Palestine Journal of Mathematics

Vol.15(3), (2026) , 471–483 © Palestine Polytechnic University-PPU 2026

Stable Range in Commutative Semirings

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Communicated by Ayman Badawi

MSC 2010 Classifications: 16Y60, 13A15, 13A99.

Keywords and phrases: (S-relative) n-stable, n_J-stable, and B-stable semirings, subtractive ideal (= k-ideal), simple semiring, Gelfand semiring, polynomial semiring, relative stable range.

The research of the second author was in part supported by grant no. 1404130015 from IPM

Abstract We extend the classical notion of stable range from ring theory to the broader setting of *commutative semirings with identity* by introducing and analyzing n*-stable*, n_J*-stable*, and B*-stable* semirings. These notions are characterized via unimodular sequences and examined under standard constructions such as *homomorphic images*, *direct products*, and *polynomial extensions*. In particular, we prove that the semiring $R[x]$ is not B-stable when R is zerosumfree or when certain ideal-theoretic conditions hold. We further introduce S*-relative* versions of these stability conditions, offering a more refined framework. Several examples and counterexamples are provided to delineate the boundaries of each notion. Our results contribute to a unified generalization of stable range theory in semiring contexts and deepen the structural understanding of stability beyond the ring-theoretic setting.

1 Introduction

The concept of *stable range*, originally introduced by H. Bass in his foundational work on algebraic K-theory [2], has emerged as an important arithmetic invariant in ring theory. It plays a central role in understanding structural properties such as cancellation, substitution, and exchange within algebraic systems. The notion of stable range one, in particular, has proven especially significant in addressing foundational problems within commutative algebra and K-theory.

In recent years, semiring theory has naturally attracted attention as a generalization of ring theory, with applications extending beyond pure algebra into computer science, optimization, automata theory, and tropical geometry. As a result, extending central concepts from ring theory to the broader setting of semirings is a worthwhile pursuit. While stable range conditions have been widely studied in ring theory [3], their systematic extension to semirings has not yet been fully developed. Despite the growing body of work on semirings, many foundational concepts from ring theory—such as stable range—have yet to be systematically extended and adapted to the semiring setting. Stable range plays a critical role in structural classification, module cancellation, and the behavior of homomorphisms, making it essential to understand its analogues in non-ring contexts. Moreover, semirings appear naturally in tropical algebra, computer science, and optimization, where substitution and exchange properties—closely tied to stability—are implicitly used. Establishing a robust theory of stable range in semirings thus not only advances abstract algebra but also lays groundwork for further connections to applied algebraic systems.

The primary goal of this paper is to study the notion of n*-stable and* n_J*-stable semirings* (Definition 3.2) in a natural way, as an extension of n*-stable rings* [3], and to investigate their algebraic properties. In particular, we explore how unimodular sequences in semirings behave with respect to stability conditions, some of which involve the Jacobson k-radical of a semiring R. To our knowledge, this is the first detailed investigation of stable range notions for semirings that parallels the deep structural results known in ring theory. Our generalization to n_J-stability, B-stability and S-relative stability provides new tools for semiring theorists studying homological and categorical aspects. For some recent studies on stable range in algebraic structures, see [1], [6], and [8].

Related Work and Background. The foundational theory of stable range in rings originates

from H. Bass [2], who introduced it to study the general linear group in algebraic K-theory. Estes and Ohm [3] further developed the theory by characterizing n-stable commutative rings using unimodular sequences. Golan's monograph [4] established a systematic treatment of semirings, including ideal theory, subtractive properties, and radical concepts, which underpins the framework used in this paper. Sen and Mandal [13] studied Jacobson-type radicals in semirings, while Rahimi [9, 12] investigated the transfer of stability properties and Gelfand-type conditions within semiring structures. Our work extends these contributions by introducing and studying new stability classes in semirings that generalize the structural behavior of rings and bring additional categorical flexibility to semiring theory.

The organization of the paper is as follows. In Section 1, we provide the necessary background on commutative semirings, including standard definitions and preliminary results, mainly following Golan [4]. Section 2 is devoted to n-stable and n_J-stable semirings. In our study of these semirings, it suffices only to consider unimodular sequences of size $n+1$ instead of arbitrary unimodular sequences of size larger than $n+1$ (Theorems 3.8, 3.9). We show that any unimodular sequence of R that meets $J_k(R)$ is stable provided its maximal ideals are subtractive (Theorem 3.5). In this section, we discuss the homomorphic image and the direct product of n-stable type semirings (Theorems 3.12 and 3.14). In the last part of this section, we focus only on semirings with stable range one (Theorem 3.15, Corollary 3.16, and Proposition 3.20) and provide some examples of stable semirings (Examples 3.18, 3.19, and 3.21). In Section 3, we introduce and study some algebraic properties of the n-stable and n_J-stable semirings with respect to a nonempty subset S of R (Definition 4.1). In the last section (Section 4), we study some algebraic properties of a subclass of (S-relative) 2-stable semirings R, namely (S-relative) B-stable semirings (Definitions 5.1 and 5.2). It is shown that $R[x]$, the semiring of polynomials over a semiring R, can not be a B-stable semiring with respect to $I[x]$ when I is a strong proper ideal of R (Theorem 5.14(a)); and $R[x]$ is not B-stable when R is a zerosumfree semiring (Theorem 5.14(b)). Finally, we end the paper with a discussion related to the B-stability condition of $R[x]$ (Example 5.19) and also construct a new semiring R by combining a semidomain D and a semifield F, $D \subseteq F$, and show that D is an n-stable, n_J-stable, or a B-stable semidomain when R is an n-stable, n_J-stable, or a B-stable semiring, respectively, (Proposition 5.20).

2 Background and Notation on Commutative Semirings

In this section we will recall some definitions and prove some results concerning semirings which will be used in the sequel. By a semiring $(R, +, \cdot)$, we will mean a nonempty set R with two binary operations of addition and multiplication defined on R such that $(R, +)$ and $(R, \cdot)$ are *commutative monoids* with identity elements 0 and 1, respectively, where multiplication distributes over addition (from either side) and $0a = 0$ for all $a \in R$ and $1 \neq 0$.

