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# On the Blow-Up Behavior of Solutions to a Nonlinear Damped Viscoelastic Petrovsky Equation with Internal Distributed Delay and Polynomial Nonlinearity Effects

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**Abstract.** This research is concerned with the following nonlinear viscoelastic Petrovsky equation

$$\begin{cases} |\omega_t(x,t)|^l\omega_{tt}(x,t) + \Delta^2\omega(x,t) - \Delta\omega_{tt}(x,t) - \int_0^t \Re(t-s)\Delta^2\omega(x,s)\,ds + \Psi_1\omega_t(x,t) \\ + \int_{\tau_1}^{\tau_2} |\Psi_2(q)|\omega_t(x,t-q)\,dq = b|\omega|^{p-2}\omega. \quad (x,t) \in \Omega \times (0,\infty) \end{cases}$$

In the presence of a distributed delay and a polynomial nonlinearity, we employ the multiplier method and, under appropriate conditions, prove that the energy of any weak solution blows up in finite time if the initial energy is negative.

## 1 Introduction

The Petrovsky equation originates from the mid-20 the century work of Ivan G. Petrovsky on parabolic and hyperbolic PDEs. Though initially focused on diffusion and wave equations, his conditions for parabolicity and regularity laid the groundwork for later viscoelastic extensions.
In applied mathematics, the Petrovsky viscoelastic equation is widely applied to model wave propagation in polymers, seismic waves in the Earth's crust, biomechanical tissues, and damped vibrations in engineering structures. The inclusion of memory terms reflects how past deformations influence present behavior, which is critical to accurately predict creep, stress relaxation, and attenuation phenomena. The theory has been further developed in the works of **Ladyzhenskaya et al. (1968)** [16], **Dafermos (1970)** [6], and **Tikhonov and Samarskii (1990)** [34], who extended Petrovsky's analytical tools to broader classes of evolution equations in continuum mechanics.
Physically, the Petrovsky equation plays a significant role in modeling heat conduction, mass diffusion, and viscous fluid flow. In contrast to the classical heat equation, it accounts for inhomogeneous materials and time-dependent sources, making it well suited for complex systems in thermodynamics, plasma physics and porous media flow [34]. Furthermore, it is well known that delay efects can destroy the stabilizing properties of a well-behaved system. In the literature, there are several examples that illustrate how time delays destabilize some internal or boundary control system [3, 9, 11, 14, 31]. Viscous materials are the opposite of elastic materials that possesss the ability to dissipate and store mechanical [1, 20]. The mechanical properties of these viscous substances are of great importance when they seem in many applications of natural sciences [5, 19, 29].

Problems involving PDEs with time-delay effects are of growing interest due to their wide application in physical, biological, and engineering systems. Time delays can destabilize otherwise stable systems.

In (2008)Nicaise and Pignotti [27] demonstrated that the energy associated with the solution of the following wave equation with distributed delay on the boundary decays at an exponential rate

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0, \text{ in } \Omega \times (0,\infty) \\ u(x,t) = 0, \quad on\ \Gamma_0 \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) + \int_{\tau_1}^{\tau_2} \mu(s) u_t(x,t-s)\, ds + \mu_0(s) u_t(x,t) = 0, \ on\ \Gamma_1 \times (0,\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,-t) = f_0(x,t), \quad (x,t) \in \Omega \times (0,\tau_2), \end{cases}$$

by assuming that

$$\mu_0 > \|a\|_\infty \int_{\tau_1}^{\tau_2} \mu(s) ds.$$

Mustafa and Kafini [24] proved the exponential stability of the following thermoelastic system with internal distributed delay

$$\begin{cases} a u_{tt}(x,t) - d u_{xx}(x,t) + \beta \theta_x = 0, \text{ in } (0,L) \times (0,\infty) \\ b\theta_x - k_1 \theta_{xx} - \int_{\tau_1}^{\tau_2} k_2(s) \theta_{xx}(x,t-s)\, ds + \beta u_{xt}(x,t) = 0, \text{ in } (0,L) \times (0,\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad x \in (0,L), \\ \theta_x(x,-t) = f_0(x,t), \quad (x,t) \in (0,L) \times (0,\tau_2). \end{cases}$$

A similar result was obtained by the authors in [25]. We also refer to the work of Apalara [2] on the Timoshenko system with second sound and internal distributed delay, as well as the work of Messaoudi et al. [22] concerning a wave equation with strong damping and a constant time delay defined by

$$u_{tt}(x,t) - \Delta u(x,t) - \mu_1 \Delta u_t(x,t) - \mu_2 \Delta u_t(x,t-\tau) = 0, \text{ in } \Omega \times (0,\infty).$$

They proved a global existence and established exponential decay result. Furthermore, they studied a general stability estimates by using some properties of convex functions. The control of PDEs with has become an active area of research, see for example [10, 12, 13, 15, 36].

