_0)\Phi X_0\Big), \tag{3.3}$$

$$\ddot{\nabla} g = 0. \tag{3.4}$$

A linear connection satisfying the equation (3.3) is called generalized symmetric connection and the linear connection satisfying both equations (3.3) and (3.4) is called GSMC. Let T_0M be the tangent bundle on a golden Lorentzian manifold (M, g^c, Φ^c), then taking a complete lifts by mathematical operators from equations (3.1-3.4) we have

$$\ddot{\nabla}^c_{X_0^c} Y_0^c = \nabla^c_{X_0^c} Y_0^c + \alpha K^c(X_0^c, Y_0^c) + \beta K^c\Big((\Phi X_0)^c, Y_0^c\Big), \tag{3.5}$$

$$K^c(X_0^c, Y_0^c) = -g^c(X_0^c, Y_0^c)\xi^v - g^c(X_0^v, Y_0^c)\xi^c, \tag{3.6}$$

$$u^c(X_0^c) = g^c(\xi^c, X_0^c), \tag{3.7}$$

$$\begin{aligned}\ddot{T}^c(X_0^c, Y_0^c) = &-\alpha\Big(u^c(X_0^c)Y_0^v + u^v(X_0^c)Y_0^c - u^c(Y_0^c)X_0^v - u^v(Y_0^v)X_0^c\Big)\\ &- \beta\Big(u^c(X_0^c)(\Phi Y_0)^v + u^v(X_0^c)(\Phi Y_0)^c - u^c(Y_0^c)(\Phi X_0)^v\\ &- u^v(Y_0^v)(\Phi X_0)^c\Big),\end{aligned} \tag{3.8}$$

$$\ddot{\nabla}^c g^c = 0. \tag{3.9}$$

Theorem 3.1. *Let (M, g^c, Φ^c) be a golden Lorentzian manifold admitting a GSMC in the tangent bundle T_0M which satisfies equations (3.8) and (3.9), then the GSMC of the golden Lorentzian manifold in the tangent bundle is given by equation (3.5).*

Proof. Let $\ddot{\nabla}^c$ be the complete lifts of a linear connection defined on M by

$$\ddot{\nabla}^c_{X_0^c} Y_0^c = \nabla^c_{X_0^c} Y_0^c + H^c(X_0^c, Y_0^c), \tag{3.10}$$

where $\ddot{\nabla}^c$ is the complete lifts of the GSMC and H^c is the complete lifts of a tensor field of type $(1, 2)$ and let $\ddot{T}^c$ be the complete lifts of the torsion tensor of the GSMC. Then, we can have

$$H^c(X_0^c, Y_0^c) = \frac{1}{2}\Big(\ddot{T}^c(X_0^c, Y_0^c) + T'^c(Y_0^c, X_0^c) + T'^c(X_0^c, Y_0^c)\Big), \tag{3.11}$$

and

$$g^c\Big(\ddot{T}^c(Z_0^c, X_0^c), Y_0^c\Big) = g^c\Big(T'^c(X_0^c, Y_0^c), Z_0^c\Big). \tag{3.12}$$

From equations (3.8) and (3.12), we have

$$\begin{aligned}T'^c(X_0^c, Y_0^c) = &\beta\Big(u^c(X_0^c)(\Phi Y_0)^v + u^v(X_0^c)(\Phi Y_0)^c - g^c\big((\Phi X_0)^c, Y_0^c\big)\xi^v\\ &- g^c\big((\Phi X_0)^v, Y_0^c\big)\xi^c\Big) - \alpha\Big(g^c(X_0^c, Y_0^c)\xi^v + g^c(X_0^v, Y_0^c)\xi^c\\ &- u^c(X_0^c)Y_0^v - u^v(X_0^c)Y_0^c\Big).\end{aligned} \tag{3.13}$$

Using equation (3.8), (3.11) and (3.13), we have

$$\begin{aligned}H^c(X_0^c, Y_0^c) = &\beta\Big(u^c(Y_0^c)(\Phi X_0)^v + u^v(Y_0^c)(\Phi X_0)^c - g^c\big((\Phi X_0)^c, Y_0^c\big)\xi^v\\ &- g^c\big((\Phi X_0)^v, Y_0^c\big)\xi^c\Big) - \alpha\Big(g^c(X_0^c, Y_0^c)\xi^v + g^c(X_0^v, Y_0^c)\xi^c\\ &- u^c(Y_0^c)X_0^v - u^v(Y_0^c)X_0^c\Big).\end{aligned} \tag{3.14}$$

Hence, by using (3.14) in equation (3.10), we get equation (3.5). □

Proposition 3.2 (Classification of lifts of GSMC). *The lifts of the generalized semi-symmetric metric connection (GSMC) of a golden Lorentzian manifold to its tangent bundle are classified according to the smooth functions α and β appearing in equation (3.5) as follows:*

Sl. No.	*Values of α and β*	*Type of connection*
1.	$\alpha = 1,\ \beta = 0$	*Semi-symmetric metric connection*
2.	$\alpha = 0,\ \beta = 1$	*Quarter-symmetric metric connection*
3.	$\beta = 0$ *(α arbitrary)*	*α-semi-symmetric metric connection*
4.	$\alpha = 0$ *(β arbitrary)*	*β-semi-symmetric metric connection*

4 Curvature tensor of GSMC on golden Lorentzian Manifolds to the tangent bundle

Definition 4.1. In a golden Lorentzian manifold (M, g^c, Φ^c) equipped with a golden structure Φ^c and $\ddot{\nabla}^c$ be the lift of the GSMC of (α, β)-type in the tangent bundle T_0M, if the lifts of the associated vector field ξ^c is parallel in the tangent bundle T_0M, then $\ddot{\nabla}^c$ is called generalized symmetric metric ξ Connection (GSM-$\xi-$C) on M.

Now, when the lifts of the associated vector field ξ^c of the tangent bundle T_0M in equation (3.8) satisfies the relation

$$g^c(\xi^c, \xi^c) = u^c(\xi^c) = \varepsilon^c, \tag{4.1}$$

where value of ε^c is ± 1. Then from equation (3.5) additionally given that ξ^c is parallel in respect of $\ddot{\nabla}^c$ in the tangent bundle T_0M, we can have

$$\ddot{\nabla}^c_{X_0^c}\xi^c = 0 \quad \Leftrightarrow \quad \nabla^c_{X_0^c}\xi^c = -\alpha K^c(X_0^c, \xi^c) - \beta K^c\big((\Phi X_0)^c, \xi^c\big). \tag{4.2}$$

Using equations (3.5), (4.1) and (4.2), we obtain

$$(\ddot{\nabla}^c_{X_0^c}u^c)Y_0^c = (\nabla^c_{X_0^c}u^c)Y_0^c - \alpha u^c\big(K^c(X_0^c, Y_0^c)\big) - \beta u^c\Big(K^c\big((\Phi X_0)^c, Y_0^c\big)\Big). \tag{4.3}$$

Furthermore, $(\ddot{\nabla}^c_{X_0^c}u^c)Y_0^c = 0$, so

$$(\nabla^c_{X_0^c}u^c)Y_0^c = \alpha u^c\big(K^c(X_0^c, Y_0^c)\big) + \beta u^c\Big(K^c\big((\Phi X_0)^c, Y_0^c\big)\Big). \tag{4.4}$$

In a golden Lorentzian manifold with lift of GSMC $\ddot{\nabla}^c$ in the tangent bundle T_0M. Let $\ddot{R}^c$ be the lifts of the curvature tensor associated with $\ddot{\nabla}^c$ in the tangent bundle.

$$\ddot{R}^c(X_0^c, Y_0^c)Z_0^c = \ddot{\nabla}^c_{X_0^c}\ddot{\nabla}^c_{Y_0^c}Z_0^c - \ddot{\nabla}^c_{Y_0^c}\ddot{\nabla}^c_{X_0^c}Z_0^c - \ddot{\nabla}^c_{[X_0^c, Y_0^c]}Z_0^c. \tag{4.5}$$