• A nonempty subset I of a semiring R will be called an *ideal* if $a, b \in I$ and r in R implies $a + b$ in I and ra in I.

Note that in [4, Chapter 5], Golan defines an ideal I of a semiring R to be different from R, but we don't follow this assumption and make it clear when there is any confusion in the context. In general, by "ideal", in contrast to Golan, we do not necessarily mean a proper ideal. We shall thus always say "proper ideal" when we mean a proper ideal.

Definition 2.1. A subtractive ideal (= k-ideal) I of a semiring S is an ideal such that if $a, a+b \in I$, then $b \in I$. An ideal I of S is said to be a strong ideal (= a strongly k-ideal) if and only if $a+b \in I$ implies that $a \in I$ and $b \in I$.

Remark 2.2. From the above definition, it is clear that (0) is a k-ideal of S. Also, every strongly k-ideal of a semiring S is a k-ideal of S. But the converse need not be true in general. For example, the set $2N$ of all nonnegative even integers is a subtractive ideal of the semiring of all nonnegative integers. But it is not a strongly k-ideal since $3 + 5 \in 2N$ while neither 3 nor 5 belong to $2N$. Note that in [4], Golan uses the term "subtractive ideal" [resp. strong] for a k-ideal [resp. strongly k-ideal], but in the literature of semirings, authors use equivalently the

term "k-ideal" [resp. strongly k-ideal] as well. Throughout this work, except for some cases in this section, we mainly follow Golan in [4]. Also, for some examples of nonsubtractive ideals in a semiring, see Chapter 5 of [4].

• We define the *Jacobson k-radical* of a semiring R, denoted by $J_k(R)$, to be the intersection of all maximal k-ideals of R. Notice that by [13, Corollary 2.2], the Jacobson k-radical of R always exists and it can easily be seen that it is a k-ideal since the intersection of any number of k-ideals is a k-ideal.

We now follow Golan [4, Chapter 8, p. 92] to define a morphism of semirings as follows.

Definition 2.3. If R and S are semirings then a function $f : R \to S$ is a morphism of semirings if and only if:

(a) $f(0_R) = 0_S$;

(b) $f(1_R) = 1_S$; and

(c) $f(r + r') = f(r) + f(r')$ and $f(rr') = f(r)f(r')$ for all r and r' in R.

We now begin considering some properties of morphisms of semirings.

Proposition 2.4. (cf. [4, Proposition 8.37]) *Let $f : R \to S$ be a morphism of semirings.*

(a) *If H is an ideal of S, then $f^{-1}(H)$ is an ideal of R. Moreover, if H is subtractive then so is $f^{-1}(H)$.*

(b) *If f is a surjective morphism and I is an ideal of R, then $f(I)$ is an ideal of S.*

(c) *If f is a surjective morphism, then the kernel of f is a subtractive ideal of R.*

(d) *If f is a surjective morphism, then u is a unit in R if and only if $f(u)$ is a unit in S.*

Proof. Parts (a) and (b) follow from [4, Proposition 8.37] and (c) follows from (a) since $\ker(f) = f^{-1}(\{0\})$. The necessary part of (d) is clear since $1_S = f(1_R) = f(uu^{-1}) = f(u)f(u^{-1})$. Conversely, let $u \in R$, $I = (u)$ an ideal of R, and $f(u)$ be a unit in S. Clearly $f(I) = S$ since $f(u)$ is a unit in S and so $I = R$. Otherwise, $I \neq R$ implies $1_R \in R \setminus I$, which implies $1_S = f(1_R) \notin f(I) = S$, yielding a contradiction. Thus u is a unit in R. □

Remark 2.5. (As defined on page 68 of [4]), an ideal I of a semiring R defines an *equivalence relation* $=_I$ on R called the *Bourne relation*, given by $r =_I r'$ if and only if there exist elements a and a' of I satisfying $r + a = r' + a'$. Note that if $r =_I r'$ and $s =_I s'$ in R, then $r + s =_I r' + s'$ and $rs =_I r's'$. We denote the set of all *equivalence classes* of elements of R under this relation by R/I and will denote the equivalence class of an element r of R by r/I. Clearly this relation is a *congruence* (i.e., an equivalence relation which is compatible with two binary operations of R) and, consequently, R/I is well-defined for any ideal I of R. Also, $a \in I$ implies $a \in 0/I$ since $a =_I 0$ by the fact that $a + 0 = 0 + a$. Thus $I \subseteq 0/I$. Moreover, if I is a subtractive ideal of R, then $0/I = I$ since $a + i = 0 + j \in I$ implies $a \in I$. Thus, for any subtractive ideal I of R, the factor semiring R/I and the surjective morphism $f : R \to R/I$, given by $r \mapsto r/I$, is well defined and its kernel is I. See also Example 9.1 and Proposition 9.10 in [4].

We end this section by recalling some more definitions from [4] and write them here for the sake of completeness as follows.

• A semiring with no nonzero zero divisors is called an *entire (= semidomain)*. A *semifield* is a semiring in which every nonzero element has a *multiplicative inverse*. A semiring R is *zerosumfree* if and only if $r + r' = 0$ implies that $r = r' = 0$. A semiring R is said to be *simple* if $1 + r = 1$ for each $r \in R$. Let R be a semiring and

$$G(R) = \{r \in R \mid 1 + r \in U(R)\}.$$

A semiring R is called a *Gelfand semiring* when $G(R) = R$. Clearly, every simple semiring is Gelfand. Of course, *bounded distributive lattices* are among Gelfand semirings. But the class of the Gelfand semirings is quite wider as Example 1.4 in [10] shows (cf. [4, Example 3.38]).