Benaissa et al. [4] proved the existence of global solution as well as a general stability result for the following wave equation with a constant weak delay and a weak internal feedback

$$u_{tt}(x,t) - \Delta u(x,t) - \int_0^t h(t-s)\, \Delta u(s)\, ds + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0, \text{ in } \Omega \times (0,\infty),$$

when $h$ is of general decay rate and $g_1, g_2$ are non-linear functions.In this context, we can mention a set of related works [7, 26].

Mezouar et al. [23] studied the following problem

$$|u_t(x,t)|^l u_{tt}(x,t) + \Delta^2 u(x,t) - \Delta u_{tt}(x,t) - \int_0^t h(t-s) \Delta^2 u(x,s)\, ds + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0, \quad (x,t) \in \Omega \times (0,\infty),$$

they showed the global existence of weak solutions using the Faedo–Galerkin method, and obtained general stability estimates by introducing Lyapunov method, then in (2019) the authors in [32] studied the following problem

$$|u_t(x,t)|^l u_{tt}(x,t) + \Delta^2 u(x,t) - \Delta u_{tt}(x,t) - \int_0^t h(t-\sigma) \Delta^2 u(x,\sigma)\, d\sigma + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x,t-s)\, ds = 0, \quad (x,t) \in \Omega \times (0,\infty).$$

They employed the energy method combined with the Faedo–Galerkin approximation, under conditions on the damping weight and the weight of distributed delay. The global existence

result was established. Also, they studied the exponential stability of the solution by introducing a suitable Lyapunov functional, see [8,21,35].

To investigate the blow-up behavior of solutions with a polynomial source term, the authors in [17]examined the following Petrovsky equation

$$u_{tt}(x,t)+\Delta^2 u(x,t)-\Delta u_t\,(x,t)-u_t\,|u_t|^{m-1}=u\,|u|^{p-1}\,,\ \text{in}\ \Omega\times(0,\infty),$$

they obtained the global existence and uniform decay of solutions if the initial data are in some stable set without any interaction between the damping mechanism $u_t\,|u_t|^{m-1}$ and the source term $u\,|u|^{p-1}$.

Moreover, they established the blow up properties of local solution in the case $p>m$where the initial energy is less than the potential well depth.

Recently, Peyravi and Tahamtani [28] considered the following nonlinear damped viscoelastic Petrovsky equation with polynomial source term

$$u_{tt}+\Delta^2 u-\Delta u_t-\Delta u_{tt}-\int_0^t g(t-\tau)\,\Delta^2 u\,(\tau)\,d\tau+u_t\,|u_t|^{m-1}=u\,|u|^{p-1}\,,\ \text{in}\ \Omega\times(0,\infty),$$

under suitable assumptions on the function [27] and some different cases on $m$, they proved the polynomial-exponential stability and blow-up solutions. In (2024) Yılmaz and Pişkin considered the following a logarithmic Petrovsky equations with variable exponents

$$u_{tt}+\Delta^2 u+u_t\,|u_t|^{p(x)-2}=|u|^{q(x)-2}\,u\ln u,\ \text{in}\ \Omega\times(0,T),$$

they proved the blow-up of solutions see [38] We may also recall the related recent works in [18,30,33,37].

In the current work, based on the work [32]. We consider the following nonlinear Petrovsky plate equation incorporating viscoelastic damping, a distributed delay term, and a polynomial-type nonlinearity.

$$\begin{cases}|\omega_t(x,t)|^l\omega_{tt}(x,t)+\Delta^2\omega(x,t)-\Delta\omega_{tt}(x,t)-\int_0^t\Re(t-s)\Delta^2\omega(x,s)\,ds+\Psi_1\omega_t(x,t)\\ +\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\omega_t(x,t-q)\,dq=b|\omega|^{p-2}\omega,\quad (x,t)\in\Omega\times(0,\infty)\\ \omega(x,t)=\dfrac{\partial\omega}{\partial\eta}(x,t)=0,\quad x\in\partial\Omega,\\ \omega(x,0)=\omega_0(x),\quad \omega_t(x,0)=\omega_1(x),\quad x\in\Omega,\\ \omega_t(x,-t)=f_0(x,t),\quad (x,t)\in\Omega\times(0,\tau_2).\end{cases}\tag{1.1}$$

