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Existence of Solutions for a $(p,q)-$triharmonic Kirchhoff Problem with Singular Potential

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Abstract We study the existence of a weak solution for the elliptic $(p,q)-$Kirchhoff system with Navier boundary conditions:

$$\begin{cases} -M(t_p)\Delta_p^3 u = \lambda F_u(x,u,v), & \text{in } \Omega, \\ -M(t_q)\Delta_q^3 v = \lambda F_v(x,u,v), & \text{in } \Omega, \\ \Delta^2 u = \Delta^2 v = \Delta u = \Delta v = u = v = 0, & \text{on } \partial\Omega. \end{cases}$$

By employing variational methods and compact embedding theorems in weighted Sobolev spaces. We prove the existence of a nontrivial weak solution.

1 Introduction

In mathematics, a partial differential equation is a type of differential equation, or a relationship involving one or more unknown functions with multiple independent variables, along with the partial derivatives of those variables.

Partial differential equations are used to formulate and solve problems involving functions with multiple variables, such as those in acoustics, heat transfer, electrostatics, fluid dynamics, elasticity, and other fields of study. Different physical phenomena can be expressed using mathematical equations of similar form. These kind of problems appears in the mathematical modeling of various physical phenomena such as the study of image restotation as well as in some model of electrolrheological fluids. Moreover, an other important application is the image processing where nonlinearity of the diffusion operator and convection terms are used to underline the borders og the distored image and to eliminate the noise, for more detail we can refer to [17, 29, 30].

The problem that we propose in this article is basically related to the following Kirchhoff equation:

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

presented by Kirchhoff in 1883 [19]. The parameters in (1.1) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In particular, in [1], the authors studied the following $p-$Laplacian elliptic problem with Neumann boundary condition:

$$\begin{cases} -\Delta_{p_i} u_i + a_i(x)|u_i|^{p_i-2}u_i = \lambda F_{u_i}(x,u_1,\dots,u_n) + \mu G_{u_i}(x,u_1,\dots,u_n) & \text{in } \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

For $1 \leq i \leq n$, where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a non-empty bounded open set with a smooth boundary $\partial\Omega$, $\Delta_{p_i} u_i := \text{div}\left(|\nabla u_i|^{p_i-2}\nabla u_i\right)$ is the p_i-Laplacian operator, $p_i > N$, $a_i \in L^\infty(\Omega)$ with $\text{ess}\inf_\Omega a_i > 0$ for $1 \leq i \leq n$, and λ and μ are positive.

Also in [6], G. Bonanno and P. Candito using Ricceri's three critical points theorem [26], established the existence of an interval $\Lambda \subseteq]0, +\infty[$ and a positive real number q such that for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a non-empty bounded open set with a boundary $\partial\Omega$ of class C^1, $a \in L^\infty(\Omega)$ with $\text{ess}\inf_\Omega a > 0$, $p > N$, $\lambda > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function, possesses at least three weak solutions whose norms in $W^{1,p}(\Omega)$ are less than q.
In [7], Bousgheiri et al investigate the existence of weak solutions of the following Dirichlet $(p_1, ..., p_N)-$polyharmonic Kirchhoff problem.

$$\begin{cases} M_i\left(\int_\Omega \frac{1}{p_i}|D_L u_i|^{p_i} dx\right)\Delta^L_{p_i} u_i = \lambda F_{u_i}(x, u_1, u_2, ..., u_N) + \mu G_{u_i}(x, u_1, u_2, ..., u_N), \text{ in } \Omega, \\ D^\alpha u_i = 0, \text{ on } \partial\Omega, \end{cases}$$

for each mult-index α, with $|\alpha| \leq L-1$, $L \in \mathbb{N}^*$, $1 \leq i \leq N$ and $N \in \mathbb{N}^*$. By applying a recent three-critical-point theorem obtained by B. Ricceri, they establish the existence of at least three weak solutions.