3 n-Stable and n_J-Stable Semirings

This section is mainly devoted to two parts, where the first part is related to n-stability condition of semirings in general; and the second part investigates the structure of semirings with stable range one. In this section, we will show that any *unimodular sequence* of a semiring R (Definition 3.1) that meets $J_k(R)$ is stable (Definition 3.1) provided that every *maximal ideal* of R is subtractive (Theorem 3.5). In our study of n-stable (Definition 3.2) [resp., n_J-stable (Definition 3.2)] semirings, for a fixed integer $n \geq 1$, we need only consider unimodular sequences [resp., unimodular J-sequences (Definition 3.3)] of size $n+1$ (Theorem 3.8 [resp. 3.9]). We will show that a semiring R is n-stable if and only if it is n_J-stable provided that every maximal ideal of R is subtractive (Theorem 3.11). It is shown that the n-stability of a semiring is preserved under the *homomorphism* of semirings (Theorem 3.12); and also shown that any *direct product* of n-stable semirings is n-stable if and only if each factor of the product is n-stable (Proposition 3.14). Finally, we conclude the section with some results and examples related to semirings with stable range one, namely *Gelfand semirings* [in particular, *simple semirings*] (Theorem 3.15, Corollary 3.16, Proposition 3.20). See also Examples 3.6, 3.18, 3.19, and 3.21 whose stable range of each is one.

Definition 3.1. Let R be a semiring (ring) and $s \geq 1$ an integer. A sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$ of elements of R is said to be *stable* if $(a_1, a_2, \ldots, a_s, a_{s+1}) = (a_1 + b_1 a_{s+1}, a_2 + b_2 a_{s+1}, \ldots, a_s + b_s a_{s+1})$ for some $b_1, b_2, \ldots, b_s \in R$. A sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$ of elements of R is said to be a *unimodular sequence* if 1 is in the ideal $(a_1, a_2, \ldots, a_s, a_{s+1})$.

Definition 3.2. Let R be a commutative semiring and $J_k(R)$ its Jacobson k-radical. A fixed integer $n \geq 1$ is said to be in the *stable range* of R (or R is n-stable for short) if every unimodular sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq n$, of elements of R is stable. The semiring R is said to be n_J*-stable* if every unimodular sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq n$, of elements of R with $(a_1, a_2, \ldots, a_{s-1}) \not\subseteq J_k(R)$ is stable.

Definition 3.3. For any integer $n \geq 1$, a (unimodular) sequence $(a_1, a_2, \ldots, a_n, a_{n+1})$ of a semiring R is said to be a (unimodular) J-sequence whenever $(a_1, a_2, \ldots, a_{n-1})$ is not contained in $J_k(R)$.

Remark 3.4. Equivalently, for any fixed integer $n \geq 1$, we can define an n-stable [resp., n_J-stable] semiring as follows: a semiring R is said to be n-stable [resp., n_J-stable] provided that any unimodular sequence [resp., unimodular J-sequence] of elements in R of size larger than n is stable. For convenience, an n-stable [resp., an n_J-stable] semiring is called stable [resp., 1_J-stable] whenever $n = 1$. It follows that if R is an n-stable [resp., an n_J-stable] semiring, then also R is m-stable [resp., m_J-stable] for any fixed integer $m \geq n$. Note that the term "R is n-stable" is used in [9] (for convenience) and is exactly the same as the statement "n is in the stable range of R", which is used by D. Estes and J. Ohm [3, page 345]. Also, as in [3], we use $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq 1$, to denote both a sequence and the ideal generated by the elements of the sequence; but the context will always make our meaning clear. Moreover, we follow [3] for the term "unimodular sequence" instead of "primitive vector" as used in [7]. For a detailed study of stable range in commutative rings; see [3], [7], [9], [12], and [5].

We now show the stability of a unimodular sequence of a semiring under some special conditions.

Theorem 3.5. *Let $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 1$, be a unimodular sequence in a semiring R such that $\{a_1, a_2, \ldots, a_n, a_{n+1}\} \cap J_k(R) \neq \emptyset$.*

(a) *Suppose that every maximal ideal of R is subtractive. Then $(a_1, a_2, \ldots, a_i + a_{n+1}, \ldots, a_n) = R$ whenever $a_i \in J_k(R)$ and $a_{n+1} \notin J_k(R)$; or $a_i \notin J_k(R)$ and $a_{n+1} \in J_k(R)$ for some $1 \leq i \leq n$.*

(b) *Suppose that R is a subtractive semiring (i.e., a semiring in which every ideal is subtractive). Then $(a_1, a_2, \ldots, a_i + a_{n+1}, \ldots, a_n) = R$ whenever $a_i \in J_k(R)$ and $a_{n+1} \notin J_k(R)$; or $a_i \notin J_k(R)$ and $a_{n+1} \in J_k(R)$ for some $1 \leq i \leq n$.*

(c) *Suppose that* $(a_1, a_2, \ldots, a_i + a_{n+1}, \ldots, a_n)$, $1 \leq i \leq n$, *is a subtractive ideal of* R. *Then* $(a_1, a_2, \ldots, a_i + a_{n+1}, \ldots, a_n) = R$ *whenever* $a_i \in J_k(R)$ *and* $a_{n+1} \notin J_k(R)$; *or* $a_i \notin J_k(R)$ *and* $a_{n+1} \in J_k(R)$ *for some* $1 \leq i \leq n$.

Proof. Parts (a) and (b) follows directly from the fact that every ideal in a semiring is contained in a maximal ideal [4, Proposition 5.47]. Part (c) follows from [13, Corollary 2.2] that states any subtractive ideal in a semiring is contained in a maximal subtractive ideal. Note that the argument regarding the proof of the above three cases will end up to a conclusion that R is contained in a maximal ideal of R, which is a contradiction. □

The following example provides a trivial instance of a class of semirings with stable range one.

