**Palestine Journal of Mathematics**

**Vol.15(3), (2026)** , 494–510 © Palestine Polytechnic University-PPU 2026

# Sharp Estimates for Coefficient Bounds and Toeplitz Determinants of Certain Orders for Inverses of Fundamental Subclasses of Univalent Functions

B. Winne, D. Vamshee Krishna and Biswajit Rath

Communicated by Sarika Verma

MSC 2020 Classifications: Primary 30C45; Secondary 30C50.

Keywords and phrases: Analytic function; Univalent (Schlicht) function; Upper bound; Hankel determinant; Toeplitz determinant; Carathéodory function; Bounded turning function; Starlike function; Convex function, Zalcman functional.

*The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.*

Corresponding Author: D. Vamshee Krishna

**Abstract** This article delineates precise bounds for the initial five coefficients, specific Toeplitz determinants, and the Zalcman functional (only for the class $\mathcal{S}^*$) of normalized analytic functions within the open unit disc $\mathbb{D}$, especially those that serve as inverses of the principal subclasses of univalent functions. The main results are derived using coefficient inequalities from the Carathéodory class $\mathcal{P}$.

## 1 Introduction

Let us denote by $\mathcal{H}$ the class of analytic functions in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. By $\mathcal{A}$, we denote the class of functions $f \in \mathcal{H}$, normalized by $f(0) = f'(0) - 1 = 0$, i.e, of the following form:

$$f(z) = \sum_{t=1}^{\infty} a_t z^t \qquad (a_1 := 1). \tag{1.1}$$

An important topic in Complex Analysis for more than a century, the assumption that $\mathcal{S}$, is the subfamily of $\mathcal{A}$, with one-to-one (schlicht) functions in $\mathbb{D}$, has piqued a lot of interest. Gaining approximations of functional defined on the complete class of univalent functions and on its subclasses, such as subclasses of convex functions, starlike functions, etc., is a common issue in Geometric Function Theory (GFT), a subfield of Complex Analysis. These functionals include, among others, the so-called Hankel determinants and the Toeplitz determinants. Here is how [38] described the $n^{th}$-Hankel determinant of order $r$, with $r,\ n \in \mathbb{N} = \{1, 2, 3, ...\}$ , for $f$ given in (1.1) as follows:

$$H_{r,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix} \quad (a_1 = 1). \tag{1.2}$$

One interesting and essential topic is the calculation of an upper bound of $H_{r,n}(f)$ with respect to $f \in \mathcal{F}$, for a subclass $\mathcal{F}$ of $\mathcal{A}$, with precise values of $r$ and $n$, (for references, see (see [37], [49]). This leads us to the second Hankel determinant, $H_{2,1}(f) = a_3 - a_2^2$, for $r = 2$ and $n = 1$ in (1.2). After that, Fekete-Szegö generalized the estimate $|a_3 - \rho a_2^2|$ with $\rho$ real, for $f \in \mathcal{S}$ (see [10], Theorem 3.8). Additionally, the second Hankel determinant, which is defined as the functional

$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$, for $r = 2$ and $n = 2$ in (1.2), was thought of by numerous authors as the functions that meet the analytic requirements; they are starlike $\mathcal{S}^*$, convex $\mathcal{C}$, and they belong to the class of bounded turning functions $\mathcal{R}$, (known as functions whose derivative have a positive real port). Regarding the unit disc $\mathbb{D}$, Janteng et al. [19, 20] determined the sharp bound for $|H_{2,2}(f)|$ for the classes $\mathcal{R}$, $\mathcal{S}^*$, and $\mathcal{C}$ with analytic conditions as $\mathrm{Re}(f'(z)) > 0$, $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$ and $\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$ as $4/9$, $1$, and $1/8$, respectively. Estimating an upper bound (especially sharp) to coefficient problems has recently been the subject of much research. The third and fourth order Hankel determinants, $H_{3,1}(f)$ and $H_{4,1}(f)$ respectively, are obtained for the combinations $r = 3$, $n = 1$ and $r = 4$, $n = 1$ in (1.2) for functions in class $\mathcal{S}$ and its subclasses, and more generally in subclasses of class $\mathcal{A}$. For up-to-date findings and research on $H_{3,1}(f)$ and $H_{4,1}(f)$, refer to (see [3],[5], [7], [8], [22], [25], [26], [40], [42], [45], [47], [54]). The use of Hankel matrices and determinants is widespread and important across many mathematical disciplines [53]. The determinants of Toepliz are closely related to the determinants of Hankel. Fixed elements along the major diagonal are present in Toeplitz matrices whereas in Hankel matrices they are located along the secondary diagonal. Toeplitz matrices have many uses in both theoretical and practical mathematics, and a fair overview of these uses may be found in [53]. The notion of the Toeplitz matrices came recently from Thomas and Halim [48], who introduced the concept of the symmetric Toepliz determinants for analytic functions $f$ of the form (1.1), defined as follows. For the natural numbers $q$ and $s$,

$$T_{q,s}(f) = \begin{vmatrix} a_s & a_{s+1} & \cdots & a_{s+q-1} \\ a_{s+1} & a_s & \cdots & a_{s+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s+q-1} & a_{s+q-2} & \cdots & a_s \end{vmatrix}. \tag{1.3}$$

The current focus of many academics is on the estimation of sharp bounds for Toeplitz determinants of precise order for functions belonging to specific subclasses of $\mathcal{A}$. Recently, the best possible bounds for the Toeplitz determinants of order three and four for functions in $\mathbb{D}$ that belong to fundamental subclasses of univalent functions were determined by Winne Bareh et al. [51]. Within a specific class of analytic functions that possess a Borel distribution, Wanas et al. [50] established upper bounds for the Toeplitz determinants. With the use of bounds, Rahmatan [41] was able to determine the fifth Toeplitz determinant for bounded turning functions. As for a class of higher-dimensional Holomorphic mappings, Giri and Kumar [13] estimated Toeplitz determinants of a given order. In their study, Farah Zulfiqar et al. [11] calculated an approximate constraint for the fourth-order Toeplitz determinant of functions belonging to the class $\mathcal{C}$ connected with the sine function. Bounds for symmetric Toeplitz determinants for a new subclass of $q$-starlike functions were computed by Isra Al-shbeil et al. [18]. For a subclass of alpha-close-to-convex functions, Jha and Sahoo [24] discovered the upper bound of the Toeplitz determinants. Bounds for the Toeplitz determinants of a novel class of prestarlike functions were found by Luminita and Wanas [30]. This family of Bazilevic functions is related with the lemniscate of Bernoulli, and Ni Made Asih et al. [35] determined bounds for the Toeplitz determinants. A recent study on bounds for Toeplitz determinants (see [1], [6], [9], [14], [15], [17], [31], [32], [33], [34], [39], [44], [49],[52]).

Let $\mathcal{P}$ be a class of all functions $g$ of the following form, which have positive real part in $\mathbb{D}$:

$$g(z) = 1 + \sum_{t=1}^{\infty} c_t z^t. \tag{1.4}$$

We refer to all of these functions as Carathéodory functions [10]. Given the analytic requirements for $\mathcal{R}$, $\mathcal{S}^*$ and $\mathcal{C}$ of $\mathcal{S}$ and (1.4), the coefficients of functions in these subclasses can be stated in terms of coefficients of functions in the class $\mathcal{P}$. For every function $f \in \mathcal{S}$, there exists an inverse denoted by $f^{-1}$, given by

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right). \tag{1.5}$$

In a study by Ali [2], the Fekete-Szegö coefficient functional of the inverse function was estimated with high precision, and significant constraints on the first four coefficients were found for functions that belong to the class of strongly starlike functions denoted by $\mathcal{SS}^*(\alpha)$, defined as $|\arg(zf'(z)/f(z))| < \pi\alpha/2,\ (0 < \alpha \leq 1)$. In a recent study, Sim et al. [43] obtained a sharp bound of $|H_{2,2}(f^{-1})|$ for the class of strongly Ozaki functions denoted by $\mathcal{F}_o(\lambda)$, where the analytic condition is $\mathrm{Re}\{1 + (zf''(z)/f'(z))\} < (1 - 2\lambda)/2,\ (1/2 \leq \lambda \leq 1)$. For each $f$ of the form (1.1), Libera et al. [28] found a relationship between the coefficients of $f$ and $f^{-1}$, when $f(\mathbb{D})$ is a convex region. Meanwhile, Kapoor and Mishra [21] built upon the work of Krzyz et al. [23], who investigated bounds on the initial coefficients of the inverse of starlike functions of order $\alpha$ $(0 \leq \alpha < 1)$. A function $f \in \mathcal{A}$, is said to be bi-univalent in $\mathbb{D}$, if $f$ is univalent in $\mathbb{D}$ and $f^{-1}$ has a univalent analytic extension in $\mathbb{D}$. For subclasses of bi-univalent functions linked to a nephroid domain, Srivastava et al. [46] recently found a bound for the second Hankel determinant. Inspired by these results, in this paper, we try to find sharp upper bounds to the initial five coefficients, the symmetric Toeplitz determinants of order two and three, for the inverse function of $f$ given in (1.1), when it belongs to the fundamental subclasses of univalent functions. In addition, we compute an exact estimate for the Zalcman conjecture $|a_n^2 - a_{2n-1}| \leq (n-1)^2$, where $n \in \mathbb{N} \setminus \{1\}$ $(n = 2, 3)$, for the inverse function of $f$ given in (1.1), when it belongs to the class $\mathcal{S}^*$.