Jaafri in [18] studies bi-nonlocal elliptic problem involving $p(x)-$Biharmonic operator. By applying variational methods and under adequate conditions, he proves the existence of nontrivial weak solutions to the following problem:

$$\begin{aligned} M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)} dx\Big)\Delta^2_{p(x)} u = \Big[\int_\Omega F(x,u)dx\Big]^r \quad \text{on } \Omega, \\ \Delta u = u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

Where, $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is bounded smooth domain, $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous functions, $F(x,u) = \int_0^u f(x,s)ds$, $\lambda \in \mathbb{R}$,$r > 0$.

In [12], Chung investigates the existence of solutions to the following nonlocal problem:

$$\begin{cases} -M\left(\int_\Omega |x|^{-ap}|\nabla u|^p\, dx\right) \text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = |x|^{-p(a+1)+c} f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 3$, and the function M is continuous and may vanish at zero.

Mazan in [21] investigates the existence of a weak solution to a class of nonlocal problems involving the variable exponent and concave-convex nonlinearities to the following problem:

$$\begin{aligned} M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx\Big)(-\Delta_{p(x)} u) = f_\lambda(x,u) \quad \text{on } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

Where, $f_\lambda = \lambda(m_q(x)|u|^{q(x)-2}u + m_h(x)|u|^{h(x)-2}u)$, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is bounded smooth domain, p is Lipschits continuous on $\overline{\Omega}$, $m_q, m_h \in C_+(\overline{\Omega})$, $m_q, m_h > 0$.

Here we study a version of the Kirchhoff-type problem, with Navier boundary condition,

where $u = u(x)$ is the lateral displacement in coordinate space and M is typically a line with positive slope, and we aim to investigate the existence of a weak solution:

$$\begin{cases} -M(t_p)\,\Delta_p^3 u = \lambda\, F_u(x,u,v), & \text{in } \Omega, \\ -M(t_q)\,\Delta_q^3 v = \lambda\, F_v(x,u,v), & \text{in } \Omega, \\ \Delta^2 u = \Delta^2 v = \Delta u = \Delta v = u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Where, $\Delta_p^3 u := \operatorname{div}\left(|\nabla \Delta u|^{p-2}\nabla \Delta u\right)$ denotes the p-triharmonic operator,

$$t_p = \int_\Omega \frac{1}{p}\, a_p(x)\, |\nabla \Delta u|^p\, dx, \quad t_q = \int_\Omega \frac{1}{q}\, a_q(x)\, |\nabla \Delta v|^q\, dx,$$

The weight functions are given by

$$a_p(x) = |x|^{-ap}, \quad a_q(x) = |x|^{-aq},$$

where, $0 \le a < \frac{N-3p}{p}$ is a fixed constant. Throughout, $\Omega \subset \mathbb{R}^N$ denotes a bounded domain of class C^1, with $N > 3$, and $\lambda \ge 0$ is a non-negative parameter.

Let $F : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ be a function such that, for every $(t_1, t_2) \in \mathbb{R}^2$, the mapping $x \mapsto F(x, t_1, t_2)$ is continuous on $\overline{\Omega}$, and for each $x \in \Omega$, the mapping $(s,t) \mapsto F(x, t_1, t_2)$ is of class C^1 on $\mathbb{R}^2$. Furthermore, for every $k > 0$, we assume

$$\sup_{\substack{|t_1|\le k \\ |t_2|\le k}} \left(|F_{t_1}(x,t_1,t_2)| + |F_{t_2}(x,t_1,t_2)|\right) \in L^1(\Omega),$$

Here F_u and F_v denote the partial derivatives of F regarding the second and third variables, respectively.

This category of systems can be regarded as a natural extension of classical Kirchhoff-type issues, integrating nonlocal elements that reflect physical processes in which diffusion or internal stress relies on the overall system state. For example, Kirchhoff's seminal model for elastic strings [19] and its modern adaptations [9, 24] demonstrate these behaviors. These nonlocal characteristics are especially pertinent in areas such as elasticity theory, population dynamics, and the modeling of electrorheological fluids [29]. Moreover, the incorporation of the p-biharmonic operator facilitates the representation of fourth-order anisotropic effects, which are crucial for examining thin structures, layered composites, and other heterogeneous materials [17, 25]. The nonlinear interactions introduced through the functions F further add realistic connections between different physical components, enhancing the mathematical framework of the problem and presenting complex challenges from the viewpoints of variational methods and critical point theory [30, 31, 32].