Example 3.6. A semifield R is stable. That is, $1 \in (a_1, a_2, \ldots, a_n, a_{n+1}) = (a_1 + ba_{n+1}, a_2, \ldots, a_n, a_{n+1})$, where $n \geq 1$, $0 \neq a_i \in R$ (a unit) for all $1 \leq i \leq n+1$, and $b = 0$. Moreover, besides some trivial examples of semifields such as semifields of nonnegative reals and nonnegative rationals, see [4, Proposition 7.8] that states: If I is a subtractive maximal ideal of a commutative semiring R, then R/I is a semifield. Actually, by virtue of the following theorem (Theorem 3.8), in order to check that R is stable, it suffices to show that any unimodular sequence of size 2 in R is stable. That is, $1 \in (a_1, a_2) = (a_1 + ba_2)$ for some $b \in R$.

Example 3.7. Let $S = (a_1, a_2, \ldots, a_n, a_{n+1})$ be a unimodular sequence in the semiring R, where $n \geq 1$ and a_i is a unit in R for at least one $1 \leq i \leq n$. Then, similar to the above example, S is stable. That is, $S = (a_1, a_2, \ldots, a_i + ba_{n+1}, \ldots, a_n)$, where $b = 0$.

Theorem 3.8. *Let* $n \geq 1$ *be a fixed integer. A commutative semiring* R *is* n*-stable if and only if all unimodular sequences of size* $n+1$ *in* R *are stable.*

Proof. Proof by induction on n. Let $(a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}) = R$, with

$$1 = \sum_{i=1}^{n+2} a_i x_i = \sum_{i=1}^{n} a_i x_i + y,$$

where $y = a_{n+1}x_{n+1} + a_{n+2}x_{n+2}$, $a_i, x_i \in R$. Hence,

$$(a_1, a_2, \ldots, a_n, y) = R$$

and by the induction hypothesis we have

$$(a_1 + b_1 y, a_2 + b_2 y, \ldots, a_n + b_n y) = R$$

for appropriate $b_1, b_2, \ldots, b_n \in R$. Thus,

$$(a_1 + b_1 x_{n+2} a_{n+2}, \ldots, a_n + b_n x_{n+2} a_{n+2}, a_{n+1} + 0 a_{n+2}) = R$$

and the proof is complete. □

Theorem 3.9. *Let* $n \geq 1$ *be a fixed integer. A commutative semiring* R *is* n_J*-stable if and only if all unimodular* J*-sequences of size* $n+1$ *in* R *are stable.*

Proof. The proof by induction on n is similar to the proof of Theorem 3.8. Let $(a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}) = R$ be a J-sequence (Definition 3.3). That is,

$$(a_1, a_2, \ldots, a_{n-1}, a_n) \not\subseteq J_k(R).$$

Now, we just argue only for the case when a_n is not in $J_k(R)$ and $a_i \in J_k(R)$ for all $1 \leq i \leq n-1$. Clearly,

$$1 = \sum_{i=1}^{n+2} a_i x_i = \sum_{i=1}^{n} a_i x_i + y,$$

where $y = a_{n+1}x_{n+1} + a_{n+2}x_{n+2}$, $a_i, x_i \in R$. Hence,

$$1 \in (a_n, a_2, \ldots, a_1, y) = R$$

is a unimodular J-sequence of size $n+1$ and by the induction hypothesis we have

$$(a_n + b_n y, a_2 + b_2 y, \ldots, a_1 + b_1 y) = R$$

for appropriate $b_1, b_2, \ldots, b_n \in R$. Thus,

$$(a_1 + b_1 x_{n+2} a_{n+2}, \ldots, a_n + b_n x_{n+2} a_{n+2}, a_{n+1} + 0 a_{n+2}) = R$$

and the proof is complete. □

Remark 3.10. In view of the above two theorems, we need only consider unimodular sequences [resp., unimodular J-sequences] of size $n+1$ in our study of n-stable [resp., n_J-stable] semirings for a fixed integer $n \geq 1$.

Theorem 3.11. *Let $n \geq 1$ be a fixed integer and R a commutative semiring in which every maximal ideal is subtractive. Then R is n-stable if and only if R is n_J-stable.*

Proof. The necessary part is clear. Now suppose R is an n_J-stable semiring and $(a_1, a_2, \ldots, a_n, a_{n+1}) = R$. Clearly, $(a_1, a_2, \ldots, a_n, a_{n+1})$ is stable by hypothesis if $(a_1, a_2, \ldots, a_{n-1}) \not\subseteq J_k(R)$. For the case $(a_1, a_2, \ldots, a_{n-1}) \subseteq J_k(R)$, the result follows from Theorem 3.5(a). Thus R is n-stable by virtue of Theorem 3.8. □

We now consider the homomorphic image of n-stable and n_J-stable semirings.

Theorem 3.12. *Let $f : R \to S$ be a surjective morphism of semirings.*

(a) *If R is an n-stable semiring, then so is S.*

(b) *If R is an n_J-stable semiring, then so is S provided that $f(J_k(R)) \subseteq J_k(S)$.*

Proof. We write a proof for Part (b) and leave the other part to the reader. By virtue of Remark 3.10, it suffices to argue only for unimodular J-sequences of size $n+1$ ($n \geq 1$ a fixed integer). Here, without loss of generality, we just illustrate the proof for the case $n+1 = 3$. Suppose R is an n_J-stable semiring and let $1_S \in (x_1, x_2, x_3)$ with $x_1 \notin J_k(S)$, where $x_1, x_2, x_3 \in S$. Thus $f(1_R) = 1_S = \sum s_i x_i$ for some $s_i \in S$, where $i = 1, 2, 3$. Therefore,

$$f(1_R) = \sum_1^3 f(r_i) f(a_i) = \sum f(r_i a_i) = f(\sum r_i a_i)$$

for some r_i and a_i in R, where $f(r_i) = s_i$ and $f(a_i) = x_i$ and $i = 1, 2, 3$. Clearly $1_R \in (a_1, a_2, a_3)$ since $1_S \in f(a_1, a_2, a_3)$ implies $1_S = f(a)$ for some $a \in (a_1, a_2, a_3)$, which implies a is a unit in R by Proposition 2.4(d). Thus, by hypothesis, $1_R \in (a_1 + b_1 a_3, a_2 + b_2 a_3)$ for some $b_1, b_2 \in R$. Consequently $f(1_R) = 1_S \in (f(a_1) + f(b_1) f(a_3), f(a_2) + f(b_2) f(a_3)) = (x_1 + t_1 x_3, x_2 + t_2 x_3)$, where $t_i = f(b_i)$ for $(1 \leq i \leq 2)$. □