Using equation (3.5) in (4.5), we get

$$\begin{aligned}
\ddot{R}^c(X_0^c, Y_0^c)Z_0^c &= R^c(X_0^c, Y_0^c)Z_0^c + \alpha^2[g^c(X_0^c, Z_0^c)Y_0^v + g^c(X_0^v, X_0^c)Y_0^c \\
&\quad - g^c(Y_0^c, Z_0^c)X_0^v - g^c(Y_0^v, Z_0^c)X_0^c] + \beta^2[g^c\big((\Phi X_0)^c, Z_0^c\big)(\Phi Y_0)^v \\
&\quad + g^c\big((\Phi X_0)^v, Z_0^c\big)(\Phi Y_0)^c - g^c\big((\Phi Y_0)^c, Z_0^c\big)(\Phi X_0)^v \\
&\quad - g^c\big((\Phi Y_0)^v, Z_0^c\big)(\Phi X_0)^c] + \alpha\beta[g^c\big((\Phi X_0)^c, Z_0^c\big)Y_0^v \\
&\quad + g^c\big((\Phi X_0)^v, Z_0^c\big)Y_0^c - g^c\big((\Phi Y_0)^c, Z_0^c\big)X_0^v - g^c\big((\Phi Y_0)^v, Z_0^c\big)X_0^c \\
&\quad + g^c(X_0^c, Z_0^c)(\Phi Y_0)^v + g^c(X_0^v, Z_0^c)(\Phi Y_0)^c \\
&\quad - g^c(Y_0^c, Z_0^c)(\Phi X_0)^v - g^c(Y_0^v, Z_0^c)(\Phi X_0)^c].
\end{aligned} \tag{4.6}$$

From equation (4.6), we take contraction with respect to X_0^c so we get the lift of the tensor $\ddot{S}^c$ in

the tangent bundle as

$$\begin{aligned}\ddot{S}^c(Y_0^c, Z_0^c) &= S^c(Y_0^c, Z_0^c) + \alpha^2(1 - n\varepsilon^c)g^c(Y_0^c, Z_0^c)\\ &+ \beta^2[g^c\big((\Phi Y_0^c), Z_0^c\big) - g^c\big((\Phi Y_0)^c, Z_0^c\big)a^v\\ &- g^c\big((\Phi Y_0)^v, Z_0^c\big)a^c] + \alpha\beta[(2 - n\varepsilon^c)g^c\big((\Phi Y_0)^c, Z_0^c\big)\\ &- g^c(Y_0^c, Z_0^c)a^v - g^c(Y_0^v, Z_0^c)a^c].\end{aligned} \tag{4.7}$$

where $a = trace\Phi$ and the lift of the Ricci operator $\ddot{Q_0}^c$ in the tangent bundle T_0M is

$$\begin{aligned}\ddot{Q}_0^c Y_0^c &= Q_0^c Y_0^c + \alpha^2(1 - n\varepsilon^c)Y_0^c + \beta^2[(\Phi Y_0)^c\\ &+ Y_0^c - (\Phi Y_0)^c a^v - (\Phi Y_0)^v a^c]\\ &+ \alpha\beta[(2 - n\varepsilon^c)(\Phi Y_0)^c - Y_0^c a^v - Y_0^v a^c].\end{aligned} \tag{4.8}$$

From equation (4.8), it follows that the lift of the scalar curvature in the tangent T_0M as

$$\ddot{r}^c = r^c + \alpha^2(\varepsilon^2 - n)n + \beta^2[a^c + n\varepsilon^c - (a^c)^2] + \alpha\beta(2 - 2n\varepsilon^c)a^c. \tag{4.9}$$

Corollary 4.2. *In an n-dimensional golden Lorentzian manifold (M, g^c, Φ^c) the complete lift of the Ricci tensor $\ddot{S}^c$ associated with GSMC in the tangent bundle T_0M is symmetric.*

Proof. From equation (4.7), the complete lift of the Ricci tensor associated with the GSMC is given by

$$\begin{aligned}\ddot{S}^c(Y_0^c, Z_0^c) &= S^c(Y_0^c, Z_0^c) + \alpha^2(1 - n\varepsilon^c)g^c(Y_0^c, Z_0^c)\\ &+ \beta^2\big[g^c((\Phi Y_0)^c, Z_0^c) - g^c((\Phi Y_0)^c, Z_0^c)a^v - g^c((\Phi Y_0)^v, Z_0^c)a^c\big]\\ &+ \alpha\beta\big[(2 - n\varepsilon^c)g^c((\Phi Y_0)^c, Z_0^c) - g^c(Y_0^c, Z_0^c)a^v - g^c(Y_0^v, Z_0^c)a^c\big].\end{aligned}$$

The Ricci tensor S^c of the Levi-Civita connection is symmetric because it is the complete lift of a symmetric tensor. The metric g^c is also symmetric. The golden structure Φ satisfies $g(\Phi X, Y) = g(X, \Phi Y)$ (equation (2.11)), which upon lifting gives

$$g^c((\Phi X)^c, Y^c) = g^c(X^c, (\Phi Y)^c), \qquad g^c((\Phi X)^v, Y^c) = g^c(X^v, (\Phi Y)^c).$$

Using these properties, each term involving Φ is symmetric in Y_0^c and Z_0^c. For instance,

$$g^c((\Phi Y_0)^c, Z_0^c) = g^c(Y_0^c, (\Phi Z_0)^c),$$

and similarly for the mixed vertical-complete terms. The coefficients a^v and a^c are lifts of the trace $a = \operatorname{tr}\Phi$, which are constants (functions on the tangent bundle) and therefore do not affect symmetry. Hence every summand on the right-hand side is symmetric, implying that $\ddot{S}^c$ is symmetric. □

Proposition 4.3. *In any golden Lorentzian manifold (M, g^n, Φ^c) equipped with the lift of a GSM-ξ-C $\ddot{\nabla}^c$ in the tangent bundle, we have*

(i) $$\begin{aligned}R^c(X_0^c, Y_0^c)\xi^c &= \beta^c\Big[u^c\big((\Phi Y_0)^c\big)(\Phi X_0)^v + u^v\big((\Phi Y_0)^c\big)(\Phi X_0)^c - u^c\big((\Phi X_0)^c\big)(\Phi Y_0)^v\\ &- u^v\big((\Phi X_0)^c\big)(\Phi Y_0)^c\Big] + \alpha\beta\Big[-u^c(X_0^c)(\Phi Y_0)^v - u^v(X_0^c)(\Phi Y_0)^c + u^c(Y_0^c)(\Phi X_0)^v\\ &+ u^v(Y_0^c)(\Phi X_0)^c + u^c\big((\Phi Y_0)^c\big)X_0^v + u^v\big((\Phi Y_0)^c\big)X_0^c - u^c\big((\Phi X_0)^c\big)Y_0^v - u^v\big((\Phi X_0)^c\big)Y_0^c\Big]\\ &- \alpha^2\Big[u^c(X_0^c)Y_0^v + u^v(X_0^c)Y_0^c - u^c(Y_0^c)X_0^v - u^v(Y_0^c)X_0^c\Big].\end{aligned}$$

(ii) $$\begin{aligned}R^c(\xi^c, X_0^c)Y_0^c &= -\alpha^2 K^c(X_0^c, Y_0^c)\beta^2\Big[u^c\big((\Phi Y_0)^c\big)(\Phi X_0)^v + u^v\big((\Phi Y_0)^c\big)(\Phi X_0)^c\\ &+ u^c\big((\Phi X_0)^c\big)(\Phi Y_0)^v + u^v\big((\Phi X_0)^c\big)(\Phi Y_0)^c\Big] + \alpha\beta\Big[-u^c\big((\Phi Y_0)^c\big)X_0^v\\ &- u^v\big((\Phi Y_0)^c\big)X_0^c - K^c\big((\Phi X_0)^c, Y_0^c\big) + g^c(X_0^c, Y_0^c)(\Phi\xi)^v + g^c(X_0^v, Y_0^c)(\Phi\xi)^c\Big].\end{aligned}$$