## 2 A Set of Lemmas

We need to incorporate certain ideas from the work of Libera and Złotkiewicz [27] and use the following lemmas as a foundation for proving our key results.

**Lemma 2.1.** *[36] If $g \in \mathcal{P}$, then $|c_t| \leq 2$, for $t \in \mathbb{N}$, equality occurs for the function $h(z) = \frac{1+z}{1-z}$, $z \in \mathbb{D}$.*

**Lemma 2.2.** *[16] If $g \in \mathcal{P}$, then the following estimate holds for the values $i, j \in \mathbb{N}$, with $i > j$.*

$$|c_i - \mu c_j c_{i-j}| \leq \begin{cases} 2, & \text{if} \quad \mu \in [0,1], \\ 2|2\mu - 1|, & \text{if} \quad \mu > 1. \end{cases}$$

With the help of Lemma 2.2, Livingston ([29]) proved that $|c_i - c_j c_{i-j}| \leq 2$.

**Lemma 2.3.** *[36] If $g \in \mathcal{P}$, then the estimate $|c_1^3 - 2c_1c_2 + c_3| \leq 2$.*

Using Lemma 2.3, Arif et al. [4] proved the generalized result as

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2(|J| + |K - 2J| + |J - K + L|),$$

where $J$, $K$, and $L$ are complex constants.

**Lemma 2.4.** *Let $g \in \mathcal{P}$, of the form in (1.4) with $c_1 \geq 0$ then*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

$$\text{and } 4c_3 = \left[c_1^3 + \left\{2c_1x - c_1x^2 + 2(1 - |x|^2)y\right\}(4 - c_1^2)\right],$$

*for some $x, y$ such that $|x| \leq 1$ and $|y| \leq 1$.*

The fomula on coefficient $c_2$ is by Pommerenke ([36], p. 166) and the formula $c_3$ due to Libera and Zlotkiewicz ([27], [28]).

## 3 Sharp Bounds for the First Five Coefficients

In the section, we will derive the exact estimates for the initial five coefficients.

**Theorem 3.1.** *If $f \in \mathcal{R}$ then*

$$(i)\ \ |t_2| \leq 1 \quad (ii)\ \ |t_3| \leq \frac{4}{3} \quad (iii)\ \ |t_4| \leq \frac{13}{6} \quad (iv)\ \ |t_5| \leq \frac{59}{15},$$

*all these bounds are sharp for the function,*

$$f_0^{-1}(z) = z - iz^2 - \frac{4}{3}z^3 + \frac{13i}{6}z^4 + \frac{59}{15}z^5 + ..., for\ which$$
$$t_2 = -i,\ t_3 = -\frac{4}{3},\ t_4 = \frac{13i}{6},\ and\ t_5 = \frac{59}{15},\ etc\ , where\ f_0'(z) = \frac{1+iz}{1-iz}.$$

*Proof.* Let $f \in \mathcal{R}$ be of the form (1.1). Then there exists an analytic function $g \in \mathcal{P}$, such that

$$f'(z) = g(z), \qquad z \in \mathbb{D}. \tag{3.1}$$

Substitute the values for $f$ and $g$ in (3.1), after simplifying, we get

$$na_n = c_{n-1}, n \geq 2. \tag{3.2}$$

For $f \in \mathcal{R}$, by using the definition in (1.5), we have

$$\begin{aligned} w = f\left(f^{-1}\right) =& f^{-1}(w) + \sum_{n=2}^{\infty} a_n \left(f^{-1}(w)\right)^n \\ =& w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n \left(w + \sum_{n=2}^{\infty} t_n w^n\right)^n. \end{aligned}$$

Thus, upon simplification, we obtain

$$\begin{aligned} &(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4 \\ &+ (t_5 + 2a_2t_4 + 2a_2t_2t_3 + 3a_3t_3 + 3a_3t_2^2 + 4a_4t_2 + a_5)w^5 + \cdots = 0. \end{aligned} \tag{3.3}$$

We now equate the coefficients of powers of $w$ in (3.3), after simplifying, we get

$$\begin{aligned} &t_2 = -a_2,\ t_3 = -a_3 + 2a_2^2,\ t_4 = -a_4 + 5a_2a_3 - 5a_2^3, \\ &\text{and } t_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{3.4}$$

Substituting the values of $a_n$ $(n = 2, 3, 4, 5)$ from (3.2) into (3.4), if we simplify the resulting expressions, we obtain

$$\begin{aligned} &t_2 = -\frac{c_1}{2},\ t_3 = -\frac{1}{6}(2c_2 - 3c_1^2),\ t_4 = -\frac{1}{24}(6c_3 - 20c_1c_2 + 15c_1^3) \\ &\text{and}\ \ t_5 = -\frac{1}{120}(24c_4 - 90c_1c_3 - 40c_2^2 + 210c_1^2c_2 - 105c_1^4). \end{aligned} \tag{3.5}$$

Taking the modulus on both sides to the expressions namely $t_2$, $t_3$, and $t_4$ in (3.5) and applying Lemmas 2.1, 2.2 and 2.3 respectively, we obtain the inequalities as follows:

$$(i)\ \ |t_2| \leq 1.\ \ (ii)\ \ |t_3| = \frac{1}{3}\left|c_2 - \frac{3}{2}c_1^2\right| \leq \frac{2}{3}\left|2\left(\frac{3}{2}\right) - 1\right| = \frac{4}{3} \tag{3.6}$$

$$\text{(iii)}\ \ |t_4| \ = \frac{1}{24}|6c_3 - 20c_1c_2 + 15c_1^3| \leq \frac{2}{24}(15 + 10 + 1) = \frac{26}{12} = \frac{13}{6}.$$

(iv) Taking modulus on both sides of $t_5$ in (3.5), re-organising the terms, applying the triangle inequality, and then upon using Lemmas 2.1, 2.2 and 2.3 appropriately, we obtain

$$\begin{aligned} |t_5| &= \frac{1}{120}\left(24c_4 - 90c_1c_3 - 40c_2^2 + 210c_1^2c_2 - 105c_1^4\right) \\ &= \frac{1}{120}\left|-40(c_2 - c_1^2)^2 + 24(c_4 - c_1c_3) - 65c_1(c_1^3 - 2c_1c_2 + c_3) - c_1c_3\right| \\ &\leq \frac{1}{120}\left[40|c_2^2 - c_1|^2 + 24|c_4 - c_1c_3| + 65|c_1||c_1^3 - 2c_1c_2 + c_3| + |c_1||c_3|\right] \\ &= \frac{1}{120}\left[40(2)^2 + 24(2) + 65(2)(2) + (2)(2)\right] = \frac{472}{120} = \frac{59}{15}. \end{aligned}$$

Let $f_0'(z) = \frac{1+iz}{1-iz} = w = u+iv$. Then $z = \frac{w-1}{i(w+1)}$. For $|z| < 1 \implies \left|\frac{w-1}{i(w+1)}\right| < 1$, which implies that $|w-1| < |w+1| \implies [(u-1)^2+v^2] < [(u+1)^2+v^2]$, it simplifies to give $4u > 0 \implies 4\text{Re}\, f_0'(z) > 0 \implies \text{Re} f_0'(z) > 0$. Now on integrated $f_0'(z)$ gives $f_0(z) = -z + 2i\log(1-iz)$.

Hence the function $f_0(z) \in \mathcal{R}$.

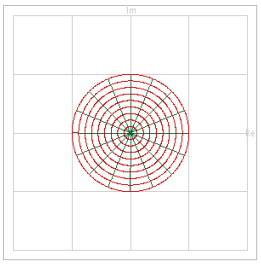


**Figure 1.** $z$-plane

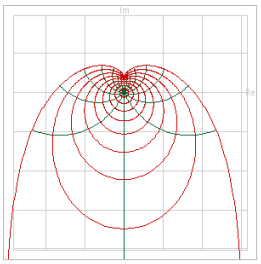


**Figure 2.** $w$-plane

The figures above demonstrate the nature of the sharp function $f_0(z)$. The function maps the open unit disc in the $z$-plane to the $w$-plane. We can see in these figures that the green and red curves in $z$-plane match up with the corresponding curves in the $w$-plane. The presence of non-intersecting red curves in the $w$-plane indicates that the function $f_0(z)$ is univalent.