Where $\omega(x,t)$is the displacement of a particle at position $x$ in a bounded domain $\Omega\subset\mathbb{R}^n$, with a smooth boundary $\partial\Omega$ ,where $n\geq 1$ and $b,l>0,p>2$. Moreover, $\Psi_1$ is a positive constant, $\Re$ is a positive non-increasing function defined on $\mathbb{R}^+$ satisfying assumptions $(A1)-(A3)$ and the initial data $(\omega_0,\omega_1,f_0)$ belonging to a suitable function space. $\tau_1,\tau_2$ are the time delay with $0\leq\tau_1<\tau_2$, $\Psi_2$ is an $L^\infty$ function $\eta$ is the unit outward normal vector. we showed that the energy of any weak solution blows up in finite time if the initial energy is negative.

This paper is structured as follows: Section 2 presents the necessary notations, preliminary lemmas, and the main results. In Section 3, we establish the blow-up result .

## 2 Preliminaries

Let $\|\cdot\|_p$ denote the $L^p$ norm, and $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$. We write $\|\cdot\|$ instead of $\|\cdot\|_2$.

We state the following assumptions:

(A1) $\Re:\mathbb{R}^+\to\mathbb{R}^+$ is decreasing, differentiable, and satisfies

$$\Re(t)\geq 0,\quad 1-\int_0^\infty\Re(s)\,ds=\ell>0.\tag{2.1}$$

(A2) There exists $\xi > 0$ such that

$$\Re'(t) \leq -\xi \Re(t), \quad t \geq 0. \tag{2.2}$$

(A3) $\Psi_2 \in L^\infty([\tau_1, \tau_2])$ such that

$$\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\Psi_2(q)|\, dq \leq \Psi_1, \quad \delta > \frac{1}{2}. \tag{2.3}$$

Let $B_p > 0$ is the smallest embedding constant satisfying,

$$\|\nabla v\|_p \leq B_p \|\Delta v\|_p. \quad v \in H_0^2(\Omega) \tag{2.4}$$

We

$$(\Re o \Delta\omega)(t) = \int_\Omega \int_0^t \Re(t-s)|\Delta\omega(t) - \Delta\omega(s)|^2\, ds\, dx. \tag{2.5}$$

Firstly, similar to [11], we introduce a new variable:

$$z(x, \rho, q, t) = \omega_t(x, t - q\rho),$$

satisfying

$$\begin{cases} qz_t + z_\rho = 0, \\ z(x, 0, q, t) = \omega_t(x, t). \end{cases} \tag{2.6}$$

Then, the original problem becomes equivalent to:

$$\begin{cases} |\omega_t|^l \omega_{tt} + \Delta^2\omega - \Delta\omega_{tt} - \displaystyle\int_0^t \Re(t-s)\Delta^2\omega(s)\, ds + \Psi_1\omega_t \\ \quad + \displaystyle\int_{\tau_1}^{\tau_2} |\Psi_2(q)||z(x, 1, q, t)|\, dq = b|\omega|^{p-2}\omega, \\ qz_t + z_\rho = 0, \end{cases} \tag{2.7}$$

with the initial and boundary conditions:

$$\begin{cases} \omega(x, t) = \dfrac{\partial\omega}{\partial\eta}(x, t) = 0, \quad x \in \partial\Omega, \\ z(x, \rho, q, 0) = f_0(x, q\rho), \\ \omega(x, 0) = \omega_0(x), \quad \omega_t(x, 0) = \omega_1(x). \end{cases} \tag{2.8}$$

Where

$$(x, \rho, q, t) \in \Omega \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

**Theorem 2.1.** *Suppose that* (2.1), (2.2), *and* (2.3) *hold. Let*

$$\begin{cases} p \geq 2, \quad n = 1, 2, 3, 4, \\ 2 < p < \dfrac{2(n-2)}{n-4}, \quad n \geq 5. \end{cases} \tag{2.9}$$

*Then, for any initial data*

$$(\omega_0, \omega_1, f_0) \in (H_0^2(\Omega) \times H_0^2(\Omega) \times L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2)),$$

*the problem* (2.7)–(2.8) *has a unique solution*

$$\omega \in C([0, T]; H_0^2(\Omega)),$$

*for some* $T > 0$.