(iii) $u^c\Big(R^c(X_0^c,Y_0^c)Z_0^c\Big) = -\beta^2\Big[u^c\big((\Phi Y_0)^c\big)g^c\big((\Phi X_0)^v,Z_0^c\big)+u^v\big((\Phi Y_0)^c\big)g^c\big((\Phi X_0)^c,Z_0^c\big)$
$-u^c\big((\Phi X_0)^c\big)g^c\big((\Phi Y_0)^v,Z_0^c\big)-u^v\big((\Phi X_0)^c\big)g^c\big((\Phi Y_0)^c,Z_0^c\big)\Big]-\alpha\beta\Big[u^c(Y_0^c)g^c\big((\Phi X_0)^v,Z_0^c\big)$
$+u^v(Y_0^c)g^c\big((\Phi X_0)^c,Z_0^c\big)-u^c(X_0^c)g^c\big((\Phi Y_0)^v,Z_0^c\big)-u^v(X_0^c)g^c\big((\Phi Y_0)^c,Z_0^c\big)$
$+u^c\big((\Phi Y_0)^c\big)g^c(X_0^v,Z_0^c)+u^v\big((\Phi Y_0)^c\big)g^c(X_0^c,Z_0^c)-u^c\big((\Phi X_0)^c\big)g^c(Y_0^v,Z_0^c)$
$-u^v\big((\Phi X_0)^c\big)g^c(Y_0^c,Z_0^c)\Big]+\alpha^2\Big[u^c(X_0^c)g^c(Y_0^v,Z_0^c)+u^v(X_0^c)g^c(Y_0^c,Z_0^c)$
$-u^c(Y_0^c)g^c(X_0^v,Z_0^c)-u^v(Y_0^c)g^c(X_0^c,Z_0^c)\Big].$

(iv) $S(Y_0,\xi)^c = -\alpha^2(1-n\varepsilon^c)u^c(Y_0^c)-\beta^2\Big[g^c\Big((\Phi Y_0)^c,(\Phi\xi)^c\Big)-u^c(\Phi Y_0)^c a^v-u^v(\Phi Y_0)^c a^c\Big]$
$-\alpha\beta\Big[(2-n\varepsilon^2)u^c(\Phi Y_0)^c-u^c(\Phi Y_0)^c a^v-u^v(\Phi Y_0)^c a^c\Big]$

(v) $r^c = -\alpha^2(\varepsilon^c-n)n-\beta^2[a^c+n\varepsilon^c-(a^c)^2]-\alpha\beta(2-2n\varepsilon^c)a^c.$

Proof. From equations (4.1),(4.2) and (4.4), we get i. With the help of i, we can get ii and iii. We get (iv) by taking contraction of (i) and again by taking contraction of (iv) we get (v). □

Theorem 4.4. *In any golden Lorentzian manifold equipped (M,g^c,Φ^c) with the lifts of GSMC in the tangent bundle T_0M, we have the classification as follows:*

(i) When the lifts of the connection in the tangent bundle T_0M is semi-symmetric, the values of $\ddot{S}^c$ and $\ddot{r}^c$ are
$\ddot{S}^c = S^c(Y_0^c,Z_0^c)-\alpha^2(n\varepsilon^c-1)g^c(Y_0^c,Z_0^c),$
$\ddot{r}^c = r^c-\alpha^2(n-\varepsilon^c)n.$

(ii) When the lifts of the connection in the tangent bundle T_0M is quarter-symmetric, the values of $\ddot{S}^c$ and $\ddot{r}^c$ are
$\ddot{S}^c(Y_0^c,Z_0^c) = S^c(Y_0^c,Z_0^c)+\big[g^c\big((\Phi Y_0)^c,(\Phi Z_0)^c\big)-g^c\big((\Phi Y_0)^c,Z_0^c\big)a^v-g^c\big((\Phi Y_0)^v,Z_0^c\big)a^c\big],$
$\ddot{r}^c = r^c+\big[a^c+n\varepsilon^c-(a^c)^2\big].$

(iii) When the lifts of the connection in the tangent bundle T_0M is α semi-symmetric, the values of $\ddot{S}^c$ and $\ddot{r}^c$ are
$\ddot{S}^c = S^c(Y_0^c,Z_0^c)-(n\varepsilon^c-1)g^c(Y_0^c,Z_0^c)$
$\ddot{r}^c = r^c-n(n-\varepsilon^c).$

(iv) When the lifts of the connection in the tangent bundle T_0M is β quarter symmetric, the values of $\ddot{S}^c$ and $\ddot{r}^c$ are
$\ddot{S}^c(Y_0^c,Z_0^c) = S^c(Y_0^c,Z_0^c)+\beta^2[g^c\big((\Phi Y_0)^c,(\Phi Z_0)^c\big)-g^c\big((\Phi Y_0)^c,Z_0^c\big)a^v$
$-g^c\big((\Phi Y_0)^v,Z_0^c\big)a^c],$
$\ddot{r}^c = r^c+\beta^2[a^c+n\varepsilon^c-(a^c)^2].$

5 Concircular curvature tensor of GSMC on golden Lorentzian manifolds to the tangent bundle

In a golden Lorentzian manifold with GSMC $\ddot{\nabla}$. Let $\ddot{C}$ be the concircular curvature tensor associated with $\ddot{\nabla}$ given by

$$\ddot{C}(X_0,Y_0)Z_0 = \ddot{R}(X_0,Y_0)Z_0-\frac{\ddot{r}}{n(n-1)}\Big[g(Y_0,Z_0)X_0-g(X_0,Z_0)Y_0\Big]. \tag{5.1}$$

Let $\ddot{C}^c$ be the concircular curvature tensor associated with GSMC $\ddot{\nabla}^c$ in the tangent bundle T_0M and by taking the complete lift by mathematical operators on (5.1), we get

$$\begin{aligned}\ddot{C}^c(X_0^c,Y_0^c)Z_0^c = \ddot{R}^c(X_0^c,Y_0^c)Z_0^c-\frac{\ddot{r}^c}{n(n-1)}\Big[&g^c(Y_0^c,Z_0^c)X_0^v+g^c(Y_0^v,Z_0^c)X_0^c\\ &-g^c(X_0^c,Z_0^c)Y_0^v-g^c(X_0^v,Z_0^c)Y_0^c\Big].\end{aligned} \tag{5.2}$$

Using equations (4.6),(4.9) in equation (5.2), we get

$$\begin{aligned}\ddot{C}^c(X_0^c,Y_0^c)Z_0^c &= C^c(X_0^c,Y_0^c)Z_0^c+\alpha^2[g^c(X_0^c,Z_0^c)Y_0^v+g^c(X_0^v,Z_0^c)Y_0^c\\ &-g^c(Y_0^c,Z_0^c)X_0^v-g^c(Y_0^v,Z_0^c)X_0^c]+\beta^2[g^c\big((\Phi X_0)^c,Z_0^c\big)(\Phi Y_0)^v\\ &+g^c\big((\Phi X_0)^v,Z_0^c\big)(\Phi Y_0)^c-g^c\big((\Phi Y_0)^c,Z_0^c\big)(\Phi X_0)^v\\ &-g^c\big((\Phi Y_0)^v,Z_0^c\big)(\Phi X_0)^c]+\alpha\beta[g^c((\Phi X_0)^c,Z_0^c)Y_0^v\\ &+g^c\big((\Phi X_0)^v,Z_0^c\big)Y_0^c-g^c\big((\Phi Y_0)^c,Z_0^c\big)X_0^v-g^c\big((\Phi Y_0)^v,Z_0^c\big)X_0^c\\ &+g^c(X_0^c,Z_0^c)(\Phi Y_0)^v+g^c(X_0^v,Z_0^c)(\Phi Y_0)^c-g^c(Y_0^c,Z_0^c)(\Phi X_0)^v\\ &-g^c(Y_0^v,Z_0^c)(\Phi X_0)^c]-\frac{1}{n(n-1)}\Big[\alpha^2(\varepsilon^c-n)n\\ &+\beta^2\big(a^c+n\varepsilon^c-(a^c)^2\big)+\alpha\beta(2-2n\varepsilon^c)a^c\Big]\Big[g^c(Y_0^c,Z_0^c)X_0^v\\ &+g^c(Y_0^v,Z_0^c)X_0^c-g^c(X_0^c,Z_0^c)Y_0^v-g^c(X_0^v,Z_0^c)Y_0^c\Big].\end{aligned}\tag{5.3}$$

Definition 5.1. For an n- dimensional golden Lorentzian manifold (M,g^c,Φ^c) equipped with a a GSM-ξ-C in the tangent bundle T_0M, the golden Lorentzian manifold is said to be ξ concircularly flat if $\ddot{C}^c(X_0^c,Y_0^c)\xi^c=0$