**Theorem 3.2.** *If $f \in \mathcal{S}^*$ then*

$$(i)\ \ |t_2| \le 2\ \ (ii)\ |t_3| \le 5\ \ (iii)\ |t_4| \le 14\ \ (iv)\ |t_5| \le 42,$$

*all these inequalities are sharp with the extremal function*

$$f_1^{-1}(z) = z - 2iz^2 - 5z^3 + 14iz^4 + 42z^5 + ..., \text{for which}$$
$$t_2 = -2i,\ t_3 = -5,\ t_4 = 14i,\ \text{and}\ t_5 = 42,\ \text{where}\ f_1(z) = \frac{z}{(1-iz)^2}.$$

*Proof.* Let $f \in \mathcal{S}^*$, be of the form (1.1). Then there exists an analytic function $g \in \mathcal{P}$, such that

$$zf'(z) = f(z)g(z), \qquad z \in \mathbb{D}. \tag{3.7}$$

Substitute the values for $f$ and $g$ in (3.7), equating the coefficients of like powers of $z$, upon simplification, we obtain

$$a_2 = c_1,\ a_3 = \frac{1}{2}(c_2 + c_1^2),\ a_4 = \frac{1}{6}(2c_3 + 3c_1c_2 + c_1^3)$$
$$\text{and}\ a_5 = \frac{1}{24}(6c_4 + 8c_1c_3 + 3c_2^2 + 6c_1^2c_2 + c_1^4). \tag{3.8}$$

Putting the values of $a_n$ $(n = 2,3,4,5)$ from (3.8) into (3.4), upon simplifying the resulting expressions, we obtain

$$t_2 = -c_1,\ t_3 = -\frac{1}{2}(c_2 - 3c_1^2),\ t_4 = -\frac{1}{3}(c_3 - 6c_1c_2 + 8c_1^3)$$
$$\text{and}\ t_5 = -\frac{1}{24}(6c_4 - 40c_1c_3 - 15c_2^2 + 150c_1^2c_2 - 125c_1^4). \tag{3.9}$$

Taking modulus on both sides of the expressions for $t_2$, $t_3$ and $t_4$ in (3.9) and applying Lemmas 2.1, 2.2 and 2.3 respectively, we obtain the following inequalities.

$$(i)\ \ |t_2| \le 2.\ \ (ii)\ |t_3| = \frac{1}{2}|c_2 - 3c_1^2| \le \frac{1}{2}(2)|2(3) - 1| = 5. \tag{3.10}$$
$$\text{(iii)}\ |t_4| = \frac{1}{3}|c_3 - 6c_1c_2 + 8c_1^3| \le \frac{2}{3}(8 + 10 + 3) = 14.$$

(iv) Taking modulus on both sides of $t_5$ in (3.9), re-arranging the terms, applying the triangle inequality, and performing Lemmas 2.1, 2.2 and 2.3, we obtain

$$\begin{aligned}
|t_5| &= \frac{1}{24}|6c_4 - 40c_1c_3 - 15c_2^2 + 150c_1^2c_2 - 125c_1^4| \\
&= \frac{1}{24}\left|70c_1^2(c_2 - c_1^2) + 6(c_4 - c_2^2) - 40c_1(c_1^3 - 2c_1c_2 + c_3) - 9c_2^2 - 15c_1^4\right| \\
&\leq \frac{1}{24}\left[70|c_1|^2|c_2 - c_1^2| + 6|c_4 - c_2^2| + 40|c_1||c_1^3 - 2c_1c_2 + c_3| + 9|c_2|^2 + 15|c_1|^4\right] \\
&= \frac{1}{24}\left[70(2)^2(2) + 6(2) + 40(2)(2) + 9(2)^2 + 15(2)^4\right] = \frac{1008}{24} = 42.
\end{aligned}$$

Let $f_1(z) = \frac{z}{(1-iz)^2} = w = u + iv$. Then $f_1'(z) = \frac{1+iz}{(1-iz)^3}$, it implies that $\frac{zf_1'(z)}{f_1(z)} = \frac{1+iz}{1-iz}$. From Theorem 3.1, we have Re $\left(\frac{1+iz}{1-iz}\right) > 0$.

Therefore, the function $\dfrac{z}{(1-iz)^2} \in \mathcal{S}^*$.

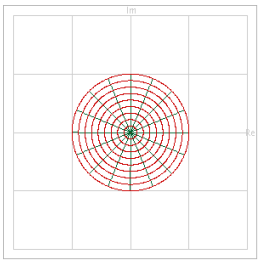


**Figure 3.** $z$-plane

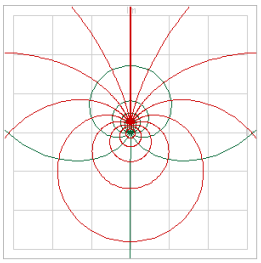


**Figure 4.** $w$-plane

The figures above illustrate how the function $f_1(z)$ maps the unit disc in $z$-plane onto the $w$-plane. The red and green curves in the $z$-plane correspond to the red and green curves in the $w$-plane, respectively. Since the red curves in the $w$-plane are not intersecting, this indicates that $f_1(z)$ is a univalent function.

**Theorem 3.3.** *If $f \in \mathcal{C}$ then*

$$(i)\ \ |t_2| \leq 1\ \ (ii)\ |t_3| \leq 1\ \ (iii)\ |t_4| \leq 1\ \ (iv)\ |t_5| \leq 1,$$

*all these bounds are sharp. The extremal function is*

$$f_2^{-1}(z) = z - iz^2 - z^3 + iz^4 + z^5 + ..., \text{for which}$$
$$t_2 = -i,\ t_3 = -1,\ t_4 = i,\ \text{and}\ \ t_5 = 1,\ \text{where}\ f_2(z) = \frac{z}{(1-iz)}.$$

*Proof.* Let $f \in \mathcal{C}$, be of the form (1.1). Then there exists an analytic function $g \in \mathcal{P}$, such that

$$f'(z) + zf''(z) = f'(z)g(z), \qquad z \in \mathbb{D}. \tag{3.11}$$

Substitute the values for $f$ and $g$ in (3.11), equating the coefficients of same powers of $z$, upon simplification, we obtain

$$a_2 = \frac{c_1}{2},\ a_3 = \frac{1}{6}(c_2 + c_1^2),\ a_4 = \frac{1}{24}(2c_3 + 3c_1c_2 + c_1^3)$$
$$\text{and } a_5 = \frac{1}{120}(6c_4 + 8c_1c_3 + 3c_2^2 + 6c_1^2c_2 + c_1^4). \tag{3.12}$$

Substituting the values of $a_n$ $(n = 2, 3, 4, 5)$ from (3.12) into (3.4), after simplifying, we get

$$t_2 = -\frac{c_2}{2},\ t_3 = -\frac{1}{6}(c_2 - 2c_1^2),\ t_4 = -\frac{1}{24}(2c_3 - 7c_1c_2 + 6c_1^3)$$
$$\text{and } t_5 = -\frac{1}{120}(6c_4 - 22c_1c_3 - 7c_2^2 + 46c_1^2c_2 - 24c_1^4). \tag{3.13}$$

Taking modulus on both sides of the expressions for $t_2$, $t_3$ and $t_4$ in (3.13) and applying Lemmas 2.1, 2.2 and 2.3, we obtain the following inequalities.

$$(i)\ \ |t_2| \le 1.\ \ (ii)\ \ |t_3| \le \frac{1}{6}\left|c_2 - 2c_1^2\right| \le \frac{1}{6}(2)\left|2(2) - 1\right| = 1. \tag{3.14}$$

$$\text{(iii)}\ \ |t_4| \ = \frac{1}{24}|2c_3 - 7c_1c_2 + 6c_1^3| \le \frac{2}{24}(6 + 5 + 1) = \frac{24}{24} = 1.$$

(iv) Taking modulus on both sides of $t_5$ in (3.13), re-arranging the terms, applying the triangle inequality, and then upon using Lemmas 2.1, 2.2 and 2.3, we obtain

$$\begin{aligned}
|t_5| &= \frac{1}{120}|6c_4 - 22c_1c_3 - 7c_2^2 + 46c_1^2c_2 - 24c_1^4| \\
&= \frac{1}{120}\left|2c_1^2(c_2 - c_1^2) + 6(c_4 - c_2^2) - 22c_1(c_1^3 - 2c_1c_2 + c_3) - c_2^2\right| \\
&\le \frac{1}{120}\left[2|c_1|^2|c_2 - c_1^2| + 6|c_4 - c_2^2| + 22|c_1||c_1^3 - 2c_1c_2 + c_3| + |c_2|^2\right] \\
&= \frac{1}{120}\left[2(2)^2(2) + 6(2) + 22(2)(2) + (2)^2\right] = \frac{120}{120} = 1.
\end{aligned}$$

Let $f_2(z) = \frac{z}{(1-iz)} = w = u + iv$. Then $f_2'(z) = \frac{1}{(1-iz)^2} \implies f_2''(z) = \frac{2i}{(1-iz)^3}$, it implies that $\left(1 + \frac{zf_2''(z)}{f_2'(z)}\right) = \frac{1+iz}{1-iz}$. From Theorem 3.1, we have Re $\left(\frac{1+iz}{1-iz}\right) > 0$.