The paper is organized as follows. The second paragraph establishes the notational framework and reviews key lemmas concerning Lebesgue and Sobolev spaces. It also summarizes the abstract critical point Theorem 3.2 and defines a weak solution for the problem (1.5). Section 3 proves the existence of at least one weak solution to problem (1.5).

2 Preliminary notes

In order to deal with the problem, We need some theory of Sobolev Space. For convenience, We only recall some basic facts which will be used later, we refer to the book of Musielak [22] and [4]. Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. Let

$$p_k^* = \frac{Np}{N-kp} \text{ if } kp < N \qquad \text{and } p_k^* = +\infty \quad \text{if} \quad kp \ge N.$$

Define the Lebesgue space $L^p(\Omega)$,

$$L^p(\Omega) = \{\, u : \Omega \longrightarrow \mathbb{R} \quad \text{mesurable} \quad : \int_\Omega |u|^p dx < \infty \} \,.$$

Then $L^p(\Omega)$ endowed with the norm :

$$|u|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}},$$

becomes a Banach separable and reflexive space.
Define the variable exponent Sobolev space $W^{k,p}(\Omega)$:

$$W^{k,p}(\Omega) = \{\, u \in L^p(\Omega) : \;\; D^\alpha u \in L^p(\Omega), |\alpha| \leq k\} \,,$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 ..... \partial^{\alpha_N} x_N}$, with $\alpha = (\alpha_1, ...; \alpha_N)$ is a multi-index, and $|\alpha| = \sum_{i=1}^N \alpha_i$.

Then $W^{k,p}(\Omega)$ endowed with the norm :

$$\|u\|_p = \sum_{|\alpha| \leq k} |D^\alpha u|_p,$$

becomes a Banach separable and reflexive space.

Proposition 2.1. *([15])*
For any $u \in L^p(\Omega), v \in L^q(\Omega)$, we have

$$\int_\Omega uv dx \leq \|u\|_p \|v\|_q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

We denote by X the space

$$X = X_p \times X_q,$$

where, $X_p = W^{3,p}(\Omega, |x|^{-ap}) \cap W_0^{1,p}(\Omega, |x|^{-ap})$, $X_q = W^{3,q}(\Omega, |x|^{-aq}) \cap W_0^{1,q}(\Omega, |x|^{-aq})$ and define the norm in X, by the formula

$$\|(u,v)\| = \|u\|_p + \|v\|_q,$$

where, $\|u\|_p = \Big(\int_\Omega [|u|^p + |\nabla u|^p + |\nabla \Delta u|^p] |x|^{-ap} dx \Big)^{\frac{1}{p}}$.

Then X_p Equipped with this norm $\|.\|_p$ is a reflexive and separable Banach space.

The following embedding result is in fact, a generalization of the classical Rellich–Kondrachov compactness theorem.

Lemma 2.2. *(Compactness embedding theorem) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain containing the origin. Assume that the variable exponents $1 < p, s < +\infty$ satisfy:*

i) $1 < 3p < N$,

ii) $1 \leq s < p_3^* := \frac{Np}{N-3p}$,

iii) $0 < \alpha < (1+a)l + N(1 - \frac{l}{p})$.

Then the embedding

$$W^{3,p}(\Omega, |x|^{-ap}) \hookrightarrow L^s(\Omega, |x|^{-\alpha}),$$

is continuous and compact. Where, $L^s(\Omega, |x|^{-\alpha})$ is the weighted $L^s(\Omega)$ space with the norm:

$$\|u\|_{L^s(\Omega, |x|^{-\alpha})} = \Big(\int_\Omega |x|^{-\alpha} |u|^s dx \Big)^{\frac{1}{s}}.$$

Proof. Firstly, we have that under conditions $i)$, $ii)$ and $iii)$ the embedding $W^{1,p}(\Omega, |x|^{ap})$ in $L^l(\Omega, |x|^{-\alpha})$ is continuous because of the Hardy inequality:

$$\Big(\int_{\mathbb{R}^N} |x|^{-\alpha}|u|^l dx\Big)^{\frac{p}{l}} \leq C \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p.$$

