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Employing Matrix Inequalities in Bounding Zeros of Polynomials

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Abstract. This paper investigates the use of operator theoretic tools, specifically the numerical radius and Cartesian decomposition, to establish new bounds for the zeros of complex polynomials. By associating a polynomial with matrix similar to its Frobenius companion matrix, we leverage the properties of numerical radius as an upper bound for the spectral radius to derive refined estimates for the location of zeros in the complex plane. Furthermore, the Cartesian decomposition of operators with self-adjoint components enables to offer more accurate bounds than those obtainable via traditional norm inequalities.

1 Introduction

The study of zeros of polynomials is a fundamental topic in mathematics, with wide applications in scientific computing, control theory, and numerical linear algebra. A classical and powerful tool for investigating the zeros of a polynomial is the Frobenius companion matrix (FCM), whose eigenvalues coincide exactly with the zeros of a polynomial. This correspondence allows algebraic questions about zeros to be reformulated as spectral questions about matrices, making it possible to apply matrix analysis and numerical techniques. Many bounds for the zeros of polynomials, whether with scalar or matrix coefficients, have been discussed by numerous researchers, as can be seen in [1], [6, 7, 8, 9, 10, 11],[14], and [20, 21]. In this context, matrices serve as a powerful and versatile tool for deriving new mathematical results and establishing innovative frameworks, see, e.g., [22] and [23].

Here, we give some basic definitions and facts that we need in our work. For $B \in M_m(\mathbb{C})$, where the symbol $M_m(\mathbb{C})$ denotes the algebra of all $m \times m$ complex matrices, let $r(B), \| B \|$, and $w(B)$ denote the spectral radius, the spectral norm, and the numerical radius of B, respectively, which are defined by:

$$r(B) = max\{|\mu| : \mu \, is \, an \, eigenvalue \, of \, B\},$$
$$\|B\| = max\{\|By\| : y \in \mathbb{C}^m, \|y\| = 1\},$$
$$w(B) = max\{|\langle By, y\rangle| : y \in \mathbb{C}^m, \|y\| = 1\},$$

It is well known that

$$r(B) \leq w(B) \leq \|B\|$$

with equality if B is normal.
A considerable number of researchers have shown great interest in investigating the relationship between inequalities involving the numerical radius and the spectral norm. In particular, many works focus on exploring these relations and deriving new inequalities in order to establish further theoretical results and potential applications. For more recent contributions concerning

numerical radius and norm inequalities, we refer the reader to [2, 3, 4] ,[15] and [19].
In this paper, we establish a new technique that enables us to present better bounds for the zeros of polynomials and we present an illustrative example. Moreover, a new bound for the zeros of polynomials with matrix coefficients is given.

2 Locating the Zeros of Polynomials with Scalar Coefficients via the Cartesian Decomposition of Matrices

In this section, we present a new approach to get new bounds for the zeros of polynomials. This approach relies primarily on the use of the Cartesian decomposition of matrices. A numerical example demonstrates that our results are superior to some known bounds found in the literature.

It was shown in [5] that if $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is such that $A_{kj} \in M_n(\mathbb{C}), i, j = 1, 2$, then

$$w\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \leq \sqrt{w\left(P_{11}^2 + Q_{11}^2\right) + w\left(P_{22}^2 + Q_{22}^2\right) + \sqrt{\left(w\left(P_{11}^2 + Q_{11}^2\right) - w\left(P_{22}^2 + Q_{22}^2\right)\right)^2 + N^2}}, \tag{2.1}$$

where $P_{kj}+iQ_{kj}$ is the Cartesian decomposition of A_{kj}, $N = \||P_{12}|+|Q_{12}|\|+\||P_{21}|+|Q_{21}|\|$.
In the following, a new algorithm has been employed to locate the zeros of polynomials via a matrix similar to the FCM partitioned using the Cartesian decomposition and then we apply our work on Inequality (2.1).
Let

$$q(z) = z^m + b_m z^{m-1} + b_{m-1} z^{m-2} + \ldots + b_2 z + b_1,$$

be a polynomial with complex coefficients $b_1, b_2, \ldots, b_m$, where $b_1 \neq 0$. The FCM of q is the $m \times m$ matrix given by

$$H(q) = \begin{bmatrix} -b_m & -b_{m-1} & -b_{m-2} & \ldots & -b_2 & -b_1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