Thus, the function $\dfrac{z}{(1-iz)} \in \mathcal{C}$.

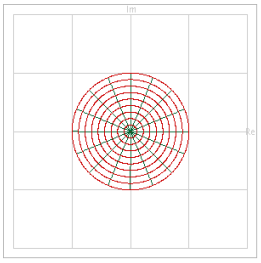


**Figure 5.** $z$-plane

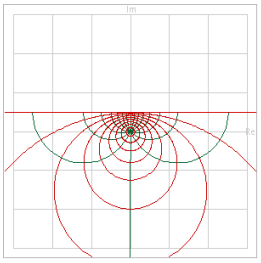


**Figure 6.** $w$-plane

The above figures illustrates that, how the function $f_2(z)$ maps the unit disc in the $z$-plane onto the $w$-plane. Each coloured line in the $z$-plane corresponds to curve of the same colour in the $w$-plane. As the red curves in the $w$-plane are not intersecting, implying that the function $f_2(z)$ is univalent.

## 4 Sharp bound for the Second-order Toeplitz determinants

In this section, we state and prove our main results associated with second-order Toeplitz determinants as follows.

**Theorem 4.1.** *If $f \in \mathcal{R}$, then*

$$(i)\ \left|T_{2,1}\left(f^{-1}\right)\right| \leq 2. \qquad (ii)\ \left|T_{2,2}\left(f^{-1}\right)\right| \leq \frac{25}{9}.$$

$$(iii)\ \left|T_{2,3}\left(f^{-1}\right)\right| \leq \frac{233}{36}. \qquad (iv)\ \left|T_{2,4}\left(f^{-1}\right)\right| \leq \frac{18149}{900},$$

*all these bounds are best possible by the same function given in Theorem 3.1.*

*Proof.* Let $f \in \mathcal{R}$ be of the form (1.1), in view of (1.3), let us now recall that

$$T_{2,1}(f^{-1}) := \begin{vmatrix} t_1 & t_2 \\ t_2 & t_1 \end{vmatrix} = 1 - t_2^2, \tag{4.1}$$

Substituting the value of $t_2$ from (3.5) in (4.1), taking modulus on both sides, applying the triangle inequality and then using Lemma 2.1, it simplifies to give

$$|T_{2,1}(f^{-1})| \leq 1 + \frac{|c_1|^2}{4} = 1 + 1 = 2.$$

Next, if $f \in \mathcal{R}$ be of the form (1.1) then in view of (1.3), we have

$$T_{2,2}(f^{-1}) := \begin{vmatrix} t_2 & t_3 \\ t_3 & t_2 \end{vmatrix} = t_2^2 - t_3^2. \tag{4.2}$$

Substituting the values of $t_2$ and $t_3$ from (3.5) in (4.2), taking modulus on both sides, applying the traingle inequality and then using Lemmas 2.1, and 2.2, upon simplification, we obtain

$$|T_{2,2}(f^{-1})| \leq \frac{|c_1|^2}{4} + \frac{1}{36}\left|2c_2 - 3c_1^2\right|^2 = 1 + \frac{16}{9} = \frac{25}{9}.$$

Again, if $f \in \mathcal{R}$ be of the form (1.1), then based on (1.3), we have

$$T_{2,3}(f^{-1}) := \begin{vmatrix} t_3 & t_4 \\ t_4 & t_3 \end{vmatrix} = t_3^2 - t_4^2.$$

Applying the same procedure carried in estimating $|T_{2,1}(f^{-1})|$ and $|T_{2,2}(f^{-1})|$, performing Lemma 2.2 and generalization of Lemma 2.3, after simplifying, we get

$$|T_{2,3}(f^{-1})| \leq \frac{1}{36}\left|2c_2 - 3c_1^2\right|^2 + \frac{1}{576}\left|15c_1^3 - 20c_1c_2 + 6c_3\right|^2 = \frac{16}{9} + \frac{2704}{576} = \frac{233}{36}.$$

Next, for $f \in \mathcal{R}$ be of the form (1.1), then

$$T_{2,4}(f^{-1}) := \begin{vmatrix} t_4 & t_5 \\ t_5 & t_4 \end{vmatrix} = t_4^2 - t_5^2. \tag{4.3}$$

Upon substituting the values of $t_4$ and $t_5$ in (4.3), we have

$$\begin{aligned} T_{2,4}(f^{-1}) :&= \frac{1}{576}\left(15c_1^3 - 20c_1c_2 + 6c_3\right)^2 \\ &\quad - \frac{1}{14400}(-24c_4 + 90c_1c_3 + 40c_2^2 - 210c_1^2c_2 + 105c_1^4)^2. \\ &= \frac{1}{576}\left(15c_1^3 - 20c_1c_2 + 6c_3\right)^2 \\ &\quad - \frac{1}{14400}\left(-24(c_4 - c_2^2) + 16(c_2 - c_1^2)^2 + c_1(89c_1^3 - 178c_1c_2 + 90c_3)\right)^2. \end{aligned}$$

Taking modulus on both sides, applying the traingle inequality and then using Lemmas 2.1, 2.2 and generalization of Lemma 2.3, upon simplification, we obtain

$$\left|T_{2,4}(f^{-1})\right| \leq \frac{(52)^2}{576} + \frac{(472)^2}{14400} = \frac{18149}{900}.$$

**Theorem 4.2.** *If $f \in \mathcal{S}^*$, then*

$$(i)\ \left|T_{2,1}\left(f^{-1}\right)\right| \leq 5. \qquad (ii)\ \left|T_{2,2}\left(f^{-1}\right)\right| \leq 29.$$
$$(iii)\ \left|T_{2,3}\left(f^{-1}\right)\right| \leq 221. \qquad (iv)\ \left|T_{2,4}\left(f^{-1}\right)\right| \leq 1960,$$

*and all these inequalities are sharp for the same function given in Theorem 3.2.*

*Proof.* Let $f \in \mathcal{S}^*$ be of the form given in (1.1). Substituting the values of $t_2, t_3, t_4$, and $t_5$ from equation (3.9), and applying the same procedure carried in the estimation of $T_{2,1}(f^{-1})$, $T_{2,2}(f^{-1})$, $T_{2,3}(f^{-1})$ and $T_{2,4}(f^{-1})$ respectively in Theorem 4.1, we obtain

$$|T_{2,1}(f^{-1})| \leq 1 + |c_1|^2 = 1 + 4 = 5.$$

$$|T_{2,2}(f^{-1})| \leq |c_1|^2 + \frac{1}{4}\left|c_2 - 3c_1^2\right|^2 = 4 + 25 = 29.$$

$$|T_{2,3}(f^{-1})| \leq \frac{1}{4}\left|c_2 - 3c_1^2\right|^2 + \frac{1}{9}\left|8c_1^3 - 6c_1c_2 + c_3\right|^2 = \frac{(10)^2}{4} + \frac{(42)^2}{9} = 221.$$

$$\begin{aligned} T_{2,4}(f^{-1}) :&= \frac{1}{9}\left(8c_1^3 - 6c_1c_2 + c_3\right)^2 - \frac{1}{576}(6c_4 - 40c_1c_3 - 15c_2^2 + 150c_1^2c_2 - 125c_1^4)^2. \\ &= \frac{1}{9}\left(8c_1^3 - 6c_1c_2 + c_3\right)^2 \\ &\quad - \frac{1}{576}\left(6(c_4 - c_2^2) - 9c_2^2 - 5c_1(25c_1^3 - 30c_1c_2 + 8c_3)\right)^2. \end{aligned}$$

$$\text{Therefore, } |T_{2,4}(f^{-1})| \leq \frac{(42)^2}{9} + \frac{(1008)^2}{576} = 196 + 1764 = 1960.$$

**Theorem 4.3.** *If $f \in \mathcal{C}$, then*

$$(i)\ \left|T_{2,1}\left(f^{-1}\right)\right| \leq 2. \qquad (ii)\ \left|T_{2,2}\left(f^{-1}\right)\right| \leq 2.$$
$$(iii)\ \left|T_{2,3}\left(f^{-1}\right)\right| \leq 2. \qquad (iv)\ \left|T_{2,4}\left(f^{-1}\right)\right| \leq 2,$$