In addition is a compact embedding see [28]. However, to prove our compact embedding, it is sufficient to show that the embedding $W^{3,p}(\Omega, |x|^{ap}) \hookrightarrow W^{1,p}(\Omega, |x|^{ap})$ is continuous, which is true because of the inequality

$$\|u\|_{W^{1,p}(\Omega,|x|^{ap})} \leq \|u\|_{W^{3,p}(\Omega,|x|^{ap})},$$

where, $\|u\|_{W^{1,p}(\Omega,|x|^{ap})} = \Big(\int_\Omega |x|^{-ap}|\nabla u|^p dx\Big)^{\frac{1}{p}}$. □

3 Main results.

Next, we will review the definition of the weak solution to the problem (1.5), as well as our main result.

Definition 3.1. A pair $(u, v) \in X$ is said to be a weak solution of the problem (1.5), if :

$$\langle A'(u), \varphi\rangle + \langle A'(v), \psi\rangle \quad = \quad \lambda \int_\Omega \Big[F_u(x, u, v)\varphi - F_v(x, u, v)\psi\Big] dx, \tag{3.1}$$

for all $(\varphi, \psi) \in X$, where, $A(u) = \tilde{M}\Big(\int_\Omega \frac{1}{p}\Big[a_p(x)|\nabla \Delta u|^p\Big] dx\Big)$.

Let define the following operators. For fixed $\lambda > 0$, let us define ϕ, Ψ, $J : X \to \mathbb{R}$ by

$$\phi(u, v) = \tilde{M}\Big(\int_\Omega \frac{1}{p}\Big[a_p(x)|\nabla \Delta u|^p\Big] dx\Big) + \tilde{M}\Big(\int_\Omega \frac{1}{q}\Big[a_q(x)|\nabla \Delta v|^q\Big] dx\Big), \tag{3.2}$$

$$\Psi(u, v) = -\int_\Omega F(x, u, v) dx, \tag{3.3}$$

where $\tilde{M}(t) = \int_0^t M(s)ds$.

$$J(u, v) = \phi(u, v) + \lambda\Psi(u, v),$$

Hereafter, we state the assumptions on $M(t)$ and $F(x, t_1, t_2)$:

(M_0) $M(t) : \mathbb{R} \longrightarrow (m_0, +\infty)$ is a continuous and increasing function, with $m_0 > 0$.

(M_1) There exist $m_2 \geq m_1 > 0$ and $\theta > 1$ such that,

$$m_1 t^{\theta-1} \leq M(t) \leq m_2 t^{\theta-1}, \quad \forall t \in \mathbb{R}^+.$$

Regarding the function F, we assume that there exist constants $c > 0$, $\tau > 0$, η, and β with $1 < \eta < p$, $1 < \beta < q$ such that

(F_0) $F(x, t_1, t_2) > 0$ for a.e. $x \in \Omega$ and all $(s, t) \in [0, \tau]^2$,

(F_1) There exist p_1, q_1 with $1 < p_1 < +\infty$ and $1 < q_1 < +\infty$ such that, $\max\{\theta p, \theta q\} < \min\{p_1, q_1\}$

$$\limsup_{(t_1,t_2)\longrightarrow(0,0)} \sup_{x\in\Omega} \frac{F(x, t_1, t_2)}{|t_1|^{p_1} + |t_2|^{q_1}} < +\infty.$$

(F_2) $|F_{t_i}(x, t_1, t_2)| \leq c_1\Big(1 + |t_1|^\eta + |t_2|^\beta\Big)$ for a.e. $x \in \Omega$ and $(t_1, t_2) \in \mathbb{R}^2$.

(F_3) There exists a positive constant $\mu > M^* = \max\{\theta p, \theta q, \frac{m_2\theta(pq)^\theta}{m_1(p^\theta+q^\theta)\min\{p^\theta,q^\theta\}}\}$ such that

$$0 < \mu F(x, t_1, t_2)t_i \leq F_{t_i}(x, t_1, t_2),$$

$i = 1, 2$, for all $x \in \Omega$ and $|(t_1, t_2)| \geq T > 0$.

