Palestine Journal of Mathematics

Vol.15(3), (2026) , 548–555 © Palestine Polytechnic University-PPU 2026

ON THE DIOPHANTINE EQUATION $(p+2)^x - p^y = z^2$, WHERE p IS PRIME

Weerayuth Nilsrakoo, Khwanjira Thongta and Achariya Nilsrakoo

Communicated by Bouchaïb Sodaïgui

MSC 2020 Classifications: Primary 11D61; Secondary 11A07.

Keywords and phrases: Diophantine equation, congruence, prime number.

The authors would like to thank the reviewers and the editor for their constructive comments and valuable suggestions, which have improved the quality of the paper.

Corresponding Author: Achariya Nilsrakoo

Abstract The purpose of this paper is to study the non-negative integer solution (x, y, z) of the Diophantine equation $(p+2)^x - p^y = z^2$ where p is a prime number. In this proof, we reasonably apply Catalan's conjecture, Pythagoras' theorem, and basic number theory to obtain the solution. The result reveals that the infinite non-negative integer solutions are $(\alpha, 2\alpha, 0)$, where α are non-negative integers for $p = 2$ and the unique solution is $(x, y, z) = (0, 0, 0)$ for a prime number p greater than 3 with some certain conditions. Our results presented improve on the corresponding ones announced by many others and conclude with certain examples.

1 Introduction

Many authors investigate finding all non-negative integer solutions of the Diophantine equation

$$a^x - b^y = z^2 \tag{1.1}$$

where a and b are fixed positive integers, see for instance [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Firstly, we begin with Catalan's conjecture considering (1.1) when $\min\{a, b, x, y\} \geq 2$ and $z = 1$, which was proved by Preda Mihăilescu [1] in 2004, the equation $a^x - b^y = 1$ has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$. In 2021, Thongnak et al. [3] proved that $(x, y, z) = (0, 0, 0)$ is the unique non-negative integer solution of the equation $7^x - 5^y = z^2$. Moreover, they showed in [11] that $(x, y, z) = (0, 0, 0)$ is the unique non-negative integer solution of the equation $55^x - 53^y = z^2$. Consequently, Tadee and Laomalaw [9] proved for a class of primes which includes 5 and 53 that $(x, y, z) = (0, 0, 0)$ is the unique non-negative integer solution of the equation

$$(p+2)^x - p^y = z^2 \tag{1.2}$$

where p is a prime number with $p \equiv 5 \pmod{24}$. Recently, Thongnak et al. [10, 12] have shown that the equation (1.2) when $p = 13$ (which is not included in [9]) has the unique non-negative integer solution $(0, 0, 0)$ and the equation (1.2) when $p = 3$ (which is not included in [9]) has exactly three non-negative integer solutions $(x, y, z) \in \{(0, 0, 0), (1, 0, 2), (2, 2, 4)\}$.

In this paper, we extend the results of [3, 9, 11, 10, 12] to an arbitrary prime number. Our approach relies on elementary techniques together with Catalan's conjecture.

2 Preliminaries

In this section, we give some results in elementary number theory (see [17]) that are required for the proofs of the main results. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{N}_0$ the sets of positive integers and non-negative integers, respectively. We need the following results.

Definition 2.1. Let a and m be positive integers with $\gcd(a,m)=1$. The *order of a modulo m*, denoted by $\mathrm{ord}_m(a)$, is the smallest positive integer h such that $a^h \equiv 1 \pmod m$.

Lemma 2.2. *Let a and m be positive integers with* $\gcd(a,m)=1$. *If j and k are positive integers such that $a^j \equiv a^k \pmod m$, then $j \equiv k \pmod{\mathrm{ord}_m(a)}$. In particular, $\mathrm{ord}_m(a) \mid j$ for every positive integer j such that $a^j \equiv 1 \pmod m$.*

Definition 2.3. Let a be a positive integer and p be a prime number with $\gcd(a,p)=1$. The *Legendre symbol*, $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } x^2 \equiv a \pmod p \text{ is solvable;} \\ -1, & \text{if } x^2 \equiv a \pmod p \text{ is unsolvable.} \end{cases}$$

Lemma 2.4. *Let p be an odd prime number. Then*

$$\left(\frac{2}{p}\right) = \begin{cases} 1, & \text{if } p \equiv \pm 1 \pmod 8; \\ -1, & \text{if } p \equiv \pm 3 \pmod 8; \end{cases}$$

and

$$\left(\frac{-1}{p}\right) = \begin{cases} 1, & \text{if } p \equiv 1 \pmod 4; \\ -1, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Lemma 2.5. *Let p be a prime number greater than* 3. *Then*

$$\left(\frac{3}{p}\right) = \begin{cases} 1, & \text{if } p \equiv \pm 1 \pmod{12}; \\ -1, & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