It is widely known that the zeros of q and the eigenvalues of $H(q)$ are the same.
Considering the invertible matrix G,

$$G = \begin{bmatrix} 1 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & -1 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \text{we have } G^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Define the matrix M as $M = GH(q)G^{-1}$, which will be called M-companion matrix of q. Thus, the matrix M can be written as

$$M = \begin{bmatrix} a_m & a_{m-1} & a_{m-2} & \dots & a_2 & a_1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

where

$$a_m = -b_m - 1,$$

$$a_{m-1} = a_m - b_{m-1},$$

$$a_{m-2} = a_{m-1} - b_{m-2},$$

$$\vdots$$

$$a_2 = a_3 - b_2,$$

$$a_1 = a_2 - b_1.$$

To apply Lemma 3.1, we note that we have a small division challenge in applying the results we obtained because the main diagonal in the presented results requires a square submatrix. To address this issue, we propose to consider the degree $q(z)$ as even integer, and to achieve this goal, we consider the polynomial:

$$q(z) = z^{2m} + b_{2m}z^{2m-1} + \ldots + b_2 z + b_1, m \geq 2$$

and we will partition the M-companion matrix as follows:

$$M = F + K,$$

where

$$F = \begin{bmatrix} a_{2m} & a_{2m-1} & a_{2m-2} & \dots & a_2 & a_1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

To simplify the calculation of the numerical radius, we will partition the matrix F as follows:

$$F = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}_{2m \times 2m},$$

where

$$B_{11} = \begin{bmatrix} a_{2m} & a_{2m-1} & a_{2m-2} & \dots & a_{m+2} & a_{m+1} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; B_{12} = \begin{bmatrix} a_m & a_{m-1} & a_{m-2} & \dots & a_2 & a_1 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; B_{22} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Performing the Cartesian decomposition of $F, Re(F) + iIm(F)$, we obtain

$$Re(F) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}_{2m \times 2m},$$

$$P_{11} = \begin{bmatrix} Re(a_{2m}) & \frac{a_{2m-1}+1}{2} & \frac{a_{2m-2}}{2} & \frac{a_{2m-3}}{2} & \dots & \frac{a_{m+2}}{2} & \frac{a_{m+1}}{2} \\ \frac{\overline{a_{2m-1}}+1}{2} & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ \frac{\overline{a_{2m-2}}}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \ddots & 0 & 0 \\ \frac{\overline{a_{2m-3}}}{2} & 0 & \frac{1}{2} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{\overline{a_{2m+2}}}{2} & 0 & 0 & 0 & \ddots & 0 & \frac{1}{2} \\ \frac{\overline{a_{2m+1}}}{2} & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 \end{bmatrix},$$

$$P_{12} = \begin{bmatrix} \frac{a_m}{2} & \frac{a_{m-1}}{2} & \frac{a_{m-2}}{2} & \frac{a_{m-3}}{2} & \dots & \frac{a_2}{2} & \frac{a_1}{2} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} \frac{\overline{a_m}}{2} & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} \\ \frac{\overline{a_{m-1}}}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{\overline{a_{m-2}}}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{\overline{a_{m-3}}}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\overline{a_2}}{2} & 0 & 0 & 0 & \ddots & 0 & 0 \\ \frac{\overline{a_1}}{2} & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \; P_{22} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \ddots & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 \end{bmatrix}.$$

and

$$Im(F) = \begin{bmatrix} Q_{11}\, Q_{12} \\ Q_{21}\, Q_{22} \end{bmatrix}_{2m \times 2m},$$