In the following, when there is no misunderstanding, we always use c_i to denote positive constants.

The primary objective is to determine the critical points of the functional ϕ. To accomplish this, we employ the following Mountain Pass-type and critical point theorem [3].

Theorem 3.2. *Let $\Phi \in C^1(X,\mathbb{R})$ be such that Φ satisfies condition (PS). If there existr > 0 and $e \in X$ such that $0 < r < \|e\|$ and $\max\{\Phi(0),\Phi(e)\} < \inf_{\|u\|=r}\Phi(u)$, then*

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}\Phi(\gamma(t))$$

is a critical value of Φ.

Where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

By applying Theorem 3.2 to the functional ϕ associated with problem (1.5), and under appropriate structural conditions, we establish the existence of a nontrivial weak solution, as stated in our main following result.

Theorem 3.3. *Suppose that (M_1), $(F_0)-(F_3)$ are satisfied and that $\theta p > 1$, $\theta q > 1$. Then problem (1.5) has a nontrivial weak solution.*

4 proof of the main result

Lemma 4.1. *Assume that (M_1), $(F_0)-(F_3)$ are satisfied. Then the functional J satisfies the Palais-Smale condition in X.*

Proof. Let $\{u_m\} = \{u_m, u_m\} \subset X$ be a sequence such that

$$J(u_m) \to c < \infty, \quad J'(u_m) \to 0 \quad \text{in } X^* \text{ as } m \to \infty, \tag{4.1}$$

where X^* is the dual space of X.

We shall show that the sequence $\{u_m\}$ is bounded in X. Indeed, from (4.1), (M_1) and (F_2), for all m large enough, we have

$$\begin{aligned}
1+\bar{c}+\|u_m\|_{\alpha,p} &\geq J(u_m) - \frac{1}{\mu}J'\langle u_m,u_m\rangle \\
&= \widetilde{M}(t_p)+\widetilde{M}(t_q)-\frac{1}{\mu}M(t_p)\int_\Omega |x|^{-ap}|\nabla\Delta u_m|^p\,dx \\
&\quad -\frac{1}{\mu}M(t_q)\int_\Omega |x|^{-aq}|\nabla\Delta u_m|^q\,dx - \lambda\int_\Omega F(x,u_m,u_m)dx \\
&\quad +\frac{\lambda}{\mu}\int_\Omega F_{u_m}(x,u_m,u_m)\,dx \\
&\geq \frac{m_1}{\theta p^\theta}\left(\int_\Omega |x|^{-ap}|\nabla\Delta u_m|^p\,dx\right)^\theta + \frac{m_1}{\theta q^\theta}\left(\int_\Omega |x|^{-aq}|\nabla\Delta u_m|^q\,dx\right)^\theta \\
&\quad -\frac{m_2}{\mu p^\theta}\left(\int_\Omega |x|^{-ap}|\nabla\Delta u_m|^p\,dx\right)^\theta - \frac{m_2}{\mu q^\theta}\left(\int_\Omega |x|^{-aq}|\nabla\Delta u_m|^q\,dx\right)^\theta \\
&\quad \frac{\lambda}{\mu}\int_\Omega \left(F_{u_m}(x,u_m,u_m)u_m - \mu F(x,u_m,u_m)\right)\,dx \\
&\geq m\Big[\Big(\int_\Omega |x|^{-ap}|\nabla\Delta u_m|^p dx\Big)^\theta + \Big(\int_\Omega |x|^{-aq}|\nabla\Delta u_m|^q dx\Big)^\theta\Big] \\
&\quad -\frac{m_2}{\mu\min\{p^\theta,q^\theta\}}\Big[\Big(\int_\Omega |x|^{-ap}|\nabla\Delta u_m|^p dx\Big)^\theta + \Big(\int_\Omega |x|^{-aq}|\nabla\Delta u_m|^q dx\Big)^\theta\Big] \\
&\quad +\frac{\lambda}{\mu}\int_\Omega \left(F_{u_m}(x,u_m,u_m)u_m - \mu F(x,u_m,u_m)\right)\,dx \\
&\geq \left(m - \frac{m_2}{\mu\min\{p^\theta,q^\theta\}}\right)\left(\|u_m\|_{X_p}^{\theta p} + \|u_m\|_{X_q}^{\theta q}\right).
\end{aligned} \tag{4.2}$$