Corollary 3.13. *Let I be a proper ideal of a semiring R. Then R/I is an n-stable semiring when R is an n-stable semiring.*

Proof. Clearly $r \mapsto r/I$ defines a surjective morphism from R to R/I, where $r \in R$. Now the proof follows directly from Part (a) of the above theorem. See also Example 9.1 and Proposition 9.10 in [4]. □

Proposition 3.14. *The direct product of any family of semirings is an n-stable semiring if and only if each factor of the product is an n-stable semiring.*

Proof. Let $\{R_i \mid i \in I\}$ be a family of semirings and $R = \Pi_{i \in I} R_i$. The necessary part is an immediate consequence of Theorem 3.12(a), which states that the homomorphic image of an n-stable semiring is n-stable. To prove the sufficiency, let R_i be an n-stable semiring for each $i \in I$. By virtue of Remark 3.10, it suffices to argue only for unimodular sequences of size

$n+1$ ($n \geq 1$ a fixed integer). Here, without loss of generality, we just illustrate the proof for the case $n+1=3$. Suppose $R=(\{a_i\},\{b_i\},\{c_i\})$. Thus $1_R=\{1_i\} \in (\{a_i\},\{b_i\},\{c_i\})$, where 1_i is the identity element of R_i for each $i \in I$. Consequently $1_i \in (a_i,b_i,c_i)$ for each $i \in I$, which implies $1_i \in (a_i+r_ic_i, b_i+s_ic_i)$, where $r_i, s_i \in R_i$ for each $i \in I$. Therefore, $1_R=\{1_i\} \in (\{a_i\}+\{r_i\}\{c_i\}, \{b_i\}+\{s_i\}\{c_i\})$. □

• The rest of this section is devoted to the study of some semirings with stable range one.

Theorem 3.15. *Let R be a Gelfand semiring and let $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 1$, be a unimodular sequence of R. Then $1 \in (a_1, a_2, \ldots, a_n + ba_{n+1})$ for some $b \in R$ (i.e., one is in the stable range of R).*

Proof. Let $1 = r_1a_1 + r_2a_2 + \cdots + r_na_n + r_{n+1}a_{n+1}$ for appropriate $r_i \in R$, where $1 \leq i \leq n+1$. Thus,

$$1 + a_n = r_1a_1 + r_2a_2 + \cdots + (1+r_n)a_n + r_{n+1}a_{n+1}$$

implies

$$(1+a_n)(1+r_n)^{-1} \in (a_1, a_2, \ldots, a_n + r_{n+1}(1+r_n)^{-1}a_{n+1}).$$

Now the result follows since $(1+a_n)(1+r_n)^{-1}$ is a unit in R by hypothesis that R is a Gelfand semiring. □

Corollary 3.16. *The stable range of each of the following semirings is one.*

(a) *R is a Gelfand semiring.*

(b) *R is a simple semiring.*

(c) *R is a semiring in which every maximal ideal is strong.*

Proof. Part (a) follows from the above theorem and (b) follows from (a) since a simple semiring is Gelfand ([4, Example 3.36]). Part (c) follows from (a) since (by [4, Proposition 5.50]) a semiring R is Gelfand if and only if every maximal ideal of R is a strong ideal. □

Remark 3.17. Note that there is a direct (set-theoretic) argument for the proof of Part (c) of the above corollary without using [4, Proposition 5.50] and assuming that R is Gelfand (see Proposition 3.20 below).

We now provide two examples of semirings with stable range one as a consequence of the above corollary.

Example 3.18. (cf. [4, Example 5.5]). If X is a Hausdorff topological space, then the set R of all continuous bounded functions from X to $\mathbb{R}^+$ (nonnegative real numbers) is a commutative semiring, as noted in [4, Example 2.1], which is in fact a Gelfand semiring and so a stable semiring by Corollary 3.16.

Example 3.19. In [4, Proposition 5.21], it is shown that if R is a semiring, then: the set ideal(R) consisting of R and all ideals of R is a semiring under the operations of addition and multiplication of ideals. Clearly for each commutative semiring R, ideal(R) is a simple semiring since $R+A=R$ for each ideal A of R and so it is a stable semiring from Part (b) of Corollary 3.16.

Finally, by a set-theoretic technique, we show that a semiring in which every maximal ideal is strong, is stable (see also Theorem 3.5 and Corollary 3.16(c)) and end the section with an example of a semiring with stable range one (Example 3.21).

Proposition 3.20. *A semiring R is stable if each of its maximal ideals is strong.*

Proof. The proof is immediate since R is a Gelfand semiring by [4, Proposition 5.50] and the fact that each Gelfand semiring is stable by Corollary 3.16. But we now give a direct proof to it by a set-theoretic argument. Let $1 \in (a_1, a_2, \ldots, a_n, a_{n+1})$ with $n \geq 1$. We claim that $1 \in (a_1, a_2, \ldots, a_n + a_{n+1})$. Otherwise, by [4, Proposition 5.47], $(a_1, a_2, \ldots, a_n + a_{n+1})$ is contained in a maximal ideal M of R. thus $a_n, a_{n+1} \in M$ by hypothesis, which implies $R \subseteq M$, yielding a contradiction. □

Example 3.21. (cf. [4, Example 5.45]). Let X be a compact Hausdorff topological space and let R be the commutative semiring of all continuous functions from X to the semiring $\mathbb{R}^+$ of nonnegative reals. Then for each $x \in X$,

$$M_x = \{f \in R \mid f(x) = 0\}$$

is a maximal ideal of R and all maximal ideals of R are of this form. Actually, every maximal ideal of R is strong since $\mathbb{R}^+$ is a zerosumfree semiring. Thus R is a stable semiring by Proposition 3.20.