*all these bounds are sharp for the same function given in Theorem 3.3.*

*Proof.* Let $f \in \mathcal{C}$, be of the form (1.1). Now, applying the same procedure carried in Theorem 4.1, using the derived values in (3.13), we obtain

$$|T_{2,1}(f^{-1})| \leq 1 + \frac{|c_1|^2}{4} = 1 + 1 = 2.$$

$$|T_{2,2}(f^{-1})| \leq \frac{|c_1|^2}{4} + \frac{1}{36}\left|c_2 - 2c_1^2\right|^2 = 1 + 1 = 2.$$

$$|T_{2,3}(f^{-1})| \leq \frac{1}{36}\left|c_2 - 2c_1^2\right|^2 + \frac{1}{576}\left|6c_1^3 - 7c_1c_2 + 2c_3\right|^2 = \frac{(6)^2}{36} + \frac{(24)^2}{576} = 2.$$

$$\begin{aligned} T_{2,4}(f^{-1}) :&= \frac{1}{576}\left(6c_1^3 - 7c_1c_2 + 2c_3\right)^2 - \frac{1}{14400}(6c_4 - 22c_1c_3 - 7c_2^2 + 46c_1^2c_2 - 24c_1^4)^2. \\ &= \frac{1}{576}\left(6c_1^3 - 7c_1c_2 + 2c_3\right)^2 \\ &\quad - \frac{1}{14400}\left(6(c_4 - c_2^2) - (c_2 - c_1^2)^2 - c_1(23c_1^3 - 44c_1c_2 + 22c_3)\right)^2. \end{aligned}$$

$$\text{Therefore, } |T_{2,4}(f^{-1})| \leq \frac{(24)^2}{576} + \frac{(120)^2}{14400} = 1 + 1 = 2.$$

**Remark 4.4.** From the results estimated in Theorem 4.3, we can generalise that $|T_{2,n}(f^{-1})| \leq 2$, where $n$ is a positive integer.

**Remark 4.5.** The result given under Remark 4.4, coincides with that of Firoz Ali et al. [12] ( see Theorem 2.11). i.e., the sharp bound is invariant. The result can be stated as if $f \in \mathcal{C}$, then $|T_{2,n}(f)| = |T_{2,n}(f^{-1})| \leq 2$, for $n \geq 1$.

## 5 Sharp bound for the Third-order Toeplitz determinants

In this section, we state and prove our main result associated with third-order Toeplitz determinants as follows.

**Theorem 5.1.** *If $f \in \mathcal{R}$, then*

$$(i)\ \left|T_{3,1}\left(f^{-1}\right)\right| \leq \frac{35}{9}. \qquad (ii)\ \left|T_{3,2}\left(f^{-1}\right)\right| \leq \frac{817}{108},$$

*these bounds are best possible for the function given in Theorem 3.1.*

*Proof.* Let $f \in \mathcal{R}$ be of the form (1.1). In view of (1.3), we have

$$T_{3,1}\left(f^{-1}\right) := \begin{vmatrix} t_1 = 1 & t_2 & t_3 \\ t_2 & 1 & t_2 \\ t_3 & t_2 & 1 \end{vmatrix}.$$

Upon expanding the determinant, after simplifying, we get

$$T_{3,1}\left(f^{-1}\right) := 1 - 2t_2^2 + 2t_2^2t_3 - t_3^2 = 1 - 2t_2^2 - t_3(t_3 - 2t_2^2). \tag{5.1}$$

Imputting the values of $t_2$ and $t_3$ from (3.5) in (5.1), upon simplification, we obtain

$$T_{3,1}\left(f^{-1}\right) := 1 - \frac{c_1^2}{2} - \frac{c_2}{9}\left(c_2 - \frac{3}{2}c_1^2\right). \tag{5.2}$$

Taking modulus on both sides, applying the trangle inequality and then using Lemmas 2.1 and 2.2, after simplifying, we get

$$\left|T_{3,1}\left(f^{-1}\right)\right| \leq 1 + 2 + \frac{8}{9} = \frac{35}{9}.$$

Next, let $f \in \mathcal{R}$, be of the form (1.1). In view of (1.3), let us now recall that

$$T_{3,2}\left(f^{-1}\right) := \begin{vmatrix} t_2 & t_3 & t_4 \\ t_3 & t_2 & t_3 \\ t_4 & t_3 & t_2 \end{vmatrix}.$$

Upon expanding the determinant, which simplifies to give

$$\begin{aligned} T_{3,2}\left(f^{-1}\right) :&= t_2^3 - 2t_2t_3^2 - t_2t_4^2 + 2t_3^2t_4 \\ &= t_2(t_2^2 - 2t_3^2) - t_4(t_2t_4 - 2t_3^2) \\ &= (t_2 - t_4)(t_2^2 - 2t_3^2 + t_2t_4). \end{aligned} \tag{5.3}$$

Taking modulus on both sides in (5.3), we have

$$|T_{3,2}\left(f^{-1}\right)| = |t_2 - t_4||t_2^2 - 2t_3^2 + t_2t_4|. \tag{5.4}$$

Now, substituting the values of $t_2$ and $t_4$ from (3.5) in the expression $(t_2 - t_4)$, taking modulus on both sides, upon applying the triangle inequality, using Lemma 2.1 and generalisation of Lemma 2.3, we get

$$|t_2 - t_4| \leq |t_2| + |t_4| = \frac{|c_1|}{2} + \frac{\left|15c_1^3 - 20c_1c_2 + 6c_3\right|}{24} = 1 + \frac{2(26)}{24} = 1 + \frac{13}{6} = \frac{19}{6}. \tag{5.5}$$

Further, upon substituting the values of $t_2$, $t_3$, and $t_4$ from (3.5) in the expression $(t_2^2 - 2t_3^2 + t_2t_4)$, upon simplification, we obtain

$$\begin{aligned} (t_2^2 - 2t_3^2 + t_2t_4) =& \frac{1}{144}(-27c_1^4 + 36c_1^2c_2 + 18c_1c_3 - 32c_2^2 + 36c_1^2). \\ =& \frac{1}{144}[-18(c_2 - c_1^2)^2 + 9c_1^2(4 - c_1^2) - 14c_2^2 + 18c_1c_3]. \end{aligned} \tag{5.6}$$

Now, taking modulus on both sides in (5.6), upon applying the triangle inequality, using Lemmas 2.1 and 2.2 on its right-hand side, which simplifies to give

$$|t_2^2 - 2t_3^2 + t_2t_4| \leq \frac{1}{144}\left\{18|c_2 - c_1^2|^2 + 9|c_1|^2|4 - c_1^2| + 14|c_2|^2 + 18|c_1||c_3|\right\}.$$
$$= \frac{1}{144}(72 + 144 + 56 + 72) = \frac{344}{144} = \frac{43}{18}.$$
$$\text{Therefore, } |t_2^2 - 2t_3^2 + t_2t_4| \leq \frac{43}{18}. \tag{5.7}$$

From the expressions (5.4), (5.5) and (5.7), upon simplification, we obtain

$$\left|T_{3,2}\left(f^{-1}\right)\right| \leq \left(\frac{19}{6}\right)\left(\frac{43}{18}\right) = \frac{817}{108}.$$

**Remark 5.2.** The result (i) given under Theorem 5.1, coincides with that of Firoz Ali et al. [12] ( see Theorem 2.12). i.e., the sharp bound is invariant. The result can be stated as, if $f \in \mathcal{R}$, then

$$|T_{3,1}(f)| = |T_{3,1}(f^{-1})| \leq \frac{35}{9}.$$

**Theorem 5.3.** *If $f \in \mathcal{S}^*$, then*

$$(i)\ \left|T_{3,1}\left(f^{-1}\right)\right| \leq 24. \qquad (ii)\ \left|T_{3,2}\left(f^{-1}\right)\right| \leq 416,$$

*both bounds are best possible for the function given in Theorem 3.2.*

*Proof.* Let $f \in \mathcal{S}^*$, be of the form (1.1). Substituting the values of $t_2$ and $t_3$ from (3.9) in (5.1), upon simplification, we obtain

$$T_{3,1}\left(f^{-1}\right) := 1 - 2c_1^2 - \frac{1}{4}(c_2 - 3c_1^2)(c_2 + c_1^2). \tag{5.8}$$