Where, $m = \frac{m_1}{\theta}(\frac{1}{p^\theta} + \frac{1}{q^\theta})$. Since $\theta p > 1$, $\theta q > 1$ and $\mu > M^*$, from (4.2) it follows that $\{ u_m\}$ is bounded.

Passing to a subsequence if necessary, there exists $u \in X$, such that $\{u_m\}$ converges weakly to u in X. By (4.1), we obtain the following:

$$\lim_{m\to\infty} J'(u_m, u_m)(u_m - u) = 0. \tag{4.3}$$

By (F_2), Lemma 2.2 and Proposition 2.1, we set $v_m = u_m - u$. Hence,

$$\begin{aligned}\left|\int_\Omega F_{u_m}(x, u_m, u_m)(v_m)\,dx\right| &\leq \int_\Omega |F_{u_m}(x, u_m, u_m)||v_m|\,dx \\ &\leq C_1\int_\Omega (1 + |u_m|^\eta + |u_m|^\beta |v_m|)\,dx \\ &\leq C(\|v_m\|_{L^{\eta'}(\Omega)} + \|v_m\|_{L^{\beta'}(\Omega)}) \\ &\quad + c(\|u_m\|_\eta^\eta \|v_m\|_{L^{\eta'}(\Omega)} + \|u_m\|_\beta^\beta \|v_m\|_{L^{\beta'}(\Omega)}),\end{aligned} \tag{4.4}$$

which tends to 0 as $m \to \infty$. Where, $\frac{1}{\eta} + \frac{1}{\eta'} = 1 = \frac{1}{\beta} + \frac{1}{\beta'}$.

By (4.3), (4.4) and the definition of the functional J. It follows that

$$\begin{aligned}B(v_m) &:= M(t_p)\int_\Omega |x|^{-ap}|\nabla\Delta u_m|^{p-2}\nabla\Delta u_m \cdot \nabla\Delta(v_m)\,dx \\ &\quad + M(t_q)\int_\Omega |x|^{-aq}|\nabla\Delta u_m|^{q-2}\nabla\Delta u_m \cdot \nabla\Delta(v_m)\,dx \longrightarrow 0.\end{aligned} \tag{4.5}$$

Since $\{u_m\}$ is bounded in X, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega |x|^{-ap}|\nabla\Delta u_m|^p\,dx \to t_p \geq 0, \quad \int_\Omega |x|^{-aq}|\nabla\Delta u_m|^q\,dx \to t_q \geq 0 \text{ as } m \to \infty.$$

If $t_p = 0$ then $\{u_m\}$ converges strongly to $u = 0$ in X and the proof is finished. If $t_p > 0$ then by (M_1) and the continuity of M, we obtain

$$M\left(\int_\Omega |x|^{-ap}|\nabla\Delta u_m|^p\,dx\right) \to M(t_p) > 0 \quad \text{as } m \to \infty,$$

and

$$M\left(\int_\Omega |x|^{-aq}|\nabla\Delta u_m|^q\,dx\right) \to M(t_q) > 0 \quad \text{as } m \to \infty.$$

Thus, for m sufficiently large, we have

$$0 < c_1 \leq M\left(\int_\Omega |x|^{-ap}|\nabla\Delta u_m|^p\,dx\right) \leq c_2, \quad 0 < c_3 \leq M\left(\int_\Omega |x|^{-ap}|\nabla\Delta u_m|^p\,dx\right) \leq c_4 \tag{4.6}$$

and From (4.5) and (4.6) and the condition (M_1), we have

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}|\nabla\Delta u_m|^{p-2}\nabla\Delta u_m \cdot (\nabla\Delta u_m - \nabla\Delta u)\,dx = 0. \tag{4.7}$$