The corresponding energy functional is defined in the following lemma:

**Lemma 2.2.** *Assume that (2.1)–(2.2) and (2.10) hold. Let $\omega$ be a solution of (2.7)–(2.8). Then the energy $E(t)$ is non-increasing and satisfies*

$$E(t) = \frac{1}{l+2}\|\omega_t(t)\|^{l+2} + \frac{1}{2}\left(1 - \int_0^t \Re(s)\,ds\right)\|\Delta\omega(t)\|^2 + \frac{1}{2}(\Re\circ\Delta\omega)(t) + \frac{1}{2}\|\nabla\omega_t(t)\|^2 \tag{2.10}$$

$$+ \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\Psi_2(q)|\,|z(x,\rho,q,t)|^2\,dq\,d\rho\,dx - \frac{b}{p}\|\omega(t)\|_p^p,$$

*and*

$$E'(t) \le -c_1\left(\|\omega_t(t)\|^2 + \int_\Omega\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\,|z(x,1,q,t)|^2\,dq\,dx\right). \tag{2.11}$$

*Proof.* Multiplying the first equation of (2.7) by $\omega_t$ and integrating over $\Omega$, we obtain:

$$\frac{d}{dt}\left\{\frac{1}{l+2}\|\omega_t\|^{l+2} + \frac{1}{2}\left(1 - \int_0^t \Re(s)\,ds\right)\|\Delta\omega\|^2 + \frac{1}{2}(\Re\circ\Delta\omega)(t) + \frac{1}{2}\|\nabla\omega_t(t)\|^2 - \frac{b}{p}\|\omega\|_p^p\right\} \tag{2.12}$$

$$= -\Psi_1\|\omega_t\|^2 - \int_\Omega \omega_t\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\,|z(x,1,q,t)|\,dq\,dx + \frac{1}{2}(\Re'\circ\Delta\omega)(t) - \frac{1}{2}\Re(t)\|\Delta\omega\|^2.$$

Similarly, we compute the derivative of the memory term:

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\Psi_2(q)|\,|z(x,\rho,q,t)|^2\,dq\,d\rho\,dx \qquad (2.13)\\
&= -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} 2|\Psi_2(q)|\,z(x,\rho,q,t)\,z_\rho(x,\rho,q,t)\,dq\,d\rho\,dx\\
&= \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\,|z(x,0,q,t)|^2\,dq\,dx - \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\,|z(x,1,q,t)|^2\,dq\,dx\\
&= \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\,dq\right)\|\omega_t\|^2 - \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\,|z(x,1,q,t)|^2\,dq\,dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{d}{dt}E(t) = &-\Psi_1\|\omega_t\|^2 - \int_\Omega\int_{\tau_1}^{\tau_2}\Psi_2(q)|\omega_t z(x,1,q,t)|\,dq\,dx\\
&- \frac{1}{2}\Re(t)\|\Delta\omega\|^2 + \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\Psi_2(q)\,dq\right)\|\omega_t\|^2\\
&+ \frac{1}{2}(\Re'\circ\Delta\omega)(t) - \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\,|z(x,1,q,t)|^2\,dq\,dx. \qquad (2.14)
\end{aligned}$$

Combining (2.13) and (2.14) , we obtain (2.11). Then, using (A1)-(A3), Young's inequality, (2.1) to (2.3) and (2.15) we deduce (2.12). □

**Lemma 2.3.** *( [5]) There exists a constant $c > 0$, depending only on $\Omega$ such that*

$$\left(\int_\Omega|\omega|^p\,dx\right)^{s/p} \le c\left(\|\nabla\omega\|^2 + \|\omega\|_p^p\right), \tag{2.15}$$

*for all $\omega \in L^{p+1}(\Omega)$ and $2 \le s \le p$.*

Using the Sobolev embedding, we derive the following:

**Corollary 2.4.** *There exists $c > 0$, depending only on $\Omega$ such that*

$$\left(\int_\Omega|\omega|^p\,dx\right)^{s/p} \le c\left(\|\Delta\omega\|^2 + \|\omega\|_p^p\right), \tag{2.16}$$