4 A Relative Stability Condition

In this section, we introduce and study some algebraic properties of the n-stable and n_J-stable semirings with respect to a nonempty subset S of R [= S-relative n-stable and n_J-stable semirings] (Definition 4.1). That is, we will study some of the results of previous section for S-relative n-stable and n_J-stable semirings.

Definition 4.1. Let S be a nonempty subset of a semiring R. R is said to be an n-stable [resp. n_J-stable] semiring with respect to S or R is an S-relative n-stable [resp. an S-relative n_J-stable] semiring if for any ideal $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 1$, of R and $a \in S$ [resp. with $(a_1, a_2, \ldots, a_{n-1}) \not\subseteq J_k(R)$] such that $1 + a \in (a_1, a_2, \ldots, a_n, a_{n+1})$, then there exist $b_1, b_2, \ldots, b_n \in R$ such that

$$1 + a \in (a_1 + b_1 a_{n+1}, a_2 + b_2 a_{n+1}, \ldots, a_{n-1} + b_{n-1} a_{n+1}, a_n + b_n a_{n+1}).$$

Remark 4.2. From the above definition, an n-stable [resp. $n_J$stable] -semiring is a $\{0\}$-relative (or simply, 0-relative) n-stable [resp. 0-relative) n_J-stable] semiring. Clearly, every n-stable [resp. n_J-stable] semiring with respect to a nonempty subset S of R is an n-stable [resp. n_J-stable] semiring provided $0 \in S$. Also, let $S \subseteq T$ be two nonempty subsets of a semiring R. Then R is an S-relative n-stable [resp. S-relative n_J-stable] semiring if R is a T-relative n-stable [resp. T-relative n_J-stable] semiring. Clearly, R is an n-stable [resp. n_J-stable] semiring if and only if R is a $G(R)$-relative n-stable [resp. $G(R)$-relative n_J-stable] semiring, where

$$G(R) = \{a \in R \mid 1 + a \in U(R)\}.$$

• From the above remark, it is clear that the class of S-relative n-stable and S-relative n_J-stable semirings are contained in the class of n-stable and n_J-stable semirings, respectively, provided that $0 \in S$.

Example 4.3. In Corollary 3.16, it is shown that a Gelfand semiring R [in particular, a simple semiring] is a stable semiring. Thus from the above remark, R is an R-relative stable semiring or equivalently an S-relative stable semiring for any nonempty subset S of R.

Next, similar to Theorems 3.8 and 3.9, we provide a criterion for the study of S-relative n-stable and S-relative n_J-stable semirings. For the definition of a (unimodular) J-sequence, see Definition 3.3.

Theorem 4.4. *Let $n \geq 1$ be a fixed integer and S be a nonempty subset of a semiring R. Then R is an S-relative n-stable [resp., an S-relative n_J-stable] semiring if and only if all sequences [resp., J-sequences] of size $n + 1$ of R are S-relative stable.*

Proof. The proof is similar to the proof of Theorem 3.8 [resp. Theorem 3.9] by replacing 1 with $1 + a$, where $a \in S$. □

Remark 4.5. In view of the above theorem, we need only consider the sequences [resp. J-sequences] of size $n + 1$ that satisfy n-stability [resp. n_J-stability] condition with respect to a nonempty subset S of R in our study of S-relative n-stable [resp. S-relative n_J-stable] semirings.

We conclude this section by extending Corollary 3.13 to an S-relative n-stable semiring.

Theorem 4.6. *Let I be a proper ideal of a semiring R and S a nonempty subset of R. Then R/I is an S/I-relative n-stable semiring when R is an $S+I$-relative n-stable semiring. Further, if $S \cap I \neq \emptyset$ [in particular, if $0 \in S$], then R/I is also an n-stable semiring.*

Proof. By virtue of Remark 4.5, it suffices to argue only for sequences of size $n+1$ ($n \geq 1$ a fixed integer) that satisfy n-stability condition with respect to $S+I$. Here, without loss of generality, we just illustrate the proof for the case $n+1=3$. Suppose $\frac{1}{I} + \frac{s}{I} \in (\frac{a_1}{I}, \frac{a_2}{I}, \frac{a_3}{I})$, where $s \in S$. Thus,

$$\frac{1+s}{I} = \sum \frac{r_i}{I}\frac{a_i}{I} = \sum \frac{r_i a_i}{I}$$

for some $r_i \in R$, where $i = 1, 2, 3$. Therefore,

$$1 + s + a = r_1 a_1 + r_2 a_2 + r_3 a_3 + a'$$

for some $a, a' \in I$ by definition. Thus $1 + s + a \in (a_1, a_2, r_3 a_3 + a')$. Now by hypothesis, there exist $b_1, b_2 \in R$ such that

$$1 + s + a \in (a_1 + b_1(r_3 a_3 + a'), a_2 + b_2(r_3 a_3 + a')).$$

Consequently,

$$\frac{1}{I} + \frac{s}{I} \in (\frac{a_1}{I} + \frac{b_1 r_3}{I}\frac{a_3}{I}, \frac{a_2}{I} + \frac{b_2 r_3}{I}\frac{a_3}{I})$$

and the proof of the first part is complete. The "further" part is clear since $\frac{a}{I} = \frac{0}{I}$ when $a \in I$ (see also Remarks 2.5 and 4.2). □

5 Some Special Cases

In this section we study some algebraic properties of a subclass of (S-relative) 2-stable semirings R, namely (S-relative) B-stable semirings (Definitions 5.1 and 5.2), where S is a nonempty subset of R. It is shown that $R[x]$, the semiring of polynomials over a semiring R, can not be a B-stable semiring with respect to $I[x]$ when I is a strong proper ideal of R (Theorem 5.14(a)); and $R[x]$ is not B-stable when R is a zerosumfree semiring (Theorem 5.14(b)). Finally, we end the paper with a discussion related to the B-stability condition of $R[x]$ (Example 5.19) and also construct a new semiring R by combining a semidomain D and a semifield F, $D \subseteq F$, and show that D is an n-stable, n_J-stable, or a B-stable semidomain when R is an n-stable, n_J-stable, or a B-stable semiring, respectively, (Proposition 5.20).