On the other hand, since $\{u_m\}$ converges weakly to u in X, we have

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}|\nabla\Delta u|^{p-2}\nabla\Delta u \cdot (\nabla\Delta u_m - \nabla\Delta u)\,dx = 0. \tag{4.8}$$

By (4.7) and (4.8)

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}(|\nabla\Delta u_m|^{p-2}\nabla\Delta u_m - |\nabla\Delta u|^{p-2}\nabla\Delta u) \cdot (\nabla\Delta u_m - \nabla\Delta u)\,dx = 0. \tag{4.9}$$

or

$$\lim_{m\to\infty}\int_\Omega (|\nabla\Delta u_m|^{p-2}\nabla\Delta u_m - |\nabla\Delta u|^{p-2}\nabla\Delta u)\cdot(\nabla\Delta u_m - \nabla\Delta u)\,dx = 0. \tag{4.10}$$

where $\nabla\Delta v_m = |x|^{-ap}\nabla\Delta u_m$, $\nabla\Delta v = |x|^{-ap}\nabla\Delta u \in L^p(\Omega)$.

We recall that the following inequalities hold:

$$\langle|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi-\eta\rangle \geq c_7\left(|\xi|+|\eta|\right)^{p-2}|\xi-\eta|^2, \quad \text{if } 1<p<2, \tag{4.11}$$

$$\langle|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi-\eta\rangle \geq c_8|\xi-\eta|^p, \quad \text{if } p\geq 2, \tag{4.12}$$

for all $\xi,\eta\in\mathbb{R}^N$, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^N$.

If $1<p<2$, using Hölder's inequality, by (4.10), we have

$$\begin{aligned}
0 &\leq \|u_m-u\|_{X_p} = \||\nabla\Delta v_m - \nabla\Delta v|\|_{L^p(\Omega)}\\
&\leq \int_\Omega |\nabla\Delta v_m - \nabla\Delta v|^p\left(|\nabla\Delta v_m|+|\nabla\Delta v|\right)^{\frac{p(2-p)}{2}}dx\\
&\leq \left(\int_\Omega |\nabla\Delta v_m-\nabla\Delta v|^2(|\nabla\Delta v_m|+|\nabla\Delta v|)^{p-2}dx\right)^{p/2}\left(\int_\Omega(|\nabla\Delta v_m|+|\nabla\Delta v|)^p dx\right)^{\frac{2-p}{2}}.
\end{aligned}$$

Since this converges to 0 as $m\to\infty$. If $p\geq 2$, hence

$$\begin{aligned}
0 &\leq \|u_m-u\|_{X_p} = \||\nabla\Delta v_m-\nabla\Delta v|\|_{L^p(\Omega)}\\
&\leq c\int_\Omega\langle|\nabla\Delta v_m|^{p-2}\nabla\Delta v_m - |\nabla\Delta v|^{p-2}\nabla\Delta v, \nabla\Delta v_m-\nabla\Delta u\rangle dx,
\end{aligned} \tag{4.13}$$

we deduce that $\{u_m\}$ converges strongly to u in X, and the functional J satisfies the (PS) condition. □

Lemma 4.2. *Suppose that (M_1), $(F_0)-(F_3)$ hold. Then we have*

(i) There exist two positive real numbers ρ and R such that $J(w)\geq R>0$ for all $w=(u,v)\in X$ with $\|w\|=\rho$.

(ii) There exists $\hat{w}\in X$ such that $\|\hat{w}\|>\rho$ and $J(\hat{w})<0$.