*for all $\omega \in L^{p+1}(\Omega)$ and $2 \le s \le p$.*

Also, using the fact that $\|\omega\|_2^2 \leq c\|\omega\|_p^2 \leq c\left(\|\omega\|_p^p\right)^{2/p}$, we have:

**Corollary 2.5.** *There exists $C > 0$, depending only on $\Omega$ such that*

$$\|\omega\|_2^2 \leq c\left(\|\Delta\omega\|_2^{4/p} + \left(\|\omega\|_p^p\right)^{2/p}\right). \tag{2.17}$$

**Lemma 2.6** ( [5]). *There exists $C > 0$, depending only on $\Omega$ such that*

$$\|\omega\|_p^s \leq C\left(\|\nabla\omega\|^2 + \|\omega\|_p^p\right), \tag{2.18}$$

*for all $\omega \in L^{p+1}(\Omega)$ and $2 \leq s \leq p$.*

This gives:

**Corollary 2.7.** *There exists $C > 0$, depending only on $\Omega$ such that*

$$\|\omega\|_p^s \leq c\left(\|\Delta\omega\|^2 + \|\omega\|_p^p\right), \tag{2.19}$$

*for all $\omega \in L^{p+1}(\Omega)$ and $2 \leq s \leq p$.*

We now define the following functional:

$$\begin{aligned} H(t) = -E(t) = \frac{b}{p}\|\omega\|_p^p - \frac{1}{l+2}\|\omega_t\|^{l+2} - \frac{1}{2}\left(1 - \int_0^t \Re(s)\,ds\right)\|\Delta\omega\|^2 - \frac{1}{2}\|\nabla\omega_t(t)\|^2 \\ - \frac{1}{2}(\Re\circ\Delta\omega)(t) - \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\Psi_2(q)|\,|z(x,\rho,q,t)|^2\,dq\,d\rho\,dx. \end{aligned} \tag{2.20}$$

## 3 Blow-up results

In this section, we establish the blow-up of solutions for the problem (2.7)–(2.8).

**Theorem 3.1.** *Suppose that assumptions (2.1)–(2.3) and (2.10) hold. Suppose further that $E(0) < 0$. Then, the solution of the problem (2.7)–(2.8) blows up in finite time.*

*Proof.* From (2.11), we have:

$$E(t) \leq E(0) < 0. \tag{3.1}$$

Hence,

$$H'(t) = -E'(t) \geq c_1\left(\|\omega_t\|^2 + \int_\Omega\int_{\tau_1}^{\tau_2} |\Psi_2(q)|\,|z(x,1,q,t)|^2\,dq\,dx\right) \geq 0, \tag{3.2}$$

and

$$0 \leq H(0) \leq H(t) \leq \frac{b}{p}\|\omega\|_p^p. \tag{3.3}$$

Set the auxiliary functional:

$$K(t) = H^{1-\alpha}(t) + \frac{\varepsilon}{l+1}\int_\Omega |\omega_t|^{l+1}\omega\,dx + \frac{\varepsilon\Psi_1}{2}\int_\Omega \omega^2\,dx + \varepsilon\int \nabla\omega_t\nabla\omega dx, \tag{3.4}$$

where $\varepsilon > 0$ is small and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{3.5}$$

We multiply the first equation of (2.7) by $\omega$ and differentiate (3.4) to obtain:

$$\begin{aligned} K'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \frac{\varepsilon}{l+1}\|\omega_t\|^{l+2} + \varepsilon\int_\Omega \Delta\omega\int_0^t \Re(t-s)\Delta\omega(s)\,ds\,dx \\ - \varepsilon\|\Delta\omega\|^2 + \varepsilon\|\nabla\omega_t\|^2 + \varepsilon b\|\omega\|_p^p\,dx - \varepsilon\int_\Omega\int_{\tau_1}^{\tau_2} |\Psi_2(q)|\,|\omega z(x,1,q,t)|\,dq\,dx. \end{aligned} \tag{3.6}$$

For the last term, by Young's inequality:

$$\varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\Psi_2(q)|\, |\omega z(x,1,q,t)|\, dq\, dx \le \varepsilon\delta_1 \left( \int_{\tau_1}^{\tau_2} |\Psi_2(q)|\, dq \right) \|\omega\|^2 + \frac{\varepsilon}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\Psi_2(q)|\, |z(x,1,q,t)|^2\, dq\, dx, \tag{3.7}$$