where

$$Q_{11} = \begin{bmatrix} Im(a_{2m}) & \frac{a_{2m-1}+1}{2i} & \frac{a_{2m-2}}{2i} & \frac{a_{2m-3}}{2i} & \dots & \frac{a_{m+2}}{2i} & \frac{a_{m+1}}{2i} \\ \frac{\overline{-a_{2m-1}}+1}{2i} & 0 & \frac{-1}{2i} & 0 & \dots & 0 & 0 \\ \frac{\overline{-a_{2m-2}}}{2i} & \frac{1}{2i} & 0 & \frac{-1}{2i} & \ddots & 0 & 0 \\ \frac{\overline{-a_{2m-3}}}{2i} & 0 & \frac{1}{2i} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{\overline{-a_{2m+2}}}{2i} & 0 & 0 & 0 & \ddots & 0 & \frac{-1}{2i} \\ \frac{\overline{-a_{2m+1}}}{2i} & 0 & 0 & 0 & \dots & \frac{1}{2i} & 0 \end{bmatrix},$$

$$Q_{12} = \begin{bmatrix} \frac{a_m}{2i} & \frac{a_{m-1}}{2i} & \frac{a_{m-2}}{2i} & \frac{a_{m-3}}{2i} & \dots & \frac{a_2}{2i} & \frac{a_1}{2i} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \frac{-1}{2i} & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

$$Q_{21} = \begin{bmatrix} \frac{\overline{-a_m}}{2i} & 0 & 0 & 0 & \dots & 0 & \frac{1}{2i} \\ \frac{\overline{-a_{m-1}}}{2i} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{\overline{-a_{m-2}}}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{\overline{-a_{m-3}}}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\overline{-a_2}}{2i} & 0 & 0 & 0 & \ddots & 0 & 0 \\ \frac{\overline{-a_1}}{2i} & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad Q_{22} = \begin{bmatrix} 0 & \frac{-1}{2i} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{2i} & 0 & \frac{-1}{2i} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{2i} & 0 & \frac{-1}{2i} & \ddots & 0 & 0 \\ 0 & 0 & \frac{1}{2i} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{-1}{2i} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2i} & 0 \end{bmatrix}.$$

Based on the previous discussion and computations, we get a new bound for the zeros of the polynomials by employing Inequality (2.1) as shown in the following example.

Example 2.1. Consider the polynomial $q(z) = z^6 - z^5 + z^4 - 2z^3 + 3z^2 - 4z + 5$. The upper bounds for the zeros of this polynomial estimated by different mathematicians such as Cauchy, Carmichael and Mason, Montel, Fuji and Kubo, Kittaneh are shown in the following table.

Bound	Value
Cauchy [17]	6
Carmichael and Mason [17]	7.55
Montel[17]	16
Fuji and Kudo [12]	5.14
Kittaneh[18]	4.61

If z is any zero of the polynomial $q(z)$, then applying Inequality (2.1) on the previous partitions gives

$$|z| \leq 3.99$$

which are better than all of the estimation mentioned above for this polynomial.

3 Locating the Zeros of Polynomials with Matrix Coefficients

In this section, new bounds for zeros of polynomials with matrix coefficients using numerical radius estimates are introduced. Consider the monic polynomial

$$Q(z) = Iz^m + B_m z^{m-1} + B_{m-1} z^{m-2} + \ldots + B_2 z + B_1,$$

where $B_1, B_2, \ldots \ldots, B_m \in M_n(\mathbb{C})$. The FCM of Q is the $nm \times nm$ matrix defined by

$$H(Q) = \begin{bmatrix} -B_m & -B_{m-1} & -B_{m-2} & \dots & -B_2 & -B_1 \\ I & 0 & 0 & \dots & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \end{bmatrix},$$

A complex number z is called a zero of $Q(z)$ if there is a non-zero vector $y \in \mathbb{C}^m$ such that $Q(z)y = 0$. As in the scalar coefficients case, the zeros of $Q(z)$ coincide with the eigenvalues of $Q(z)$. Thus, if z is a zero of $Q(z)$, then $|z| \leq r(Q(z))$.
To get our bounds for the zeros of Q, Consider the invertible matrix

$$\breve{G} = \begin{bmatrix} I & -I & 0 & 0 & \dots & 0 \\ 0 & I & -I & 0 & \dots & 0 \\ 0 & 0 & I & -I & \dots & 0 \\ 0 & 0 & 0 & I & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & -I \\ 0 & 0 & 0 & 0 & \dots & I \end{bmatrix}, where\ \breve{G}^{-1} = \begin{bmatrix} I & I & I & I & \dots & I \\ 0 & I & I & I & \dots & I \\ 0 & 0 & I & I & \dots & I \\ 0 & 0 & 0 & I & \dots & I \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