Taking modulus on both sides, applying the traingle inequality and then using Lemmas 2.1 and 2.2 in (5.8), which simplifies to give

$$\left|T_{3,1}\left(f^{-1}\right)\right| \leq 1 + 8 + 15 = 24.$$

Now, for $f \in \mathcal{S}^*$, taking modulus on both sides and then applying the triangle inequality in (5.3), we have

$$\left|T_{3,2}\left(f^{-1}\right)\right| \leq |t_2||t_2^2 - 2t_3^2| + |t_4||t_2t_4 - 2t_3^2|. \tag{5.9}$$

Now, plugging the values of $t_2$ and $t_3$, from (3.9) in the expression $t_2(t_2^2 - 2t_3^2)$, upon taking modulus and then applying Lemmas 2.1 and 2.2, after simplifying, we get

$$|t_2||t_2^2 - 2t_3^2| \leq 108. \tag{5.10}$$

Further, substituting $t_2$, $t_3$ and $t_4$, values from (3.9) in the expression $t_2t_4 - 2t_3^2$, it simplifies to give

$$t_2t_4 - 2t_3^2 := \frac{1}{6}(2c_1c_3 + 6c_1^2c_2 - 3c_2^2 - 11c_1^4), \tag{5.11}$$

which is equivalent to

$$t_2t_4 - 2t_3^2 := \frac{1}{6} \times \left(d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right), \tag{5.12}$$

$$\text{where}\quad d_1 = 2;\ d_2 = 6;\ d_3 = -3;\ d_4 = -11. \tag{5.13}$$

Substituting $c_2$ and $c_3$ values from Lemma 2.4 into the right-hand side of the expression (5.12), upon simplification, we obtain

$$4\left(d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right) = \left[(d_1 + 2d_2 + d_3 + 4d_4)c_1^4\right.$$
$$+2(d_1 + d_2 + d_3)c_1^2(4 - c_1^2)x - d_1c_1^2(4 - c_1^2)x^2 + d_3(4 - c_1^2)^2x^2$$
$$\left.+2d_1c_1(4 - c_1^2)(1 - |x|^2)y\right].$$

Taking modulus on both sides and then applying the triangle inequality in the above expression with $|y| \leq 1$, it takes the form

$$4\left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right| \leq \left[|d_1 + 2d_2 + d_3 + 4d_4||c_1|^4 + 2|d_1 + d_2 + d_3||c_1|^2|4 - c_1^2||x| + \left\{(|d_1| - |d_3|)|c_1|^2 - 2|d_1||c_1| + 4|d_3|\right\}|4 - c_1^2||x|^2 + 2|d_1||c_1||4 - c_1^2|\right]. \quad (5.14)$$

From (5.13), we can now write

$$|d_1 + 2d_2 + d_3 + 4d_4| = 33;\ |d_1 + d_2 + d_3| = 5;$$
$$\left\{(|d_1| - |d_3|)|c_1|^2 - 2|d_1||c_1| + 4|d_3|\right\} = (-|c_1|^2 - 4|c_1| + 12). \quad (5.15)$$

Putting the calculated values from (5.15) and the value of $d_1$ from (5.13) in the right hand side of the expression (5.14), after simplifying, we get

$$4\left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right| \leq \left[33|c_1|^4 + 4|c_1||4 - c_1^2| + 10|c_1|^2|4 - c_1^2||x| + (-|c_1|^2 - 4|c_1| + 12)|4 - c_1^2||x|^2\right]. \quad (5.16)$$

Choosing $|4 - c_1^2| = 4 - c^2$, $|c_1| = c \in [0, 2]$, and replacing $|x|$ by $\rho$, on the right side of (5.16), we obtain

$$4\left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right| \leq \left[33c^4 + 4c(4 - c^2) + 10c^2(4 - c^2)\rho + (2 - c)(c + 6)(4 - c^2)\rho^2\right] = B(c, \rho), \text{ for } |x| = \rho \in [0, 1]. \quad (5.17)$$

$$\text{Here } B(c, \rho) = 33c^4 + \left[4c + 10c^2\rho + (2 - c)(c + 6)\rho^2\right](4 - c^2). \quad (5.18)$$

To determine the maximum value of $B(c, \rho)$ over the rectanglular region $[0, 2] \times [0, 1]$, from (5.18), we consider the partial differential coefficient of $B(c, \rho)$, with respect to $\rho$, given by

$$\frac{\partial B}{\partial \rho} = [10c^2 + 2(2 - c)(c + 6)\rho](4 - c^2). \quad (5.19)$$

For $\rho \in (0, 1)$ and fixed $c \in (0, 2)$, from (5.19), we notice that $\frac{\partial B}{\partial \rho} > 0$, which indicates that $B(c, \rho)$ turns out to be an increasing function of $\rho$, and hence its maximum value occurrs only on the boundary, i.e., when $\rho = 1$. Therefore, for $\rho = 1$ in (5.18), which simplifies to give

$$F(c) = B(c, 1) = 24(c^4 + c^2 + 2). \quad (5.20)$$

In the expression (5.20), we observe that all the coefficients of $F(c)$ are positive, hence it becomes an increasing function of $c$, so that it attains maximum value at $c = 2$ only, which upon simplification gives

$$F(2) = B(2, 1) = 528. \quad (5.21)$$

Now, simplifying the expressions (5.17) and (5.21), in view of (5.12), we obtain

$$|t_2t_4 - 2t_3^2| \leq 22. \quad (5.22)$$

Substituting the derived values from (5.10), (5.22) and $|t_4|$ value from Theorem 3.2 on the right side of the expression (5.9), after simplifying, we get

$$\left|T_{3,2}\left(f^{-1}\right)\right| \leq 416.$$

**Remark 5.4.** For the extremal function $g_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + 2z^4 + ...$, we have $c_1 = c_2 = c_3 = 2$, consequently from (5.12) and (5.13), upon simplification, we obtain

$$\left(d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right) = -132.$$

**Remark 5.5.** The result (i) given under Theorem 5.3, coincides with that of Firoz Ali et al. [12] ( see Theorem 2.1). i.e., the sharp bound is invariant. The result can be stated as, if $f \in \mathcal{S}^*$, then

$$|T_{3,1}(f)| = |T_{3,1}(f^{-1})| \leq 24.$$

**Theorem 5.6.** *If $f \in \mathcal{C}$, then*

$$(i)\ \left|T_{3,1}\left(f^{-1}\right)\right| \leq 4. \qquad (ii)\ \left|T_{3,2}\left(f^{-1}\right)\right| \leq 4,$$

*and the bounds are best possible for the function given in Theorem 3.3.*

*Proof.* For $f \in \mathcal{C}$, upon substituting the values of $t_2$ and $t_3$ from (3.13) in (5.1), after simplifying, we get

$$T_{3,1}\left(f^{-1}\right) := 1 - \frac{c_1^2}{2} - \frac{1}{36}(c_2 - 2c_1^2)(c_2 + c_1^2). \tag{5.23}$$

Taking modulus on both sides, applying the traingle inequality and then using Lemmas 2.1 and 2.2, upon simplification, we obtain

$$\left|T_{3,1}\left(f^{-1}\right)\right| \leq 1 + 2 + \frac{36}{36} = 4.$$

Taking modulus on both sides in (5.3), we have

$$|T_{3,2}\left(f^{-1}\right)| = |t_2 - t_4||t_2^2 - 2t_3^2 + t_2 t_4|. \tag{5.24}$$

Now, performing the same procedure carried in Theorem 5.1, for $f \in \mathcal{C}$, we obtain

$$|t_2 - t_4| \leq |t_2| + |t_4| = \frac{|c_1|}{2} + \frac{\left|6c_1^3 - 7c_1c_2 + 2c_3\right|}{24} = 1 + \frac{2(12)}{24} = 1 + 1 = 2. \tag{5.25}$$

Further, for $f \in \mathcal{C}$, substituting $t_2$, $t_3$, and $t_4$ values from (3.13) in the expression $(t_2^2 - 2t_3^2 + t_2t_4)$, after simplifying, we get

$$\begin{aligned}(t_2^2 - 2t_3^2 + t_2t_4) =& \frac{1}{144}(-11c_1^4 + 11c_1^2c_2 + 6c_1c_3 - 8c_2^2 - 3c_1^4 + 36c_1^2).\\ =& \frac{1}{144}[c_1(-11c_1^3 + 11c_1c_2 + 6c_3) - 8c_2^2 + c_1^2(36 - 3c_1^2)]\end{aligned} \tag{5.26}$$

Now, taking modulus on both sides in (5.26), upon applying the triangle inequality, using Lemma 2.1 and generalisation of Lemma 2.3 on its right-hand side, we get

$$\begin{aligned}|t_2^2 - 2t_3^2 + t_2t_4| \leq& \frac{1}{144}\left\{|c_1||-11c_1^3 + 11c_1c_2 + 6c_3| + 8|c_2|^2 + |c_1|^2|36 - 3c_1^2|\right\}.\\ =& \frac{1}{144}(2 \times 2(28) + 32 + 4 \times 36) = \frac{(112 + 32 + 144)}{144} = 2\end{aligned}$$

Therefore,

$$|t_2^2 - 2t_3^2 + t_2t_4| \leq 2. \tag{5.27}$$

From the expressions (5.24), (5.25) and (5.27), we obtain

$$\left|T_{3,2}\left(f^{-1}\right)\right| \leq 4.$$

**Remark 5.7.** The results (i) and (ii) given under Theorem 5.6, coincides with that of Firoz Ali et al. [12] ( see Theorem 2.11). i.e., the sharp bound is invariant. The results can be stated as, if $f \in \mathcal{C}$, then

$$(i)\ \ |T_{3,1}(f)| = |T_{3,1}(f^{-1})| \leq 4. \qquad (ii)\ \ |T_{3,2}(f)| = |T_{3,2}(f^{-1})| \leq 4.$$

## 6 Sharp estimates for the Zalcman conjecture

In this section, we will prove two inequalities associated with the Zalcman conjecture for $f \in \mathcal{S}^*$.