Definition 5.1. A commutative semiring R is said to be a *B-stable semiring* whenever for any unimodular sequence $(a_1, a_2, \ldots, a_n, a_{n+1})$ of elements in R, $n \geq 2$ an integer, there exists an element b in R such that $(a_1, a_2, \ldots, a_n + b a_{n+1}) = R$.

Definition 5.2. Let S be a nonempty subset of a semiring R and $n \geq 2$ an integer. The semiring R is said to be a B-stable semiring with respect to S; or R is an S-relative B-stable semiring if for any ideal $(a_1, a_2, \ldots, a_n, a_{n+1})$ of R and $a \in S$ such that $1 + a \in (a_1, a_2, \ldots, a_n, a_{n+1})$, then there exists $b \in R$ such that

$$1 + a \in (a_1, a_2, \ldots, a_{n-1}, a_n + b a_{n+1}).$$

Remark 5.3. In the two preceding definitions, it is clear that any (S-relative) B-stable semiring is a (S-relative) 2-stable semiring. In other words, the class of (S-relative) B-stable semirings is a subclass of the (S-relative) 2-stable semirings.

Example 5.4. A semifield R is a B-stable semiring. That is, for any integer $n \geq 2$, $1 \in (a_1, a_2, \ldots, a_n, a_{n+1}) = (a_1, a_2, \ldots, a_n + b a_{n+1})$, where $b = 0$ and $0 \neq a_i$ is a unit in R for each $1 \leq i \leq n+1$. Moreover, besides some trivial examples of semifields such as semifields of nonnegative reals and nonnegative rationals, see [4, Proposition 7.8] that states: If I is a subtractive maximal ideal of a commutative semiring R, then R/I is a semifield.

The following theorem shows that a Gelfand semiring is B-stable.

Theorem 5.5. *Let R be a Gelfand semiring [in particular, a simple semiring] and let $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 2$, be a unimodular sequence of R. Then $1 \in (a_1, a_2, \ldots, a_n + ba_{n+1})$ for some $b \in R$ (i.e., R is a B-stable semiring).*

Proof. The proof is similar to the proof of Theorem 3.15. □

Theorem 5.6. *A commutative semiring R is a B-stable semiring if and only if for any unimodular sequence (a_1, a_2, a_3) of R, there exists an element $b \in R$ such that $(a_1, a_2 + ba_3) = R$.*

Proof. The proof is essentially similar to the proof of Theorem 3.8. The necessary part is quite clear. To prove the sufficient part, let $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 2$, be a unimodular sequence in R. Now, $1 \in (a_1, a_2, \ldots, a_n, a_{n+1})$ implies $1 = \sum_{i=1}^{n+1} a_i x_i$ for some $x_1, x_2, \ldots, x_n, x_{n+1} \in R$. Thus,

$$1 \in (a_1, a_n, l),$$

where $l = a_2x_2 + a_3x_3 + \cdots + a_{n-1}x_{n-1} + a_{n+1}x_{n+1}$. Now by the hypothesis, there exists $b \in R$ such that

$$1 \in (a_1, a_n + bl) \subseteq (a_1, a_2, \ldots, a_{n-1}, a_n + bx_{n+1}a_{n+1}).$$

□

Remark 5.7. In view of the preceding theorem, we need only consider the unimodular triples instead of arbitrary unimodular sequences of size $n + 1 \geq 4$ in our study of B-stable semirings.

We now (similar to Theorem 5.6) provide a criterion for the study of S-relative B-stable semirings.

Theorem 5.8. *Let S be a nonempty subset of a semiring R. A semiring R is an S-relative B-stable semiring if and only if for every $a \in S$ and $c_1, c_2, c_3 \in R$ with $1 + a \in (c_1, c_2, c_3)$, it follows that*

$$1 + a \in (c_1, c_2 + bc_3)$$

for some $b \in R$.

Proof. The proof is similar to the proof of Theorem 5.6 by replacing 1 with $1 + a$. □

Remark 5.9. In view of the above theorem, we need only consider the sequences of size three that satisfy B-stability condition with respect to a nonempty subset S of R in our study of S-relative B-stable semirings.

Theorem 5.10. *Let $f : R \to S$ be a surjective morphism of semirings. If R is a B-stable semiring, then so is S.*

Proof. By virtue of Theorem 5.6, it suffices to argue only for unimodular sequences of size three. The proof is similar to the proof of Theorem 3.12 and we leave it to the reader. □

Corollary 5.11. *Let I be a proper ideal of a semiring R. Then R/I is a B-stable semiring when R is a B-stable semiring.*

Proof. Clearly $r \mapsto r/I$ defines a surjective morphism from R to R/I, where $r \in R$. Now the proof follows directly from the above theorem. See also Example 9.1 and Proposition 9.10 in [4]. □

We now extend the above corollary to an S-relative B-stable semiring.

Theorem 5.12. *Let I be a proper ideal of a semiring R and S a nonempty subset of R. Then R/I is an S/I-relative B-stable semiring when R is an $S + I$-relative B-stable semiring. Further, if $S \cap I \neq \emptyset$ [in particular, if $0 \in S$], then R/I is also a B-stable semiring.*

Proof. The proof is similar to the proof of Theorem 4.6 and by virtue of Theorem 5.8, it suffices to argue only for sequences of size three that satisfy B-stability condition with respect to $S + I$. □

Proposition 5.13. *The direct product of any family of semirings is a B-stable semiring if and only if each factor of the product is a B-stable semiring.*

Proof. The proof is similar to the proof of Proposition 3.14. □

In the following theorem, we partially characterize the B-stability condition of $R[x]$ (the semiring of polynomials over a semiring R).