In this paper, the $\breve{M}$-companion matrix of Q is defined by $\breve{M} = \breve{G}H(Q)\breve{G}^{-1}$. Thus,

$$\breve{M} = \begin{bmatrix} A_m & A_{m-1} & A_{m-1} & \dots & A_2 & A_1 \\ I & 0 & 0 & \dots & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & I & I \end{bmatrix},$$

$$A_m = -B_m - I,$$

$$A_{m-1} = A_m - B_{m-1},$$

$$A_{m-2} = A_{m-1} - B_{m-2},$$

$$\vdots$$

$$A_2 = A_3 - B_2,$$

$$A_1 = A_2 - B_1.$$

Since $\breve{M}$ and $H(Q)$ have the same eigenvalues, we have

$$|z| \leq r(\breve{M}) \leq w(\breve{M}).$$

Our objective of determining new bounds for the zeros of Q is acheived by estimating the numerical radius of $\breve{M}$ utilizing the following lemmas. The first lemma can be found in [13] while the second and the third lemmas can be found in [16]. Here, $M_{n\times m}(\mathbb{C})$ denotes the space of all $n \times m$ complex matrices.

Lemma 3.1. *If $B = [B_{ij}]$ is an $m \times m$ block matrix with $B_{ij} \in M_{n_i\times n_j}(\mathbb{C})$ and $\Sigma_{i=1}^k n_i = m$, then*

$$w(B) \leq \frac{1}{2}\Sigma_{i=1}^k \left(w\left(B_{ii}\right) + \sqrt{w^2\left(B_{ii}\right) + \Sigma_{j=1,j\neq i}^k \|B_{ii}\|^2} \right).$$

Lemma 3.2. *Let $T \in M_m(\mathbb{C})$ be the block matrix given by*

$$T = \begin{bmatrix} 0 & \frac{1}{2}I & 0 & \dots & 0 \\ \frac{1}{2}I & 0 & \frac{1}{2}I & \dots & 0 \\ 0 & \frac{1}{2}I & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{2}I \\ 0 & 0 & \dots & \frac{1}{2}I & 0 \end{bmatrix}.$$

Then the eigenvalues of T are $\mu_j = cos\frac{\pi j}{m+1}$ for $j = 1, 2, \dots, m$.

Lemma 3.3. *Let $L = [L_{ij}] \in M_m(\mathbb{C})$ be the block matrix with $L_{ij} \in M_{n_i \times n_j}(\mathbb{C})$ and $\Sigma_{i=1}^{k} n_i = m$.*
Then $w(L) \le w(l_{ij})$, where $l_{ij} = w\left(\begin{bmatrix} 0 & L_{ij} \\ L_{ji} & 0 \end{bmatrix}\right)$. In particular $l_{ij} = w(L_{ij})$ for $i = j$.

Theorem 3.4. *Let z be a zero of $Q(z) = Iz^m + B_m z^{m-1} + B_{m-1}z^{m-2} + \dots + B_2 z + B_1$. Then*

$$|z| \le \frac{1}{2}\left(1 + cos\left(\frac{\pi}{m-1}\right) + w\left(A_m\right) + \sqrt{w^2\left(A_m\right) + \Sigma_{k=1}^{m-1}|c_k|^2}\right.$$
$$\left. + \sqrt{cos\left(\frac{\pi}{m-1}\right)^2 + \frac{1}{4} + \Sigma_{k=2}^{m-1}|c_k|^2} + \sqrt{\frac{5}{4} + |c_1|^2}\right),$$

where $c_k = \dfrac{\|A_k\|}{2}, k = 1, 2, 3, \dots m-2, c_{m-1} = w\left(\begin{bmatrix} 0 & A_{m-1} \\ 1 & 0 \end{bmatrix}\right)$,

$$A_m = -B_m - I, A_{m-1} = A_m - B_{m-1}, \dots, A_2 = A_3 - B_2, A_1 = A_2 - B_1.$$