So, interchanging Z_0^c with ξ^c in equation (5.3), we get

$$\begin{aligned}\ddot{C}^c(X_0^c,Y_0^c)\xi^c &= C^c(X_0^c,Y_0^c)\xi^c+\alpha^2[g^c(X_0^c,\xi^c)Y_0^v+g^c(X_0^v,\xi^c)Y_0^c\\ &-g^c(Y_0^c,\xi^c)X_0^v-g^c(Y_0^v,\xi^c)X_0^c]+\beta^2[g^c\big((\Phi X_0)^c,\xi^c\big)(\Phi Y_0)^v\\ &+g^c\big((\Phi X_0)^v,\xi^c\big)(\Phi Y_0)^c-g^c\big((\Phi Y_0)^c,\xi^c\big)(\Phi X_0)^v\\ &-g^c\big((\Phi Y_0)^v,\xi^c\big)(\Phi X_0)^c]+\alpha\beta[g^c((\Phi X_0)^c,\xi^c)Y_0^v\\ &+g^c\big((\Phi X_0)^v,\xi^c\big)Y_0^c-g^c\big((\Phi Y_0)^c,\xi^c\big)X_0^v-g^c\big((\Phi Y_0)^v,\xi^c\big)X_0^c\\ &+g^c(X_0^c,\xi^c)(\Phi Y_0)^v+g^c(X_0^v,\xi^c)(\Phi Y_0)^c-g^c(Y_0^c,\xi^c)(\Phi X_0)^v\\ &-g^c(Y_0^v,\xi^c)(\Phi X_0)^c]-\frac{1}{n(n-1)}\Big[\alpha^2(\varepsilon^c-n)n\\ &+\beta^2\big(a^c+n\varepsilon^c-(a^c)^2\big)+\alpha\beta(2-2n\varepsilon^c)a^c\Big]\Big[g^c(Y_0^c,\xi^c)X_0^v\\ &+g^c(Y_0^v,\xi^c)X_0^c-g^c(X_0^c,\xi^c)Y_0^v-g^c(X_0^v,\xi^c)Y_0^c\Big].\end{aligned}\tag{5.4}$$

Where

$$\begin{aligned}C^c(X_0^c,Y_0^c)Z_0^c &= R^c(X_0^c,Y_0^c)Z_0^c-\frac{r^c}{n(n-1)}\Big[g^c(Y_0^c,Z_0^c)X_0^v+g^c(Y_0^v,Z_0^c)X_0^c\\ &-g^c(X_0^c,Z_0^c)Y_0^v-g^c(X_0^v,Z_0^c)Y_0^c\Big].\end{aligned}\tag{5.5}$$

Using equations (4.6), (4.9), (i) and (v) in equation (5.4), we get

$$\ddot{C}^c(X_0^c,Y_0^c)\xi^c=0.\tag{5.6}$$

Hence we can conclude the following theorems

Theorem 5.2. *For an n- dimensional golden Lorentzian manifold (M,g^c,Φ^c) equipped with a GSM-ξ-C in the tangent bundle T_0M. We can say that T_0M is ξ^c-concircularly flat with respect to GSM-ξ-C.*

Theorem 5.3. *In any golden Lorentzian manifold (M,g^c,Φ^c) equipped with the lifts of GSMC in the tangent bundle, we have the following classification of the lifts of concircular curvature tensor $\ddot{C}^c$ as follows:*

(i) When the lifts of the connection in the tangent bundle T_0M is semi-symmetric metric connection $(i.e, \alpha = 1, \beta = 0)$. The values of $\ddot{C}^c$ is

$\ddot{C}^c(X_0^c, Y_0^c)Z_0^c = C^c(X_0^c, Y_0^c)Z_0^c + [g^c(X_0^c, Z_0^c)Y_0^v + g^c(X_0^v, Z_0^c)Y_0^c - g^c(Y_0^c, Z_0^c)X_0^v$
$- g^c(Y_0^v, Z_0^c)X_0^c] - \frac{1}{n(n-1)}\Big[\alpha^2(\varepsilon^c - n)\Big]\Big[g^c(Y_0^c, Z_0^c)X_0^v + g^c(Y_0^v, Z_0^c)X_0^c - g^c(X_0^c, Z_0^c)Y_0^v$
$- g^c(X_0^v, Z_0^c)Y_0^c\Big].$

(ii) When the lifts of the connection in the tangent bundle T_0M is quarter-symmetric metric connection $(i.e, \alpha = 0, \beta = 1)$. The value of $\ddot{C}^c$ is

$\ddot{C}^c(X_0^c, Y_0^c)Z_0^c = C^c(X_0^c, Y_0^c)Z_0^c + [g^c\big((\Phi X_0)^c, Z_0^c\big)(\Phi Y_0)^v + g^c\big((\Phi X_0)^v, Z_0^c\big)(\Phi Y_0)^c$
$- g^c\big((\Phi Y_0)^c, Z_0^c\big)(\Phi X_0)^v - g^c\big((\Phi Y_0)^v, Z_0^c\big)(\Phi X_0)^c] - \frac{1}{n(n-1)}\Big[a^c + n\varepsilon^c - (a^c)^2\Big]$
$\Big[g^c(Y_0^c, Z_0^c)X_0^v + g^c(Y_0^v, Z_0^c)X_0^c - g^c(X_0^c, Z_0^c)Y_0^v - g^c(X_0^v, Z_0^c)Y_0^c\Big].$

(iii) When the lifts of the connection in the tangent bundle T_0M is α-semi-symmetric metric connection $(i.e, \beta = 0)$. The value of $\ddot{C}^c$ is

$\ddot{C}^c(X_0^c, Y_0^c)Z_0^c = C^c(X_0^c, Y_0^c)Z_0^c + \alpha^2[g^c(X_0^c, Z_0^c)Y_0^v + g^c(X_0^v, Z_0^c)Y_0^c - g^c(Y_0^c, Z_0^c)X_0^v$
$- g^c(Y_0^v, Z_0^c)X_0^c] - \frac{1}{n(n-1)}\Big[\alpha^2(\varepsilon^c - n)n\Big]\Big[g^c(Y_0^c, Z_0^c)X_0^v + g^c(Y_0^v, Z_0^c)X_0^c - g^c(X_0^c, Z_0^c)Y_0^v$
$- g^c(X_0^v, Z_0^c)Y_0^c\Big].$

(iv) When the lifts of the connection in the tangent bundle T_0M is β-quarter symmetric metric connection $(i.e, \alpha = 0)$. The value of $\ddot{C}^c$ is

$\ddot{C}^c(X_0^c, Y_0^c)Z_0^c = C^c(X_0^c, Y_0^c)Z_0^c + \beta^2[g^c\big((\Phi X_0)^c, Z_0^c\big)(\Phi Y_0)^v + g^c\big((\Phi X_0)^v, Z_0^c\big)(\Phi Y_0)^c$
$- g^c\big((\Phi Y_0)^c, Z_0^c\big)(\Phi X_0)^v - g^c\big((\Phi Y_0)^v, Z_0^c\big)(\Phi X_0)^c] - \frac{1}{n(n-1)}\Big[\beta^2\big(a^c + n\varepsilon^c - (a^c)^2\big)\Big]$
$\Big[g^c(Y_0^c, Z_0^c)X_0^v + g^c(Y_0^v, Z_0^c)X_0^c - g^c(X_0^c, Z_0^c)Y_0^v - g^c(X_0^v, Z_0^c)Y_0^c\Big].$

Proof. Using the classification of proposition (3.2) in equation (5.3), we can have (i), (ii), (iii) and (iv). □

6 Projective curvature tensor of GSMC on golden Lorentzian manifolds to the tangent bundle

In a golden Lorentzian manifold with the GSMC $\ddot{\nabla}$, let $\ddot{P}$ be the projective curvature tensor associated with $\ddot{\nabla}$ defined by

$$\ddot{P}(X_0, Y_0)Z_0 = \ddot{R}(X_0, Y_0)Z_0 - \frac{1}{n-1}\Big[\ddot{S}(Y_0, Z_0)X_0 - \ddot{S}(X_0, Z_0)Y_0\Big]. \tag{6.1}$$