Lemma 2.6 (Pythagorean theorem). *Let $x,y,z \in \mathbb{N}$. Then (x,y,z) with y is even,* $\gcd(x,y,z)=1$ *and $x^2+y^2=z^2$ if and only if there are $u,v \in \mathbb{N}$ with $u>v$,* $\gcd(u,v)=1$ *and $u \not\equiv v \pmod 2$ such that $x=u^2-v^2$, $y=2uv$ and $z=u^2+v^2$.*

Lemma 2.7 ([1, Catalan's conjecture, Mihăilescu's theorem]). *Let $a,b,x,y \in \mathbb{N}$ be such that* $\min\{a,b,x,y\} \geq 2$. *The equation $a^x - b^y = 1$ has the unique solution $(a,b,x,y)=(3,2,2,3)$.*

3 Main results

It is natural to start with the case where p is an even prime number, i.e., $p=2$. All solutions for this case easily follow from some known in many papers. For the sake completeness, we give the simple proof.

Theorem 3.1. *The Diophantine equation*

$$4^x - 2^y = z^2 \tag{3.1}$$

has non-negative integer solutions $(x,y,z)=(\alpha,2\alpha,0)$, where $\alpha \in \mathbb{N}_0$.

Proof. From (3.1), we obtain that $(2^x)^2 - 2^y = z^2$ and so $2^y = (2^x - z)(2^x + z)$. This implies that

$$2^\alpha = 2^x - z \quad \text{and} \quad 2^\beta = 2^x + z,$$

where $\alpha, \beta \in \mathbb{N}_0$ with $\alpha \leq \beta$ and $\alpha + \beta = y$. Consequently, we have

$$2^{x+1} = 2^x + 2^x = (2^x - z) + (2^x + z) = 2^\alpha + 2^\beta = 2^\alpha(1 + 2^{\beta-\alpha}).$$

This implies that $1 + 2^{\beta-\alpha}$ is a power of 2. Therefore, $\alpha = \beta$. So, we get $x=\alpha$, $y=2\alpha$ and $z=0$. This completes the proof. □

In the case where $p = 3$, the Diophantine equation

$$5^x - 3^y = z^2 \tag{3.2}$$

is studied by Thongnak et al. [12], they showed that the equation (3.2) has exactly three non-negative integer solutions $(x, y, z) \in \{(0, 0, 0), (1, 0, 2), (2, 2, 4)\}$.

From now on, we assume that p is a prime number greater than 3. We first consider the solutions of (1.2) when $p \equiv 1 \pmod 3$ as follows.

Theorem 3.2. *If p is a prime number such that $p \equiv 1 \pmod 3$, then the Diophantine equation (1.2) has the unique non-negative integer solution $(0, 0, 0)$.*

Proof. It is known that $(p + 2)^x \equiv 0 \pmod 3$ for all $x > 0$ and $p^y \equiv 1 \pmod 3$ for all $y \geq 0$. Then, by (1.2), $z^2 \equiv -1 \equiv 2 \pmod 3$, which is impossible for $x > 0$. Then $x = 0$ and so $y = z = 0$. This completes the proof. □

Since 7, 13 and 19 are prime numbers with $7 \equiv 13 \equiv 19 \equiv 1 \pmod 3$, we have the following.

Corollary 3.3. *The Diophantine equation $9^x - 7^y = z^2$ has the unique non-negative integer solution $(0, 0, 0)$.*

Corollary 3.4 ([10, Theorem 3.1]). *The Diophantine equation $15^x - 13^y = z^2$ has the unique non-negative integer solution $(0, 0, 0)$.*

Corollary 3.5. *The Diophantine equation $21^x - 19^y = z^2$ has the unique non-negative integer solution $(0, 0, 0)$.*

We next consider the equation (1.2) when $p \equiv 2 \pmod 3$ as by dividing in specific congruence classes modulo 4.