Theorem 5.14. *Let $R[x]$ be the semiring of polynomials over a commutative semiring R.*

(a) *Let I be a proper ideal of R. If $R[x]$ is a B-stable semiring with respect to the ideal $I[x]$ of $R[x]$, then I is not a strong ideal of R. In other words, if I is a strong proper ideal of R, then $R[x]$ can not be a B-stable semiring with respect to the ideal $I[x]$ in $R[x]$.*

(b) *If $R[x]$ is a B-stable semiring, then R is not a zerosumfree semiring.*

Proof. (a): Suppose to the contrary that I is a strong proper ideal of R. Let $a \in I$ and $1 + ax \in (x^2, x, 1 + ax)$. If $R[x]$ is a B-semiring with respect to $I[x]$, then $1 + ax \in (x^2, x + b(x)(1 + ax))$ for some $b(x) \in R[x]$ by definition. Let $1 + ax = x^2 f(x) + (x + b(x)(1 + ax))g(x)$, where $f(x), g(x) \in R[x]$. Let f_i, g_i, and b_i represent the coefficient of x^i in the polynomials $f(x)$, $g(x)$, and $b(x)$, respectively. Now by equating the corresponding coefficients in the above equation, we get $(x + b(x) + axb(x))g(x) = (x + b_0 + b_1x + \cdots + ab_0x + \cdots)g(x)$, which implies $g_0x + b_0g_0 + b_1g_0x + ab_0g_0x + b_0g_1x$ and so $1 = g_0b_0$ and $a = g_0 + b_1g_0 + ab_0g_0 + b_0g_1$. Thus if I is strong in R, then $g_0 \in I$ since $a \in I$ by the assumption, which implies $1 \in I$ and leads to a contradiction.

(b): The proof follows directly from Part (a) by setting $I = \{0\}$ and using the fact that $\{0\}$ is a strong ideal of R if and only if R is a zerosumfree semiring. □

Example 5.15. Let R be the semiring of nonnegative reals, or nonnegative rationals, or nonnegative integers, respectively, with usual addition and multiplication. Clearly, R is a commutative, zerosumfree semiring which is not additively idempotent and by Theorem 5.14(b), $R[x]$ is not a B-stable semiring. For more examples of zerosumfree semirings, see [4].

Corollary 5.16. *If R is an additively idempotent semiring [in particular, a simple semiring], then $R[x]$ is not a B-stable semiring.*

Proof. The proof is immediate from Part (b) of the above theorem since every additively idempotent semiring [in particular, a simple semiring], which is not a ring, is zerosumfree. Note that $a + b = 0$ implies $a = a + a + b = a + b + b = b$ in any additively idempotent semiring and also a simple semiring is additively idempotent since $1 + 1 = 1$. □

Finally, we end the paper with some examples related to the B-stability condition of $R[x]$, the semiring of polynomials over a semiring R, and also construct a new semiring R by combining a semidomain D and a semifield F to study some of its stability properties in general (Proposition 5.20).

Example 5.17. Let R be a semiring (ring) and $S = \text{ideal}(R)$ be the semiring of ideals of R under the addition and multiplication of the ideals of R. Then by the above Corollary, $S[x]$ is not a B-semiring since S is a simple semiring (i.e., $R + A = R$ for any ideal A of R).

Example 5.18. (cf. [4, Example 5.1]) If A is an infinite set, then the family $\text{fsub}(A)$ of all finite subsets of A is a strong ideal of the semiring $(\text{sub(A)}, \cup, \cap)$. Thus, by Theorem 5.14(a), $R[x]$ is not a B-semiring with respect to $I[x]$ when $I = \text{fsub}(A)$ and $R = (\text{sub(A)}, \cup, \cap)$.

Example 5.19. Clearly, from Theorem 5.10, if $R[x]$ (the semiring of polynomials over a semiring R) is a B-stable semiring, then so is R under the morphism $\phi : R[x] \to R$ given by $a_0 + a_1x + \cdots + a_nx^n \mapsto a_0$, where $a_i \in R$ for each $0 \leq i \leq n$. See also Theorem 5.14 that shows $R[x]$

is not a B-stable semiring when R is zerosumfree. Further, since a simple semiring R is a B-stable semiring (Theorem 5.5), the converse of this example need not be true in general since by Corollary 5.16, $R[x]$ is not a B-stable semiring when R is a simple semiring. We also, by a trivial example of a B-stable semiring, show that the converse of this example need not be true in general. Let R be the semiring of nonnegative reals, which is a B-stable semiring by Example 5.4. Now, by Theorem 5.14, $R[x]$ is not a B-stable semiring since R is zerosumfree.

We now conclude the paper by a discussion related to the stability condition of a special semiring in general.

Proposition 5.20. (cf. [7, Theorem 2.6] and [11, Theorem 5]) *Let D be a semidomain and F a semifield containing D. Let $R = \{(a_1, \ldots, a_k, a, a, \ldots) \mid a_i \in F, a \in D\}$, where k is a nonnegative integer (k may be different for distinct elements of R). The operations in R are componentwise addition and multiplication. Let $\phi : R \to D$ be a map given by $(a_1, a_2, \ldots, a_k, a, a, \ldots) \mapsto a$. Then:*

(a) *If R is an n-stable semiring, then D is an n-stable semidomain.*

(b) *If R is an n_J-stable semiring, then D is an n_J-stable semidomain provided $\phi(J_k(R)) \subseteq J_k(D)$.*

(c) *If R is a B-stable semiring, then D is a B-stable semidomain.*

Proof. Clearly $1_F = dd^{-1}$ for each $0 \neq d \in D$ and hence $1_D = 1_F 1_D = (dd^{-1})1_D = d^{-1}(d1_D) = d^{-1}d = 1_F$. So $1_R = (1, 1, 1, \ldots)$. It can be easily seen that D is a homomorphic image of R under the map given by $(a_1, a_2, \ldots, a_k, a, a, \ldots) \mapsto a$. Now the proof of Parts (a) and (b) [resp., Part (c)] follows directly from Theorem 3.12 [resp., Theorem 5.10]. □

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Received: 2025-10-06
Accepted: 2026-07-01