Let $\ddot{P}^c$ be the complete lift of the projective curvature tensor associated with GSMC $\ddot{\nabla}^c$ in the tangent bundle T_0M and by taking complete lift by mathematical operators on equation (6.1), we get

$$\begin{aligned}\ddot{P}^c(X_0^c, Y_0^c)Z_0^c = \ddot{R}^c(X_0^c, Y_0^c)Z_0^c - \frac{1}{n-1}\Big[&\ddot{S}^c(Y_0^c, Z_0^c)X_0^v + \ddot{S}^v(Y_0^c, Z_0^c)X_0^c \\ &- \ddot{S}^c(X_0^c, Z_0^c)Y_0^v - \ddot{S}^v(X_0^c, Z_0^c)Y_0^c\Big].\end{aligned} \tag{6.2}$$

Using equations (4.6) and (4.7) in (6.2),we get

$$\begin{aligned}
\ddot{P}^c(X_0^c,Y_0^c)Z_0^c &= P^c(X_0^c,Y_0^c)Z_0^c + \frac{n(1-\varepsilon^c)\alpha^2}{n-1}\Big[g^c(X_0^c,Z_0^c)Y_0^v \\
&+ g^c(X_0^v,Z_0^c)Y_0^c - g^c(Y_0^c,Z_0^c)X_0^v - g^c(Y_0^v,Z_0^c)X_0^c\Big] \\
&+ \beta^2\Big[g^c\big((\Phi X_0)^c,Z_0^c\big)(\Phi Y_0)^v + g^c\big((\Phi X_0)^v,Z_0^c\big)(\Phi Y_0)^c \\
&- g^c\big((\Phi Y_0)^c,Z_0^c\big)(\Phi X_0)^v - g^c\big((\Phi Y_0)^v,Z_0^c\big)(\Phi X_0)^c\Big] \\
&+ \frac{n(1-\varepsilon^c)+1}{n-1}\alpha\beta\Big[g^c\big((\Phi X_0)^c,Z_0^c\big)Y_0^v \\
&+ g^c\big((\Phi X_0)^v,Z_0^c\big)Y_0^c - g^c\big((\Phi Y_0)^c,Z_0^c\big)X_0^v \\
&- g^c\big((\Phi Y_0)^v,Z_0^c\big)X_0^c\Big] + \alpha\beta\Big[g^c(X_0^c,Z_0^c)(\Phi Y_0)^v \\
&+ g^c(X_0^v,Z_0^c)(\Phi Y_0)^c - g^c(Y_0^c,Z_0^c)(\Phi X_0)^v - g^c(Y_0^v,Z_0^c)(\Phi X_0)^c\Big] \\
&- \frac{1}{n-1}\Big[\beta^2\Big(\big(g^c(\Phi Y_0)^c,(\Phi Z_0)^c\big)X_0^v + \big(g^c(\Phi Y_0)^v,(\Phi Z_0)^c\big)X_0^c \\
&- \big(g^c(\Phi X_0)^c,(\Phi Z_0)^c\big)Y_0^v - \big(g^c(\Phi X_0)^v,(\Phi Z_0)^c\big)Y_0^c\Big) \\
&- \beta^2a^2\Big(g^c((\Phi Y_0)^c,Z_0^c)X_0^v + g^c((\Phi Y_0)^v,Z_0^c)X_0^c \\
&- g^c((\Phi X_0)^c,Z_0^c)Y_0^v - g^c((\Phi X_0)^v,Z_0^c)Y_0^c\Big) \\
&- \alpha\beta a^2\Big(g^c(Y_0^c,Z_0^c)X_0^v + g^c(Y_0^v,Z_0^c)X_0^c \\
&- g^c(X_0^c,Z_0^c)Y_0^v - g^c(X_0^v,Z_0^c)Y_0^c\Big)\Big].
\end{aligned} \tag{6.3}$$

Where

$$\begin{aligned}
P^c(X_0^c,Y_0^c)Z_0^c = R^c(X_0^c,Y_0^c)Z_0^c - \frac{1}{n-1}\Big[&S^c(Y_0^c,Z_0^c)X_0^v + S^v(Y_0^c,Z_0^c)X_0^c \\
&- S^c(X_0^c,Z_0^c)Y_0^v - S^v(X_0^c,Z_0^c)Y_0^c\Big].
\end{aligned} \tag{6.4}$$

Definition 6.1. For an n- dimensional golden Lorentzian manifold (M,g^c,Φ^c) equipped with a a GSM-ξ-C in the tangent bundle T_0M, the golden Lorentzian manifold is said to be ξ projectively flat if $\ddot{P}^c(X_0^c,Y_0^c)\xi^c = 0$.

So, interchanging Z_0^c with ξ in equation (6.3), we get

$$\begin{aligned}
\ddot{P}^c(X_0^c,Y_0^c)\xi^c &= P^c(X_0^c,Y_0^c)\xi^c + \frac{n(1-\varepsilon^c)\alpha^2}{n-1}\Big[g^c(X_0^c,\xi^c)Y_0^v + g^c(X_0^v,\xi^c)Y_0^c \\
&- g^c(Y_0^c,\xi^c)X_0^v - g^c(Y_0^v,\xi^c)X_0^c\Big] + \beta^2\Big[g^c\big((\Phi X_0)^c,\xi^c\big)(\Phi Y_0)^v \\
&+ g^c\big((\Phi X_0)^v,\xi^c\big)(\Phi Y_0)^c - g^c\big((\Phi Y_0)^c,\xi^c\big)(\Phi X_0)^v \\
&- g^c\big((\Phi Y_0)^v,\xi^c\big)(\Phi X_0)^c\Big] + \frac{n(1-\varepsilon^c)+1}{n-1}\alpha\beta\Big[g^c\big((\Phi X_0)^c,\xi^c\big)Y_0^v \\
&+ g^c\big((\Phi X_0)^v,\xi^c\big)Y_0^c - g^c\big((\Phi Y_0)^c,\xi^c\big)X_0^v - g^c\big((\Phi Y_0)^v,\xi^c\big)X_0^c\Big] \\
&+ \alpha\beta\Big[g^c(X_0^c,\xi^c)(\Phi Y_0)^v + g^c(X_0^v,\xi^c)(\Phi Y_0)^c - g^c(Y_0^c,\xi^c)(\Phi X_0)^v \\
&- g^c(Y_0^v,\xi^c)(\Phi X_0)^c\Big] - \frac{1}{n-1}\Big[\beta^2\Big(\big(g^c(\Phi Y_0)^c,(\Phi\xi)^c\big)X_0^v
\end{aligned}$$

$$+\big(g^c(\Phi Y_0)^v,(\Phi\xi)^c\big)X_0^c-\big(g^c(\Phi X_0)^c,(\Phi\xi)^c\big)Y_0^v$$
$$-\big(g^c(\Phi X_0)^v,(\Phi\xi)^c\big)Y_0^c\Big)-\beta^2a^2\Big(g^c((\Phi Y_0)^c,\xi^c)X_0^v$$
$$+g^c((\Phi Y_0)^v,\xi^c)X_0^c-g^c((\Phi X_0)^c,\xi^c)Y_0^v-g^c((\Phi X_0)^v,\xi^c)Y_0^c\Big)$$
$$-\alpha\beta a^2\Big(g^c(Y_0^c,\xi^c)X_0^v+g^c(Y_0^v,\xi^c)X_0^c-g^c(X_0^c,\xi^c)Y_0^v$$
$$-g^c(X_0^v,\xi^c)Y_0^c\Big)\Big]. \tag{6.5}$$

Using equations (4.6), (4.7), (i) and (iv) in equation (6.5), we get

$$\ddot{P}^c(X_0^c,Y_0^c)\xi^c=0. \tag{6.6}$$

So, we can conclude the theorems

Theorem 6.2. *For an n- dimensional golden Lorentzian manifold (M,g^c,Φ^c) equipped with a GSM-ξ-C in the tangent bundle T_0M. We can say that T_0M is ξ^c-projectively flat with respect to GSM-ξ-C.*

Theorem 6.3. *In any golden Lorentzian manifold (M,g^c,Φ^c) equipped with the lifts of GSMC in the tangent bundle T_0M, we have the classification of the lifts of projective curvature tensor $\ddot{P}^c$ in the tangent bundle as*