Also, using the inequality:

$$\varepsilon \int_\Omega \Delta\omega \int_0^t \Re(t-s)\Delta\omega(s)\, ds\, dx \le \frac{\varepsilon}{2} \int_0^t \Re(s)\, ds\, \|\Delta\omega\|^2 + \frac{\varepsilon}{2}(\Re \circ \Delta\omega)(t). \tag{3.8}$$

combining with (3.6), we obtain

$$\begin{aligned} K'(t) \ge{}& (1-\alpha)H^{-\alpha}(t)H'(t) + \frac{\varepsilon}{l+1}\|\omega_t\|^{l+2} - \varepsilon\left(1 - \frac{1}{2}\int_0^t \Re(s)ds\right)\|\Delta\omega\|^2 \\ &+ \varepsilon\|\nabla\omega_t\|^2| + \varepsilon b\|\omega\|_p^p\, dx - \varepsilon\delta_1 \left( \int_{\tau_1}^{\tau_2} |\Psi_2(q)|\, dq \right) \|\omega\|^2 \\ &- \frac{\varepsilon}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\Psi_2(q)|\, |z(x,1,q,t)|^2\, dq\, dx + \frac{\varepsilon}{2}(\Re \circ \Delta\omega)(t). \end{aligned} \tag{3.9}$$

By using (3.2) and setting $\delta$ such that $\frac{1}{4\delta c_1}=\kappa H^{-\alpha}(t)$, we obtain

$$\begin{aligned} K'(t) \ge{}& \left[(1-\alpha) - \varepsilon\kappa\right] H^{-\alpha}(t)H'(t) + \frac{\varepsilon}{l+1}\|\omega_t\|^{l+2} \\ &- \varepsilon\left(1 - \frac{1}{2}\int_0^t \Re(s)ds\right)\|\Delta\omega\|^2 + \varepsilon\|\nabla\omega_t\|^2 + \varepsilon b\|\omega\|_p^p \\ &- \varepsilon\frac{H^\alpha(t)}{4c_1\kappa}\left( \int_{\tau_1}^{\tau_2} |\Psi_2(q)|\, dq \right)\|\omega\|^2 + \frac{\varepsilon}{2}(\Re \circ \Delta\omega)(t). \end{aligned} \tag{3.10}$$

From (2.20), For $0 < a < 1$,

$$\begin{aligned} \varepsilon b\|\omega\|_p^p = {}& \varepsilon p(1-a)H(t) + \frac{\varepsilon p(1-a)}{l+2}\|\omega_t\|^{l+2} + \varepsilon ba\|\omega\|_p^p \\ &+ \frac{\varepsilon p(1-a)}{2}\left(1 - \int_0^t \Re(s)\, ds\right)\|\Delta\omega\|^2 \\ &+ \frac{\varepsilon}{2}p(1-a)(\Re \circ \Delta\omega)(t) \quad + \frac{\varepsilon p(1-a)}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\Psi_2(q)|\, |z(x,\rho,q,t)|^2\, dq\, d\rho\, dx. \end{aligned} \tag{3.11}$$

With (3.10) it follows that

$$\begin{aligned} K'(t) \ge{}& \left[(1-\alpha) - \varepsilon\kappa\right] H^{-\alpha}(t)H'(t) + \varepsilon\left[\frac{p(1-a)}{l+2} + \frac{1}{l+1}\right]\|\omega_t\|^{l+2} \\ &+ \varepsilon\left(1 + \frac{p(1-a)}{2}\right)\|\nabla\omega_t\|^2 \\ &+ \varepsilon\left[\left(\frac{p(1-a)}{2}\right)\left(1 - \int_0^t \Re(s)\, ds\right) - \left(1 - \frac{1}{2}\int_0^t \Re(s)\, ds\right)\right]\|\Delta\omega\|^2 \\ &- \frac{\varepsilon H^\alpha(t)}{4c_1\kappa}\left( \int_{\tau_1}^{\tau_2} |\Psi_2(q)|\, dq \right)\|\omega\|^2 + \varepsilon p(1-a)H(t) + \varepsilon ba\|\omega\|_p^p \\ &+ \frac{\varepsilon p(1-a)}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\Psi_2(q)|\, |z(x,\rho,q,t)|^2\, dq\, d\rho\, dx + \frac{\varepsilon}{2}\left[p(1-a)+1\right](\Re \circ \Delta\omega)(t). \end{aligned} \tag{3.12}$$