**Theorem 6.1.** *If $f \in \mathcal{S}^*$, then*

$$(i)\ \ \left|t_2^2 - t_3\right| \leq 1 \qquad (ii)\ \ \left|t_3^2 - t_5\right| \leq 17,$$

*these bounds are sharp for the same function given in Theorem 3.2.*

*Proof.* (i) Let $f \in \mathcal{S}^*$, be of the form (1.1). Substituting the values of $t_2$ and $t_3$ from (3.9) into the expression $(t_3 - t_2^2)$, taking modulus on both sides and applying Lemma 2.2, we obtain

$$|t_3 - t_2^2| = \frac{1}{2}|c_2 - c_1^2| \leq \frac{2}{2} = 1.$$

(ii) Upon substituting the values of of $t_3$ and $t_5$ from (3.9) in the expression $(t_3^2 - t_5)$, it simplifies to give

$$t_3^2 - t_5 = \frac{1}{24}\left(6c_4 - 9c_2^2 - 71c_1^4 - 40c_1c_3 + 114c_1^2c_2\right).$$

After rearranging the terms in the above exression in order to apply Lemmas specified in this paper, we have

$$t_3^2 - t_5 = \frac{1}{24}\left[6(c_4 - c_2^2) - 40c_1(c_1^3 - 2c_1c_2 + c_3) - 3(c_2 - c_1^2)\left(c_2 - \frac{31}{3}c_1^2\right)\right]. \tag{6.1}$$

Taking modulus on both sides in (6.1), applying the triangle inequality and using Lemmas 2.1, 2.2 and 2.3, we arrive at

$$\begin{aligned}
|t_3^2 - t_5| &\leq \frac{1}{24}\left[6|c_4 - c_2^2| + 40|c_1||c_1^3 - 2c_1c_2 + c_3| + 3|c_2 - c_1^2|\left|c_2 - \frac{31}{3}c_1^2\right|\right]. \\
&= \frac{1}{24}\left[6(2) + 40(2)(2) + 3(2)(2)\left|2\left(\frac{31}{3}\right) - 1\right|\right]. \\
&= \frac{1}{24}[12 + 160 + 236] = \frac{408}{24} = 17.
\end{aligned}$$

□

## 7 Concluding Remarks and Observations

The problem of finding sharp estimates to the Toeplitz determinants for the inverse function of $f$, when it is a member of certain subclass of analytic functions is technically difficult in the case when $q = 3$ and $s = 1, 2$ in (1.3) defining $T_{q,s}(f^{-1})$, than that in the case when $q = 2; s = 1, 2, 3, 4$. Here, in our present investigation, we have successfully derived the sharp bound of the third-order Toeplitz determinants namely $T_{3,1}(f^{-1})$ and $T_{3,2}(f^{-1})$ for functions $f^{-1}$ in a subclass of normalized analytic functions in the open unit disc $\mathbb{D}$, which are the inverses of the functions belonging to the class of bounded turning, starlike and convex functions. We have also described several earlier developments on the subject, which are related to our main results. Moreover, The results obtained perhaps give an opportunity for researchers to determine an upper bound to determinants of the Toeplitz matrices ( may be of higher order) for functions of other families. As future research directions, the contents of the paper on the symmetric Toeplitz matrices associated with inverse function of $f$ given in (1.1) could inspire further research related to other classes.

## References


[1] O. P. Ahuja, K. Khatter, and V. Ravichandran, Toeplitz determinants associated with Ma-Minda classes of starlike and convex functions, Iran. J. Sci. Technol. Trans. A Sci., 45(6)(2021), pp. 2021–2027.

[2] R. M. Ali, Coefficients of the inverse of strongly starlike functions, Bull. Malays. Math. Sci. Soc. (second series), 26(2003), pp. 63–71.

[3] M. Arif, L. Rani, M. Raza, and P. Zaprawa, Fourth Hankel determinant for the set of star-like functions, Math. Probl. Eng., (2021), Article ID 6674010, DOI: https://doi.org/10.1155/2021/6674010.

[4] M. Arif, M. Raza, Huo Tang, Shehzad Hussain and Hassan Khan, Hankel determinant of order three for familiar subsets of analytic functions related with sine function, Open Math., 17(1)(2019), pp. 1615–1630.

[5] M. Arif, L. Rani, M. Raza, and P. Zaprawa, Fourth Hankel determinant for the family of functions with bounded turning, Bull. Korean Math. Soc., 55 (6) (2018), pp. 1703–1711.

[6] R. Ayinla, and R. Bello, Toeplitz determinants for a subclass of analytic functions, J. Progress. Res. Math., 18(1)(2021), pp. 99–106.

[7] K. O. Babalola, On $H_3(1)$ Hankel determinant for some classes of univalent functions, In: Inequality Theory and Applications, Vol. 6 (Y. J. Cho, Editor), Nova Science Publishers, New York, 2010, pp. 1–7.

[8] S. Banga and S. S. Kumar, The sharp bounds of the second and third Hankel determinants for the class $\mathcal{SL}^*$, Math. Slovaca, 70 (2020), pp. 849–862.

[9] Daud Mohamad, and Nur Hazwani Aqilah Abdul Wahid, Bounds on Toeplitz determinant for starlike functions with respect to conjugate points, Int. J. Anal. Appl., 19 (3) (2021), pp. 477–493, DOI: https://doi.org/ 10.28924/2291-8639-19-2021-477.

[10] P. L. Duren, Univalent functions (Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo), Springer-Verlag, 1983.

[11] Farah Zulfiqar, Sarfraz Nawaz Malik, Mohsan Raza, and Md. Shajib Ali, Fourth-Order Hankel Determinants and Toeplitz Determinants for Convex Functions Connected with Sine Functions, J. Math., (2022), Article ID 2871511, pp. 1–12, DOI: https://doi.org/10.1155/2022/2871511.

[12] MD. Firoz Ali, D. K. Thomas, and A. Vasudevarao, Toeplitz determinants whose elements are the coefficients of analytic and univalent functions, Bull. Aust. Math. Soc., 97(2)(2018), pp. 253–264, DOI: https://doi.org/10.1017/S0004972717001174.

[13] S. Giri and S. S. Kumar, Toeplitz Determinants for a Class of Holomorphic Mappings in Higher Dimensions, Complex Anal. Oper. Theory, 14(6)(2023), pp. 1–16.

[14] Hai-Yan Zhang, and Huo Tang, Fourth Toeplitz determinants for starlike functions defined by using the sine function, J. Funct. Spaces,( 2021), Article ID 4103772, pp.1–7, DOI: https://doi.org/10.1155/2021/4103772.

[15] Hai-Yan Zhang, Rekha Srivastava, and Huo Tang, Third-order Hankel and Toeplitz determinants for starlike functions connected with the sine function, Mathematics, 7(5)(2019), pp.1–10, DOI: https://doi.org/ 10.3390/math7050404.

[16] T. Hayami and S. Owa, Generalized Hankel determinant for certain classes, Int. J. Math. Anal., 4(52)(2010), pp. 2573–2585.

[17] Huo Tang, Ihtesham Gul, Saqib Hussain, and Saima Noor, Bounds for Toeplitz Determinants and Related Inequalities for a New Subclass of Analytic Functions, Mathematics, 2023, 11, 3966, pp.1–15, DOI: https://doi.org/ 10.3390/math11183966.

[18] Isra Al-shbeil, Jianhua Gong, Shahid Khan, Nazar Khan, Ajmal Khan, Mohammad Faisal Khan, and Anjali Goswami, Hankel and symmetric Toeplitz determinants for a new subclass of $q$-starlike functions, Fractal Fract., 2022, 6, 658(2022), pp. 1–16, DOI: https://doi.org/10.3390/fractalfract6110658.