(i) When the lifts of the connection in the tangent bundle T_0M is semi-symmetric metric connection $(i.e,\alpha=1,\beta=0)$. The value of $\ddot{P}^c$ is
$$\ddot{P}^c(X_0^c,Y_0^c)Z_0^c=P^c(X_0^c,Y_0^c)Z_0^c+\tfrac{n(1-\varepsilon^c)}{n-1}\Big[g^c(X_0^c,Z_0^c)Y_0^v+g^c(X_0^v,Z_0^c)Y_0^c$$
$$-g^c(Y_0^c,Z_0^c)X_0^v-g^c(Y_0^v,Z_0^c)X_0^c\Big].$$

(ii) When the lifts of the connection in the tangent bundle T_0M is quarter-symmetric metric connection $(i.e,\alpha=0,\beta=1)$. The value of $\ddot{P}^c$ is
$$\ddot{P}^c(X_0^c,Y_0^c)Z_0^c=P^c(X_0^c,Y_0^c)Z_0^c+g^c\big((\Phi X_0)^c,Z_0^c\big)(\Phi Y_0)^v+g^c\big((\Phi X_0)^v,Z_0^c\big)(\Phi Y_0)^c$$
$$-g^c\big((\Phi Y_0)^c,Z_0^c\big)(\Phi X_0)^v-g^c\big((\Phi Y_0)^v,Z_0^c\big)(\Phi X_0)^c-\tfrac{1}{n-1}\Big[g^c\big((\Phi Y_0)^c,(\Phi Z_0)^c\big)X_0^v$$
$$+g^c\big((\Phi Y_0)^v,(\Phi Z_0)^c\big)X_0^c-a^c\Big(g^c\big((\Phi Y_0)^c,Z_0^c\big)X_0^v+g^c\big((\Phi Y_0)^v,Z_0^c\big)X_0^c-g^c\big((\Phi X_0)^c,Z_0^c\big)Y_0^v$$
$$-g^c\big((\Phi X_0)^v,Z_0^c\big)Y_0^c\Big].$$

(iii) When the lifts of the connection in the tangent bundle T_0M is α-semi-symmetric metric connection $(\beta=0)$, the value of $\ddot{P}^c$ is
$$\ddot{P}^c(X_0^c,Y_0^c)Z_0^c=P^c(X_0^c,Y_0^c)Z_0^c+\tfrac{n(1-\varepsilon^c)}{n+1}\alpha^2\Big[g^c(X_0^c,Z_0^c)Y_0^c+g^c(X_0^v,Z_0^c)Y_0^v$$
$$-g^c(Y_0^c,Z_0^c)X_0^v-g^c(Y_0^v,Z_0^c)X_0^c\Big].$$

(iv) When the lifts of the connection in the tangent bundle T_0M is β-quarter symmetric metric connection $(i.e,\alpha=0,)$, then the value of $\ddot{P}^c$ is
$$\ddot{P}^c(X_0^c,Y_0^c)Z_0^c=P^c(X_0^c,Y_0^c)Z_0^c+\beta^2\Big[g^c\big((\Phi X_0)^c,Z_0^c\big)(\Phi Y_0)^v+g^c\big((\Phi X_0)^v,Z_0^c\big)(\Phi Y_0)^c$$
$$-g^c\big((\Phi Y_0)^c,Z_0^c\big)(\Phi X_0)^v-g^c\big((\Phi Y_0)^v,Z_0^c\big)(\Phi X_0)^c\Big]-\tfrac{1}{n-1}\Big[\beta^2\Big(g^c\big((\Phi Y_0)^c,(\Phi Z_0)^c\big)X_0^v$$
$$+g^c\big((\Phi Y_0)^v,(\Phi Z_0)^c\big)X_0^c-g^c\big((\Phi X_0)^c,(\Phi Z_0)^c\big)Y_0^v-g^c\big((\Phi X_0)^v,(\Phi Z_0)^c\big)Y_0^c\Big)$$
$$-\beta^2a^c\Big(g^c\big((\Phi Y_0)^c,Z_0^c\big)X_0^v+g^c\big((\Phi Y_0)^v,Z_0^c\big)X_0^c-g^c\big((\Phi X_0)^c,Z_0^c\big)Y_0^v-g^c\big((\Phi X_0)^v,Z_0^c\big)Y_0^c\Big].$$

Proof. Using the classification of proposition (3.2), we obtain (i), (ii), (iii) and (iv). □

7 Conformal curvature tensor of GSMC on golden Lorentzian manifolds to the tangent bundle

Let K and $\ddot{K}$ be the curvature tensor of type $(0,4)$ given by

$$K(X_0,Y_0,Z_0,A) = g\big(R(X_0,Y_0)Z_0,A\big), \tag{7.1}$$

$$\ddot{K}(X_0,Y_0,Z_0,A) = g\big(\ddot{R}(X_0,Y_0)Z_0,A\big). \tag{7.2}$$

For all vector fields X_0, Y_0, Z_0, A on M. In a golden Lorentzian manifold equipped with GSMC $\ddot{\nabla}$. Let $\ddot{W}$ be the conformal curvature tensor associated with $\ddot{\nabla}$ given by

$$\begin{aligned}\ddot{W}(X_0,Y_0,Z_0,A) &= \ddot{K}(X_0,Y_0,Z_0,A) - \frac{1}{n-2}\Big[g(Y_0,Z_0)\ddot{S}(X_0,A)\\ &\quad - g(X_0,Z_0)\ddot{S}(Y_0,A) + \ddot{S}(Y_0,Z_0)g(X_0,A)\\ &\quad - \ddot{S}(X_0,Z_0)g(Y_0,A)\Big] + \frac{\ddot{r}}{(n-1)(n-2)}\Big[g(Y_0,Z_0)g(X_0,A)\\ &\quad - g(X_0,Z_0)g(Y_0,A)].\end{aligned} \tag{7.3}$$

Let K and $\ddot{K}$ be the lift of the curvature tensor of type $(0,4)$ in the tangent bundle T_0M and taking complete lifts by mathematical operators on equations (7.1) and (7.2), we get

$$K^c(X_0^c,Y_0^c,Z_0^c,A^c) = g^c\big(R^c(X_0^c,Y_0^c)Z_0^c,A^c\big), \tag{7.4}$$

$$\ddot{K}^c(X_0^c,Y_0^c,Z_0^c,A^c) = g^c\big(\ddot{R}^c(X_0^c,Y_0^c)Z_0^c,A^c\big). \tag{7.5}$$

Again let $\ddot{W}^c$ be the lift of the conformal curvature tensor associated with GSMC $\ddot{\nabla}^c$ in the tangent bundle T_0M and taking a complete lift of equation (7.3), we get

$$\begin{aligned}\ddot{W}^c(X_0^c,Y_0^c,Z_0^c,A^c) &= \ddot{K}^c(X_0^c,Y_0^c,Z_0^c,A^c) - \frac{1}{n-2}\Big[g^c(Y_0^c,Z_0^c)\ddot{S}^v(X_0^c,A^c)\\ &\quad + g^c(Y_0^v,Z_0^c)\ddot{S}^c(X_0^c,A^c) - g^c(X_0^c,Z_0^c)\ddot{S}^v(Y_0^c,A^c)\\ &\quad - g^c(X_0^v,Z_0^c)\ddot{S}^c(Y_0^c,A^c) + \ddot{S}^c(Y_0^c,Z_0^c)g^c(X_0^v,A^c)\\ &\quad + \ddot{S}^v(Y_0^c,Z_0^c)g^c(X_0^c,A^c) - \ddot{S}^c(X_0^c,Z_0^c)g^c(Y_0^v,A^c)\\ &\quad - \ddot{S}^v(X_0^c,Z_0^c)g^c(Y_0^c,A^c)\Big]\\ &\quad + \frac{\ddot{r}^c}{(n-1)(n-2)}\Big[g^c(Y_0^c,Z_0^c)g^c(X_0^v,A^c)\\ &\quad + g^c(Y_0^v,Z_0^c)g^c(X_0^c,A^c) - g^c(X_0^c,Z_0^c)g^c(Y_0^v,A^c)\\ &\quad - g^c(X_0^v,Z_0^c)g^c(Y_0^c,A^c)].\end{aligned} \tag{7.6}$$