Proof. (i) By (M_1), we have

$$\begin{aligned}
J(u,v) &= M\left(\int_\Omega \frac{1}{p}|x|^{-ap}|\nabla\Delta u|^p dx\right) + M\left(\int_\Omega\frac{1}{q}|x|^{-aq}|\nabla\Delta v|^q dx\right)\\
&\quad -\lambda\int_\Omega F(x,u,v)dx.
\end{aligned} \tag{4.14}$$

Since $\theta p<q<p^*$, the embeddings

$$X\hookrightarrow L^{\theta p}(\Omega)\times L^{\theta p}(\Omega), \quad X\hookrightarrow L^{\theta q}(\Omega)\times L^{\theta q}(\Omega),$$

are compact. Then, there are constants $c_5,c_6>0$ such that

$$\|u\|_{L^{\theta p}(\Omega)}\leq c_5\|u\|_{X_p}. \tag{4.15}$$

$$\|u\|_{L^{\theta q}(\Omega}\leq c_6\|u\|_{X_q}. \tag{4.16}$$

Therefore, by (4.14) and (F_1), we have

$$\begin{aligned}
J(u,v) &\geq m_1\left(\frac{1}{\theta p}\|u\|^{\theta p}_{X_p}+\frac{1}{\theta q}\|u\|^{\theta q}_{X_q}\right) - \int_\Omega F(x,u,v)dx\\
&\geq m_1\left(\frac{1}{\theta p}\|u\|^{\theta p}_{X_p}+\frac{1}{\theta q}\|u\|^{\theta q}_{X_q}\right) - c\left(|\Omega| + \frac{1}{p_1+1}\|u\|^{p_1+1}_{p_1+1}+\frac{1}{q_1+1}\|u\|^{q_1+1}_{q_1+1}\right)
\end{aligned}$$

Since $\max\{\theta p, \theta q\} < \min\{p_1, q_1\}$, there exist real numbers $\rho, R > 0$ such that $J(u) \geq R$ for all $u \in X$ with $\|w\| = \rho$.

According to (F_3), $\exists C > 0$ such that

$$F(x, t_1, t_2) \geq C|(t_1, t_2)|^{\mu} \quad \text{for all } x \in \Omega \text{ and } |(t_1, t_2)| \geq T. \tag{4.17}$$

For $w = (u, v) \in X \setminus \{(0, 0)\}$ and $t > 0$, it follows from (F_3)

$$\begin{aligned} J(tw) &= \frac{1}{p}\tilde{M}\left(\int_{\Omega} |\nabla \Delta tw|^p dx\right) + \frac{1}{q}\tilde{M}\left(\int_{\Omega} |\nabla \Delta tw|^q dx\right) - \lambda \int_{\Omega} F(x, tw) dx \\ &\leq \frac{m_1 t^{\theta p}}{\theta p}\|w\|_{X_p}^{\theta p} + \frac{m_1 t^{\theta q}}{\theta q}\|w\|_{X_q}^{\theta q} - c_9 t^{\mu} \int_{\Omega} F(x, w) dx, \end{aligned} \tag{4.18}$$

which tends to $-\infty$ as $t \to +\infty$ since $\max\{\theta p, \theta q\} < \mu$. Then, there exist $t_0 > 0$ such that $J(t_0 w) < 0$ and $\|t_0 w\| > \rho$. We set $\hat{u} = t_0 w$, then Lemma 4.2 is proved. □

Proof. By Lemmas 4.1 and 4.2, all assumptions of the mountain pass theorem in [3] are satisfied. Then the functional J has a nontrivial critical point in X, and thus problem (1.5) has a non trivial weak solution. □

Example 4.3. We will provide a practical example of the problem (1.5) we addressed in this article. We will take, $a(x) = \frac{0.5}{|x|}$, $F = 1$, $p = q = 2$ on the domain $\Omega = (0, 1)^7 \subset \mathbb{R}^7$, such that $0 \in \Omega$.

$$\begin{cases} -M(t_p)\Delta_p^3 u = 0, & \text{in } \Omega, \\ -M(t_q)\Delta_q^3 v = 0, & \text{in } \Omega, \\ \Delta^2 u = \Delta^2 v = \Delta u = \Delta v = u = v = 0, & \text{on } \partial\Omega. \end{cases}$$

$t_p = t_q = t_2 = \int_{\Omega} \frac{1}{2}|\Delta_2^3 u|^3 dx$ and $M(t) = a + b\frac{t^c}{1+\delta t^c}$, $a > 0$, $b \geq 0$, $c > 0$
The functions M, F satisfies the conditions F_i, $i = 0, 1, 2$, M_0 and M_1

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