Theorem 3.6. *If p is a prime number such that $p \equiv 1 \pmod 4$, then the Diophantine equation (1.2) has the unique non-negative integer solution $(0, 0, 0)$.*

Proof. Let (x, y, z) be a non-negative integer solution of (1.2). In the cases where $x = 0$ or $z = 0$, we get $(x, y, z) = (0, 0, 0)$. As in Theorem 3.2, we now prove only the cases of $p \equiv 1 \pmod 4$ with $p \equiv 2 \pmod 3$ and $x, z \in \mathbb{N}$. Since p and $p + 2$ are odd, we get z is even. Then $z^2 \equiv 0 \pmod 4$. It is known that

$$(p+2)^x - p^y \equiv (-1)^x - 1 \equiv \begin{cases} 0 \pmod 4, & \text{if } x \text{ is even;} \\ 2 \pmod 4, & \text{if } x \text{ is odd.} \end{cases}$$

Therefore, x is even. Let $x = 2m$ where $m \in \mathbb{N}$. Moreover, it is known that $z^2 \equiv 0, 1 \pmod 3$ and

$$(p+2)^x - p^y \equiv 1 - (-1)^y \equiv \begin{cases} 0 \pmod 3, & \text{if } y \text{ is even;} \\ 2 \pmod 3, & \text{if } y \text{ is odd.} \end{cases}$$

Therefore, y is even. Let $y = 2n$ where $n \in \mathbb{N}_0$. From (1.2), we have

$$(p^n)^2 + z^2 = ((p+2)^m)^2. \tag{3.3}$$

We can see from $\gcd(p, p+2) = 1$ that $\gcd(p^n, z, (p+2)^m) = 1$. By (3.3) and Lemma 2.6, there are $u, v \in \mathbb{N}$ with $u > v$, $\gcd(u, v) = 1$ and $u \not\equiv v \pmod 2$ such that $p^n = u^2 - v^2$, $z = 2uv$ and

$$(p+2)^m = u^2 + v^2. \tag{3.4}$$

We know from $u \not\equiv v \pmod 2$ that $u^2 + v^2 \equiv 0 + 1 \equiv 1 \pmod 4$. This implies that

$$(p+2)^m \equiv (-1)^m \equiv 1 \pmod 4.$$

Therefore, m is even, i.e., $m = 2m_1$ where $m_1 \in \mathbb{N}$. Then $m > m_1 > 0$ and, by (3.4),

$$((p+2)^{m_1})^2 = u^2 + v^2, \tag{3.5}$$

where $\gcd(u, v, (p+2)^{m_1}) = 1$. Without loss of generality, we assume that v is even. By (3.5) and Lemma 2.6, there are $u_1, v_1 \in \mathbb{N}$ with $u_1 > v_1$, $\gcd(u_1, v_1) = 1$ and $u_1 \not\equiv v_1 \pmod 2$ such that $u = u_1^2 - v_1^2$, $v = 2u_1v_1$ and

$$(p+2)^{m_1} = u_1^2 + v_1^2. \tag{3.6}$$

We know from $u_1 \not\equiv v_2 \pmod 2$ that $u_1^2 + v_1^2 \equiv 1 \pmod 4$. This implies that

$$(p+2)^{m_1} \equiv (-1)^{m_1} \equiv 1 \pmod 4.$$

Therefore, m_1 is even, i.e., $m_1 = 2m_2$ where $m_2 \in \mathbb{N}$. Then $m > m_1 > m_2 > 0$ and, by (3.6),

$$\left((p+2)^{m_2}\right)^2 = u_1^2 + v_1^2,$$

where $\gcd(u_1, v_1, (p+2)^{m_2}) = 1$. This process can be carried out as many times as desired to produce an infinite decreasing sequence of positive integers

$$m > m_1 > m_2 > \cdots > 0.$$

Because there is only a finite supply of positive integers less than m, a contradiction occurs. This completes the proof. □

Since 5, 17 and 53 are prime numbers with $5 \equiv 17 \equiv 53 \equiv 1 \pmod 4$, we have the following.

Corollary 3.7 ([3, Theorem 3.1]). *The Diophantine equation $7^x - 5^y = z^2$ has the unique non-negative integer solution $(0, 0, 0)$.*

Corollary 3.8 ([9, Corollary 3.3]). *The Diophantine equation $19^x - 17^y = z^2$ has the unique non-negative integer solution $(0, 0, 0)$.*

Corollary 3.9 ([11, Theorem 3.1]). *The Diophantine equation $55^x - 53^y = z^2$ has the unique non-negative integer solution $(0, 0, 0)$.*

Remark 3.10. Since $p \equiv 5 \pmod{24}$ implies $p \equiv 1 \pmod 4$, it is easily seen that Theorem 3.1 and Theorem 3.4 of [9] are special cases of Theorem 3.6. More precisely, the Diophantine equation (1.2) has the unique non-negative integer solution $(0, 0, 0)$ when p is a prime number greater than 3 except $p \equiv 11 \pmod{12}$. For such a case, the assumption on p is added to guarantee that equation (1.2) has the unique non-negative integer solution $(0, 0, 0)$.