Theorem 7.1. *In any golden Lorentzian manifold equipped with the lifts of GSMC in the tangent bundle T_0M, we have the classification of the lifts of conformal curvature tensor $\ddot{W}^c$ in the tangent bundle as*

(i) When the lifts of the connection in the tangent bundle T_0M is semi-symmetric metric connection $(i.e, \alpha = 1, \beta = 0)$. The value of $\ddot{W}^c$ is
$\ddot{W}^c(X_0^c,Y_0^c,Z_0^c,A^c) = W^c(X_0^c,Y_0^c,Z_0^c,A^c) + g^c(X_0^c,Z_0^c)g^c(Y_0^v,A^c)$
$+ g^c(X_0^v,Z_0^c)g^c(Y_0^c,A^c) - g^c(Y_0^c,Z_0^c)g^c(X_0^v,A^c) - g^c(Y_0^v,Z_0^c)g^c(X_0^c,A^c)$
$- \frac{2(1-n\varepsilon^c)}{n-2}\Big[g^c(X_0^c,A^c)g^c(Y_0^v,Z_0^c) + g^c(X_0^v,A^c)g^c(Y_0^c,Z_0^c) - g^c(X_0^c,Z_0^c)g^c(Y_0^v,A^c)$
$- g^c(X_0^v,Z_0^c)g^c(Y_0^c,A^c)\Big] + \frac{1}{(n-1)(n-2)}\Big[(\varepsilon^c - n)n\Big]\Big[g^c(Y_0^c,Z_0^c)g^c(X_0^v,A^c)$
$+ g^c(Y_0^v,Z_0^c)g^c(X_0^c,A^c) - g^c(X_0^c,Z_0^c)g^c(Y_0^v,A^c) - g^c(X_0^v,Z_0^c)g^c(Y_0^c,A^c)\Big]$.

(ii) When the lifts of the connection in the tangent bundle T_0M is quarter-symmetric metric connection $(i.e, \alpha = 0, \beta = 1)$. The value of $\ddot{W}^c$ is

$$\ddot{W}^c(X_0^c, Y_0^c, Z_0^c, A^c) = W^c(X_0^c, Y_0^c, Z_0^c, A^c) + g^c\big((\Phi X_0)^c, Z_0^c\big)g^c\big((\Phi Y_0)^v, A^c\big)$$
$$+ g^c\big((\Phi X_0)^v, Z_0^c\big)g^c\big((\Phi Y_0)^c, A^c\big) - g^c\big((\Phi Y_0)^c, Z_0^c\big)g^c\big((\Phi X_0)^v, A^c\big)$$
$$- g^c\big((\Phi Y_0)^v, Z_0^c\big)g^c\big((\Phi X_0)^c, A^c\big) - \tfrac{1}{n-2}\Big[g^c(Y_0^c, Z_0^c)g^c\big((\Phi X_0)^v, A^c\big)$$
$$+ g^c(Y_0^v, Z_0^c)g^c\big((\Phi X_0)^c, A^c\big) - g^c(Y_0^c, Z_0^c)g^c\big((\Phi X_0)^c, A^c\big)a^v$$
$$- g^c(Y_0^c, Z_0^c)g^c\big((\Phi X_0)^v, A^c\big)a^c - g^c(Y_0^v, Z_0^c)g^c\big((\Phi X_0)^c, A^c\big)a^c$$
$$- g^c(X_0^c, Z_0^c)g^c\big((\Phi Y_0)^c, (\Phi A)^v\big) - g^c(X_0^c, Z_0^c)g^c\big((\Phi Y_0)^v, (\Phi A)^c\big)$$
$$+ g^c(X_0^v, Z_0^c)g^c\big((\Phi Y_0)^c, (\Phi A)^c\big)g^c(X_0^c, Z_0^c)g^c\big((\Phi Y_0)^c, A^c\big)a^v$$
$$+ g^c(X_0^c, Z_0^c)g^c\big((\Phi Y_0)^v, A^c\big)a^c$$
$$+ g^c(X_0^v, Z_0^c)g^c\big((\Phi Y_0)^c, A^c\big)a^c + g^c(X_0^c, A^c)g^c\big((\Phi Y_0)^v, (\Phi Z_0)^c\big)$$
$$+ g^c(X_0^v, A^c)g^c\big((\Phi Y_0)^c, (\Phi Z_0)^c\big) - g^c(X_0^c, A^c)g^c\big((\Phi Y_0)^c, Z_0^c\big)a^v$$
$$- g^c(X_0^c, A^c)g^c\big((\Phi Y_0)^v, Z_0^c\big)a^c - g^c(X_0^v, A^c)g^c\big((\Phi Y_0)^c, Z_0^c\big)a^c$$
$$- g^c(Y_0^c, A^c)g^c\big((\Phi X_0)^v, (\Phi Z_0)^c\big) - g^c(Y_0^v, A^c)g^c\big((\Phi X_0)^c, (\Phi Z_0)^c\big)$$
$$+ g^c(Y_0^v, A^c)g^c\big((\Phi X_0)^c, Z_0^c\big)a^v + g^c(Y_0^c, A^c)g^c\big((\Phi X_0)^v, Z_0^c\big)a^c$$
$$+ g^c(Y_0^v, A^c)g^c\big((\Phi X_0)^c, Z_0^c\big)a^c\Big] + \tfrac{1}{(n-1)(n-2)}\Big[a^c + n\varepsilon^c - (a^c)^2\Big]\Big[g^c(Y_0^c, Z_0^c)g^c(X_0^v, A^c)$$
$$+ g^c(Y_0^v, Z_0^c)g^c(X_0^c, A^c) - g^c(X_0^c, Z_0^c)g^c(Y_0^v, A^c) - g^c(X_0^v, Z_0^c)g^c(Y_0^c, A^c)\Big].$$

(iii) When the lifts of the connection in the tangent bundle T_0M is α-symmetric metric connection $(i.e, \beta = 0)$. The value of $\ddot{W}^c$ is

$$\ddot{W}^c(X_0^c, Y_0^c, Z_0^c, A^c) = W_0^c(X_0^c, Y_0^c, Z_0^c, A^c) + \alpha^2\Big[g^c(X_0^c, Z^c)g^c(Y_0^v, A^c)$$
$$+ g^c(X_0^v, Z_0^c)g^c(Y_0^c, A^c) - g^c(Y_0^c, Z_0^c)g^c(X_0^v, A^c) - g^c(Y_0^v, Z_0^c)g^c(X_0^c, A^c)\Big]$$
$$- \tfrac{(1-n\varepsilon^c)\alpha^2}{(n-2)}\Big[g^c(X_0^c, A^c)g^c(Y_0^v, Z_0^c) + g^c(X_0^v, A^c)g^c(Y_0^c, Z_0^c) - g^c(X_0^c, Z_0^c)g^c(Y_0^v, A^c)$$
$$- g^c(X_0^v, Z_0^c)g^c(Y_0^c, A^c)\Big] + \tfrac{1}{(n-1)(n-2)}\Big[(\varepsilon^c - n)n\Big]\Big[g^c(Y_0^c, Z_0^c)g^c(X_0^v, A^c)$$
$$+ g^c(Y_0^v, Z_0^c)g^c(X_0^c, A^c) - g^c(X_0^c, Z_0^c)g^c(Y_0^v, A^c) - g^c(X_0^v, Z_0^c)g^c(Y_0^c, A^c)\Big].$$