By using (2.18), (3.3) and Young's inequality, we obtain

$$\begin{aligned} H^{\alpha}(t)\|\omega\|_2^2 &\leq \left(b\int_{\Omega}|\omega|^p dx\right)^{\alpha}\|\omega\|_2^2 \\ &\leq c\left[\left(\int_{\Omega}|\omega|^p dx\right)^{\alpha+\frac{2}{p}}+\left(\int_{\Omega}|\omega|^p dx\right)^{\alpha}\|\Delta\omega\|^{\frac{4}{p}}\right] \\ &\leq c\left[\left(\int_{\Omega}|\omega|^p dx\right)^{\frac{p\alpha+2}{p}}+\|\Delta\omega\|^2+\left(\int_{\Omega}|\omega|^p dx\right)^{\frac{p\alpha}{p-2}}\right]. \end{aligned} \tag{3.13}$$

By exploiting (3.5), we obtain

$$2<\alpha p+2\leq p \quad \text{and} \quad 2<\frac{\alpha p^2}{p-2}\leq p.$$

Consequently, by Lemma (2.2)

$$H^{\alpha}(t)\|\omega\|_2^2 \leq c\left(\|\omega\|_p^p+\|\Delta\omega\|^2\right). \tag{3.14}$$

By combining (3.12) and (3.14), we have

$$\begin{aligned} K'(t) \geq{}& \left[(1-\alpha)-\varepsilon\kappa\right]H^{-\alpha}(t)H'(t)+\varepsilon\left[\frac{p(1-a)}{l+2}+\frac{1}{l+1}\right]\|\omega_t\|^{l+2} \\ &+\varepsilon\left(1+\frac{p(1-a)}{2}\right)\|\nabla\omega_t\|^2 \\ &+\varepsilon\left\{\left(\frac{p(1-a)}{2}-1\right)-\int_0^t\Re(s)\,ds\left(\frac{p(1-a)-1}{2}\right)-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\,dq\right)\right\}\|\Delta\omega\|^2 \\ &+\varepsilon\left[ab-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\,dq\right)\right]\|\omega\|_p^p+\varepsilon p(1-a)H(t) \\ &+\frac{\varepsilon p(1-a)}{2}\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2}q|\Psi_2(q)|\,|z(x,\rho,q,t)|^2\,dq\,d\rho\,dx+\frac{\varepsilon}{2}\left[p(1-a)+1\right](h\circ\Delta\omega)(t). \end{aligned} \tag{3.15}$$

Taking $a>0$ small enough such that

$$\alpha_1=\frac{p(1-a)}{l+2}+\frac{1}{l+1}>0,$$

and

$$\alpha_2=1+\frac{p(1-a)}{2}>0.$$

Assume also that

$$\int_0^{\infty}\Re(s)\,ds<\frac{\frac{p(1-a)}{2}-1}{\frac{p(1-a)}{2}-\frac{1}{2}}=\frac{2\alpha_1}{2\alpha_1+1}. \tag{3.16}$$

Now choose $\kappa$ such that the coefficient of $\|\Delta\omega\|^2$ in (3.15) is positive. Which mean that,

$$\alpha_3:=\left(\frac{p(1-a)}{2}-1\right)-\left(\int_0^t\Re(s)\,ds\right)\left(\frac{p(1-a)-1}{2}\right)-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\,dq\right)>0. \tag{3.17}$$

Likewise, ensure the coefficient of $\|\omega\|_p^p$ is positive:

$$\alpha_4:=ab-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\Psi_2(q)|\,dq\right)>0. \tag{3.18}$$

Fixing such $\kappa$ and $a$, we then choose $\varepsilon>0$ small enough so that:

$$\alpha_5:=(1-\alpha)-\varepsilon\kappa>0. \tag{3.19}$$

Then from inequality (3.15), there exists a constant $\beta > 0$ such that:

$$K'(t) \geq \beta \Bigg\{ H(t) + \|\omega_t\|^{l+2} + \|\Delta\omega\|^2 + (h \circ \Delta\omega)(t) + \|\omega\|_p^p + \|\nabla\omega_t\|^2 + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q|\Psi_2(q)|\,|z(x,\rho,q,t)|^2\,dq\,d\rho\,dx \Bigg\}. \tag{3.20}$$