[19] A. Janteng, S. A. Halim and M. Darus, Hankel Determinant for starlike and convex functions, Int. J. Math. Anal., 1(13)(2007), pp. 619–625.

[20] A. Janteng, S. A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, J. Inequal. Pure Appl. Math., 7(2)(2006), pp. 1–5.

[21] G. P. Kapoor and A. K. Mishra, Coefficient estimates for inverses of starlike functions of positive order, J. Math. Anal. Appl., 329(2007), pp. 922–934.

[22] G. Kaur, G. Singh, M. Arif, R. Chinram, and J. Iqbal, A study of third and fourth Hankel determinant problem for a particular class of bounded turning functions, Math. Probl. Eng. (2021), Article ID 6687805, DOI: https://doi.org/10.1155/2021/6687805.

[23] J. G. Krzyz, R. J. Libera and E. J. Zlotkiewicz, Coefficients of inverse of regular starlike functions, Ann. Univ. Marie Curie-Sklodowska Sect A., 33(10)(1979), pp. 103–109.

[24] A. K. Jha, and P. Sahoo, Upper bounds of Toeplitz determinants for a subclass of alpha-close-to-convex functions, Creat. Math. Inform., 31(1)(2022), pp. 81–90.

[25] O. S. Kwon, A. Lecko and Y. J. Sim, The bound of the Hankel determinant of the third kind for starlike functions, Bull. Malays. Math. Sci. Soc., 42(2019), pp. 767–780.

[26] B. Kowalczyk, A. Lecko and Y. J. Sim, The sharp bound for the Hankel determinant of the third kind for convex functions, Bull. Austral. Math. Soc., 97(2018), pp. 435–445.

[27] R. J. Libera, and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$, Proc. Amer. Math. Soc., 87(2)(1983), pp. 251–257.

[28] R. J. Libera, and E. J. Zlotkiewicz, Early Coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc., 85(2)(1982), pp. 225–230.

[29] A. E. Livingston, The coefficients of multivalent close-to-convex functions, Proc. Amer. Math. Soc., 21(3)(1969), pp. 545–552.

[30] Luminita I. Cotirla, and A. K. Wanas, Symmetric Toeplitz matrices for a new family of prestarlike functions, Symmetry 14(7)(2022), p. 1413.

[31] Muhammat Kamali, and Alina Riskulova, On bounds of Toeplitz determinants for a subclass of analytic functions, Bull. Math. Anal. Appl., 14(3)(2022), pp. 36–48.

[32] Musa Josiah Marut, Shamani Supramaniam and Maisarah Haji Mohd Marut, Symmetric Toeplitz determinants for starlike and convex functions defined by subordination, AIP Conf. Proc., 2423, 060003 (2021), pp. 1-9, DOI: https://doi.org/10.1063/5.0075331.

[33] B. Nandhini and B. S. Keerthi, Toeplitz determinant for sakaguchi type functions under petal shaped domain, Aust. J. Math. Anal. Appl.,19(2)(2022), pp. 4–8.

[34] Nur Hazwani Aqilah Abdul Wahid, Daud Mohamad, Nur Maziah Kamarozzaman and Ainurraziqin Ahamad Shahminan, Toeplitz Determinants for the Class of Functions with Bounded Turning, European J. Pure Appl. Math., 15(4)(2022), pp. 1937–1947.

[35] Ni Made Asih, Saadatul Fitri, Ratno Bagus Edy Wibowo and Marjono, Hankel Determinant and Toeplitz Determinant on the Class of Bazilevic Functions Related to the Bernoulli Lemniscate, European J. Pure Appl. Math., 16(2)(2023), pp. 1290–1301.

[36] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[37] Ch. Pommerenke, On the Hankel determinants of Univalent functions, Mathematika, 14(1967), pp. 108–112.

[38] Ch. Pommerenke, On the coefficients and Hankel determinants of univalent functions, J. Lond. Math. Soc., 41(s-1)(1966), pp. 111–122.

[39] V. Radhika, Jay M. Jahangiri, S. Sivasubramanian, and G. Murugusundaramoorthy, Toeplitz matrices whose elements are coefficients of Bazilevic functions, Open Math., 16(1)(2018), pp. 1161–1169.

[40] B. Rath, K. S. Kumar, D. V. Krishna and A. Lecko, The sharp bound of the third Hankel determinant for starlike functions of order 1/2, Complex Anal. Oper. Theory, 16(2022), Article ID 65, pp. 1–8.

[41] H. Rahmatan, Bounds of the fifth Toeplitz determinant for the classes of functions with bounded turnings, Int. J. Nonlinear Anal. Appl., 14(7) (2023), pp.99–106, DOI: https://doi.org/10.22075/IJNAA.2022.24382.2732.

[42] Rubab Nawaz, Rabia Fayyaz, Daniel Breaz, and L. I. Cotirla, Sharp coefficient estimates for analytic functions associated with lemniscate of Bernoulli, Mathematics (2024) 12, DOI: https://doi.org/10.3390/math12152309.

[43] Y. J. Sim, A. Lecko and D. K. Thomas, The second Hankel determinant for strongly convex and Ozaki close-to-convex functions, Ann. Mat. Pura Appl., (4) (1923 -), 200(6)(2021), pp. 2515–2533, DOI: https://doi.org/10.1007/s10231-021-01089-3.

[44] S. Sivasubramanian, M. Govindaraj, and G. Murugusundaramoorthy. Toeplitz matrices whose elements are the coefficients of analytic functions belonging to certain conic domain, Int. J. Pure Appl. Math., 109(10)(2016), pp. 39–49.

[45] H. M. Srivastava, Biswajit Rath, K. Sanjay Kumar, and D. Vamshee Krishna, Some sharp bounds of the third-order Hankel determinant for the inverses of the Ozaki type close-to-convex functions, Bull. Sci. Math. 191 (2024), pp. 1–19. Article ID 103381. DOI: https://doi.org/10.1016/j.bulsci.2023.103381.

[46] H. M. Srivastava, G. Murugusundaramoorthy, and T. Bulboacă, The second Hankel determinant for subclasses of bi-univalent functions associated with a nephroid domain, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) (116)(2022), Article ID 145, pp. 1–21.

[47] H. M. Srivastava, G. Kaur, and G. Singh, Estimates of the fourth Hankel determinant for a class of analytic functions with bounded turnings involving cardiod domains, J. Nonlinear Convex Anal., 22 (2021), pp. 511–526.

[48] D. K. Thomas and S. A. Halim, Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions, Bull. Malays. Math. Sci. Soc., 40(4) (2017), pp.1781–1790.

[49] A. Vasudevarao, A. Lecko and D. K. Thomas, Hankel, Toeplitz and Hermitian-Toeplitz Determinants for Ozaki Close-to-convex Functions, Mediterr. J. Math.,(2022), DOI: https://doi.org/10.1007/s00009-021-01934-y.

[50] A. K. Wanas, F. M. Sakar, G. I. Oros, and L. I. Cotirla, Toeplitz determinants for a certain family of analytic functions endowed with Borel distribution, Symmetry, 15,262 (2023), pp.1–9, DOI: https://doi.org/10.3390/sym15020262.

[51] Winne Bareh, D. Vamshee Krishna and B. Rath,Sharp estimates of the Toeplitz determinants of certain order for the primary subclasses of univalent functions, Mat. Stud., 64(1)(2025), pp.15–22.

[52] Winne Bareh, D. Vamshee Krishna and B. Rath, Exact bounds to the Toeplitz determinants of certain order, Zalcman conjecture and Krushkals inequalities for the functions associated with the lemniscate of Bernoulli, Bull. Sci. math., 199 (2025), Article ID 103585, pp.1–16.

[53] K. Ye and L. H. Lim, Every matrix is a product of Toeplitz matrices, Found. Comput. Math. 16 (2016), pp. 577–598.

[54] P. Zaprawa, M. Obradović and N. Tuneski, Third Hankel determinant for univalent starlike functions, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 115(2021), Article ID 49, pp.1–6.

## Author information

B. Winne, Department of Mathematics, School of Physical Sciences, North-Eastern Hill University (NEHU), Umshing Mawkynroh, Shillong 793022, Meghalaya, Bharath (India), India.
E-mail: winnebareh5@gmail.com

D. Vamshee Krishna, Department of Mathematics, School of Physical Sciences, North-Eastern Hill University (NEHU), Umshing Mawkynroh, Shillong 793022, Meghalaya, Bharath (India), India.
E-mail: vamsheekrishna1972@gmail.com

Biswajit Rath, Department of Mathematics, GITAM School of Science, GITAM University, Visakhapatnam 530045, Andhra Pradesh, Bharath (India), India.
E-mail: urwithbr@gmail.com

Received: 2025-10-25
Accepted: 2026-02-13