(iv) When the lifts of the connection in the tangent bundle T_0M is β-quarter symmetricmetric connection $(i.e, \alpha = 0)$. The value of $\ddot{W}^c$ is

$$\ddot{W}^c(X_0^c, Y_0^c, Z_0^c, A^c) = W^c(X_0^c, Y_0^c, Z_0^c, A^c) + \beta^2 g^c\big((\Phi X_0)^c, Z_0^c\big)g^c\big((\Phi Y_0)^v, A^c\big)$$
$$+ \beta^2 g^c\big((\Phi X_0)^v, Z_0^c\big)g^c\big((\Phi Y_0)^c, A^c\big) - \beta^2 g^c\big((\Phi Y_0)^c, Z_0^c\big)g^c\big((\Phi X_0)^v, A^c\big)$$
$$- \beta^2 g^c\big((\Phi Y_0)^v, Z_0^c\big)g^c\big((\Phi X_0)^c, A^c\big) - \tfrac{\beta^2}{n-2}\Big[g^c(Y_0^c, Z_0^c)g^c\big((\Phi X_0)^v, A^c\big)$$
$$+ g^c(Y_0^v, Z_0^c)g^c\big((\Phi X_0)^c, A^c\big) - g^c(Y_0^c, Z_0^c)g^c\big((\Phi X_0)^c, A^c\big)a^v$$
$$- g^c(Y_0^c, Z_0^c)g^c\big((\Phi X_0)^v, A^c\big)a^c - g^c(Y_0^v, Z_0^c)g^c\big((\Phi X_0)^c, A^c\big)a^c$$
$$- g^c(X_0^c, Z_0^c)g^c\big((\Phi Y_0)^v, (\Phi A)^c\big) - g^c(X_0^v, Z_0^c)g^c\big((\Phi Y_0)^c, (\Phi A)^c\big)$$
$$+ g^c(X_0^c, Z_0^c)g^c\big((\Phi Y_0)^c, A^c\big)a^v + g^c(X_0^c, Z_0^c)g^c\big((\Phi Y_0)^v, A^c\big)a^c$$
$$+ g^c(X_0^v, Z_0^c)g^c\big((\Phi Y_0)^c, A^c\big)a^c + g^c(X_0^c, A^c)g^c\big((\Phi Y_0)^v, (\Phi Z_0)^c\big)$$
$$+ g^c(X_0^v, A^c)g^c\big((\Phi Y_0)^c, (\Phi Z_0)^c\big) - g^c(X_0^c, A^c)g^c\big((\Phi Y_0)^c, Z_0^c\big)a^v$$
$$- g^c(X_0^c, A^c)g^c\big((\Phi Y_0)^v, Z_0^c\big)a^c - g^c(X_0^v, A^c)g^c\big((\Phi Y_0)^c, Z_0^c\big)a^c$$
$$- g^c(Y_0^c, A^c)g^c\big((\Phi X_0)^v, (\Phi Z_0)^c\big) - g^c(Y_0^v, A^c)g^c\big((\Phi X_0)^c, (\Phi Z_0)^c\big)$$
$$+ g^c(Y_0^c, A^c)g^c\big((\Phi X_0)^c, Z_0^c\big)a^v + g^c(Y_0^c, A^c)g^c\big((\Phi X_0)^v, Z_0^c\big)a^c$$
$$+ g^c(Y_0^v, A^c)g^c\big((\Phi X_0)^c, Z_0^c\big)a^c\Big] + \tfrac{\beta^2}{(n-1)(n-2)}\Big[a^c + n\varepsilon^c - (a^c)^2\Big]\Big[g^c(Y_0^c, Z_0^c)g^c(X_0^v, A^c)$$
$$+ g^c(Y_0^v, Z_0^c)g^c(X_0^c, A^c) - g^c(X_0^c, Z_0^c)g^c(Y_0^v, A^c) - g^c(X_0^v, Z_0^c)g^c(Y_0^c, A^c)\Big].$$

Where

$$W^c(X_0^c, Y_0^c, Z_0^c, A^c) = K^c(X_0^c, Y_0^c, Z_0^c, A^c) - \frac{1}{n-2}\Big[g^c(Y_0^c, Z_0^c)S^v(X_0^c, A^c)$$
$$+ g^c(Y_0^v, Z_0^c)S^c(X_0^c, A^c) - g^c(X_0^c, Z_0^c)S^v(Y_0^c, A^c)$$
$$- g^c(X_0^v, Z_0^c)S^c(Y_0^c, A^c) + S^c(Y_0^c, Z_0^c)g^c(X_0^v, A^c)$$

$$
\begin{aligned}
&+ S^v(Y_0^c, Z_0^c)g^c(X_0^c, A^c) - S^c(X_0^c, Z_0^c)g^c(Y_0^v, A^c) \\
&- S^v(X_0^c, Z_0^c)g^c(Y_0^c, A^c)\Big] \\
&+ \frac{r^c}{(n-1)(n-2)}\Big[g^c(Y_0^c, Z_0^c)g^c(X_0^v, A^c) \\
&+ g^c(Y_0^v, Z_0^c)g^c(X_0^c, A^c) - g^c(X_0^c, Z_0^c)g^c(Y_0^v, A^c) \\
&- g^c(X_0^v, Z_0^c)g^c(Y_0^c, A^c)]. \qquad (7.7)
\end{aligned}
$$

Proof. Using equations (7.4), (7.5), (4.6), (4.7), (4.9) and classification of proposition (3.2) in equation (7.6), we get (i), (ii), (iii), (iv). □

8 Example

In this section, we present an explicit example to illustrate the main results obtained in the previous sections.

Example 8.1. Let $M = \mathbb{R}^3$ be the 3–dimensional semi-Euclidean space endowed with the Lorentzian metric

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

which has signature $(-,+,+)$. Define a $(1,1)$ tensor field Φ on M by

$$\Phi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{1+\sqrt{5}}{2}\frac{\partial}{\partial x_1}, \frac{1+\sqrt{5}}{2}\frac{\partial}{\partial x_2}, \frac{1+\sqrt{5}}{2}\frac{\partial}{\partial x_3}\right).$$

Then Φ satisfies

$$\Phi^2 = \Phi + I,$$

where I denotes the identity transformation. Moreover, it can be directly verified that

$$g(\Phi X_0, Y_0) = g(X_0, \Phi Y_0),$$

for all vector fields X_0, Y_0 on M. Hence (M, g, Φ) is a golden Lorentzian manifold.

Let $\xi = \frac{\partial}{\partial x_3}$ be a unit spacelike vector field on M, so that

$$g(\xi, \xi) = 1,$$

and let u be the associated 1–form defined by $u(X_0) = g(X_0, \xi)$. Consider the generalized symmetric metric connection $\breve{\nabla}$ of (α, β)–type on M given by

$$\breve{\nabla}_{X_0}Y_0 = \nabla_{X_0}Y_0 + \alpha\{-g(X_0, Y_0)\xi + u(Y_0)X_0\} + \beta\{-g(\Phi X_0, Y_0)\xi + u(Y_0)\Phi X_0\},$$

where ∇ is the Levi–Civita connection of g.

Now consider the tangent bundle T_0M endowed with the complete lift metric g^c and the complete lift golden structure Φ^c. The complete lift $\breve{\nabla}^c$ of the GSMC $\breve{\nabla}$ defines a generalized symmetric metric connection on T_0M satisfying

$$\breve{\nabla}^c g^c = 0.$$

Furthermore, if the lifted vector field ξ^c is parallel with respect to $\breve{\nabla}^c$, then $\breve{\nabla}^c$ becomes a GSM–ξ–connection on T_0M.

By direct computation, using equations (4.6) and (4.9), we observe that the concircular curvature tensor $\breve{C}^c$ satisfies

$$\breve{C}^c(X_0^c, Y_0^c)\xi^c = 0,$$

and the projective curvature tensor $\breve{P}^c$ satisfies

$$\breve{P}^c(X_0^c, Y_0^c)\xi^c = 0,$$

for all lifted vector fields X_0^c, Y_0^c on T_0M.

Hence, the tangent bundle T_0M is both ξ^c–concirculary flat and ξ^c–projectively flat with respect to the generalized symmetric metric ξ–connection $\breve{\nabla}^c$. This example confirms the theoretical results obtained in Theorems 5.2 and 6.2.

9 Conclusion/Discussion

The present study deals with the liftings of the golden Lorentzian manifolds endowed with generalized symmetric metric connection in the tangent bundles. We have investigated the curvature tensor and Ricci tensor of the golden Lorentzian manifolds with respect to GSMC in the tangent bundles. Moreover some properties of the curvature tensor, concircular curvature tensor, projective curvature tensor and conformal curvature tensor with GSMC in the tangent bundle T_0M are studied and we find that T_0M is ξ^c-concirculary and ξ^c-projectively flat with respect to generalized symmetric metric ξ connection in golden Lorentzian manifolds to the tangent bundles.

Conflict of Interest

The authors declare that they have no conflict of interset

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