Proof. For any two matrices $A, B \in M_n(\mathbb{C})$, let $L_{(A,B)} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. Applying Lemma 3.3 on $\overset{\smile}{M}$, we get

$$w(\overset{\smile}{M}) \le w(S),$$

where the matrix S is given by

$$\begin{bmatrix} w(A_m) & w(L(A_{m-1}, I)) & w(L(A_{m-2}, 0)) & \dots\dots\dots & w(L(A_3, 0)) & w(L(A_2, 0)) & w(L(A_1, 0)) \\ w(L(A_{m-1}, I)) & w(0) & w(L(I, 0)) & \dots\dots\dots & w(0) & w(0) & w(0) \\ w(L(0, A_{m-2})) & w(L(I, 0)) & w(0) & \dots\dots\dots & w(0) & w(0) & w(0) \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ w(L(0, A_3)) & w(0) & w(0) & \vdots & w(0) & w(L(I, 0)) & w(0) \\ w(L(0, A_2)) & w(0) & w(0) & \vdots & w(L(I, 0)) & w(0) & w(L(0, I) \\ w(L(0, A_1)) & w(0) & w(0) & w(0) & w(0) & w(L(I, 0)) & w(I) \end{bmatrix}.$$

Employing the fact that $w\left(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right) = \frac{\|B\|}{2}$ for every $B \in M_n(\mathbb{C})$, we have

$$S = \begin{bmatrix} w(A_m) & c_{m-1} & c_{m-2} & \dots & c_3 & c_2 & c_1 \\ c_{m-1} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ c_{m-2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_3 & 0 & 0 & \ddots & 0 & \frac{1}{2} & 0 \\ c_2 & 0 & 0 & \vdots & \frac{1}{2} & 0 & \frac{1}{2} \\ c_1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix},$$

where $c_k = \frac{\|A_k\|}{2}, k = 1, 3, \dots, m-2, c_{m-1} = w\left(\begin{bmatrix} 0 & A_{m-1} \\ I & 0 \end{bmatrix}\right)$.

To find $w(S)$, we need to partition the matrix S as $S = \left(\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}\right)$, where

$$L_{11} = [w(A_m)], L_{12} = \begin{bmatrix} c_{m-1} & c_{m-2} & c_{m-3} & \dots & c_2 \end{bmatrix}, L_{13} = [c_1],$$

$$L_{21} = \begin{bmatrix} c_{m-1} \\ c_{m-2} \\ c_{m-3} \\ \vdots \\ c_2 \end{bmatrix}, L_{22} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{2} \\ 0 & 0 & \dots & \frac{1}{2} & 0 \end{bmatrix}, L_{23} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{bmatrix},$$

$$L_{31} = [c_1], L_{32} = \begin{bmatrix} 0 & 0 & \dots & 0 & \frac{1}{2} \end{bmatrix}, L_{33} = [1].$$

Now, the definitions of the numerical radius and spectral norm with Lemma 3.2 yield that

$$w(L_{11}) = w(A_m), \|L_{12}\|^2 = \Sigma_{k=2}^{m-1}|c_k|^2, \|L_{13}\| = |c_1|,$$

$$\|L_{21}\|^2 = \Sigma_{k=2}^{m-1}|c_k|^2, w(L_{22}) = cos\left(\frac{\pi}{m-1}\right), \|L_{23}\| = \frac{1}{2},$$

$$\|L_{31}\| = |c_1|, \|L_{32}\| = \frac{1}{2}, w(L_{33}) = 1.$$

Applying Lemma 3.1 on the partition of S, we have

$$w(S) \leq \frac{1}{2}\left(1 + cos\left(\frac{\pi}{m-1}\right) + w\left(A_m\right) + \sqrt{w^2\left(A_m\right) + \Sigma_{k=1}^{m-1}|c_k|^2}\right.$$
$$\left. + \sqrt{cos^2\left(\frac{\pi}{m-1}\right) + \frac{1}{4} + \Sigma_{k=2}^{m-1}|c_k|^2} + \sqrt{\frac{5}{4} + |c_1|^2}\right).$$

Since $|z| \leq r(H(Q)) = r(\widetilde{M}) \leq w(\widetilde{M}) \leq w(S)$, we get the desired result. □

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