Since $H(t) \geq 0$, this implies:

$$K(t) \geq K(0) > 0, \quad \forall t > 0. \tag{3.21}$$

Now, using Hölder's and Young's inequalities, we estimate:

$$\|\omega\|_2 = \left( \int_\Omega \omega^2\,dx \right)^{1/2} \leq \left[ \left( \int_\Omega |\omega|^p\,dx \right)^{2/p} \cdot \left( \int_\Omega 1\,dx \right)^{1-\frac{2}{p}} \right]^{1/2} \leq C\|\omega\|_p. \tag{3.22}$$

Also,

$$\left| \int_\Omega |\omega_t|^{l+1}\omega\,dx \right| \leq \|\omega_t\|_{l+2}^{l+1}\|\omega\|_{l+2} \leq c\|\omega_t\|_{l+2}^{l+1}\|\omega\|_p. \tag{3.23}$$

Then, using Young's inequality:

$$\left| \int_\Omega |\omega_t|^{l+1}\omega\,dx \right|^{\frac{1}{1-\alpha}} \leq c\|\omega_t\|^{\frac{l+1}{1-\alpha}}\|\omega\|_p^{\frac{1}{1-\alpha}} \leq c\left( \|\omega_t\|_{l+2}^{\frac{\theta(l+1)}{1-\alpha}} + \|\omega\|_p^{\frac{\mu}{1-\alpha}} \right), \tag{3.24}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$. Taking $\theta = \frac{(l+2)(1-\alpha)}{l+1}$, we get:

$$\frac{l+2}{(l+2)(1-\alpha) - (l+1)} \leq p.$$

Thus, set $s = \frac{l+2}{(l+2)(1-\alpha)-(l+1)}$, we obtain:

$$\left| \int_\Omega |\omega_t|^{l+1}\omega\,dx \right|^{\frac{1}{1-\alpha}} \leq c\left( \|\omega_t\|_{l+2}^{l+2} + \|\omega\|_p^s \right).$$

By Corollary 2.7, we then conclude:

$$\left| \int_\Omega |\omega_t|^{l+1}\omega\,dx \right|^{\frac{1}{1-\alpha}} \leq c\left( \|\omega_t\|_{l+2}^{l+2} + \|\omega\|_p^p + \|\Delta\omega\|_2^2 + (\Re \circ \Delta\omega)(t) + \|\nabla\omega_t\|^2 \right). \tag{3.25}$$

Therefore, from (3.25) and the definition of $K(t)$, we obtain:

$$\begin{aligned} K^{\frac{1}{1-\alpha}}(t) &= \left( H^{1-\alpha}(t) + \frac{\varepsilon}{l+1}\int_\Omega |\omega_t|^{l+1}\omega\,dx + \frac{\varepsilon\mu_1}{2}\int_\Omega \omega^2\,dx \right)^{\frac{1}{1-\alpha}} \\ &\leq c\left( H(t) + \left| \int_\Omega |\omega_t|^{l+1}\omega\,dx \right|^{\frac{1}{1-\alpha}} + \|\omega\|_2^{\frac{2}{1-\alpha}} + \|\Delta\omega\|_2^{\frac{2}{1-\alpha}} \right) \\ &\leq c\left( H(t) + \|\omega_t\|_{l+2}^{l+2} + \|\omega\|_p^p + \|\Delta\omega\|_2^2 + (\Re \circ \Delta\omega)(t) + \|\nabla\omega_t\|^2 \right). \end{aligned} \tag{3.26}$$

Substituting (3.26) into (3.20), we obtain:

$$K'(t) \geq \lambda K^{\frac{1}{1-\alpha}}(t), \tag{3.27}$$

Integrating (3.27) yields:

$$\frac{d}{dt}K^{-\frac{\alpha}{1-\alpha}}(t) \leq -\lambda\alpha \quad \Rightarrow \quad K^{-\frac{\alpha}{1-\alpha}}(t) \leq K^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\lambda\alpha}{1-\alpha}t.$$

Hence, $K(t) \to \infty$ as $t \to T^*$, where

$$T \leq T^* := \frac{1-\alpha}{\lambda\alpha}K^{\frac{\alpha}{1-\alpha}}(0). \tag{3.28}$$

This completes the proof. □

## 4 Conclusion

Through the analysis of the nonlinear viscoelastic Petrovsky equation considered in this work, we highlight the significant influence of the distributed delay and the nonlinear source on the dynamic behavior of the system, which leads to the loss of global-in-time solutions under certain initial conditions, as demonstrated by the obtained results.

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