We now present about the solutions of (1.2) when $p \equiv 11 \pmod{24}$ as follows.

Theorem 3.11. *Let p be a prime number such that $p \equiv 11 \pmod{24}$. Then the Diophantine equation (1.2) has the unique non-negative integer solution $(0, 0, 0)$ if one of the following is satisfied:*

(i) there exists a prime number q with $q \equiv \pm 5 \pmod{12}$ such that $p \equiv 1 \pmod q$;

(ii) there exists a prime number q with $q \equiv 3 \pmod 4$ such that $p \equiv -2 \pmod q$;

(iii) $p \equiv 2 \pmod 5$;

(iv) $p \equiv 3 \pmod 5$ and $8 \mid \operatorname{ord}_{p+2}(2)$.

Proof. Let (x, y, z) be a non-negative integer solution of (1.2). For $x = 0$ or $z = 0$, we have $(x, y, z) = (0, 0, 0)$. For $y = 0$, we have $(p+2)^x - z^2 = 1$. Apply Lemma 2.7, we get $x \le 1$ or $z \le 1$. This implies from $p \neq 3$ that $(x, y, z) = (0, 0, 0)$. We now assume that $x, y, z \in \mathbb{N}$. Notice that $p \equiv 3 \pmod 4 \equiv 3 \pmod 8$ and z is even. Then $z^2 \equiv 0 \pmod 4$. It is known that

$$(p+2)^x - p^y \equiv 1 - (-1)^y \equiv \begin{cases} 0 \pmod 4, & \text{if } y \text{ is even;} \\ 2 \pmod 4, & \text{if } y \text{ is odd.} \end{cases}$$

Therefore, y is even and so $y = 2n$ where $n \in \mathbb{N}$. Observe that $z^2 \equiv (p+2)^x - p^{2n} \equiv 2^x \pmod p$. Then $\left(\frac{2^x}{p}\right) = 1$. But $p \equiv 3 \pmod 8$ and Lemma 2.4, we have $\left(\frac{2^x}{p}\right) = \left(\frac{2}{p}\right) = -1$ for x is odd.

This implies that x must be even. Let $x = 2m$ where $m \in \mathbb{N}$. From (1.2), we obtain that $((p+2)^m)^2 - p^y = z^2$ and so

$$p^{2n} = ((p+2)^m - z)\,((p+2)^m + z)\,.$$

This implies that

$$p^\alpha = (p+2)^m - z \quad \text{and} \quad p^\beta = (p+2)^m + z,$$

where $\alpha, \beta \in \mathbb{N}_0$ with $\alpha \leq \beta$ and $\alpha + \beta = 2n$. Consequently, we have

$$2(p+2)^m = p^\alpha(1 + p^{\beta-\alpha})$$

This implies that $\alpha = 0$. So, we get $\beta = 2n$ and

$$2(p+2)^m = 1 + p^{2n}. \tag{3.7}$$

Considering modulo p, we have $2^{m+1} \equiv 1 \equiv 1^2 \pmod{p}$. Then $\left(\frac{2^{m+1}}{p}\right) = 1$. Since $p \equiv 3 \pmod 8$, it follows from Lemma 2.4 that m must be odd.

(i) Assume that there exists a prime number q with $q \equiv \pm 5 \pmod{12}$ such that $p \equiv 1 \pmod q$. Considering modulo q in (3.7), we have $2(3^m) \equiv 1+1 \equiv 2 \pmod q$. Then $3^m \equiv 1 \pmod q$. It follows that $\left(\frac{3^m}{q}\right) = \left(\frac{3}{q}\right) = 1$. But $q \equiv \pm 5 \pmod{12}$ and Lemma 2.5, we have $\left(\frac{3}{q}\right) = -1$, which is impossible.

(ii) Assume that there exists a prime number q with $q \equiv 3 \pmod 4$ such that $p \equiv -2 \pmod q$. Considering modulo q in (3.7), we have $1+(-2)^{2n} \equiv 0 \pmod q$. Then $(2^n)^2 \equiv -1 \pmod q$. It follows that $\left(\frac{-1}{q}\right) = 1$, which is impossible by using Lemma 2.4 with $q \equiv 3 \pmod 4$.

(iii) Assume $p \equiv 2 \pmod 5$. It is known from (3.7) that $2(4)^m \equiv 1 + 4^n \pmod 5$. Using

$$4^k \equiv \begin{cases} 4 \pmod 5, & \text{if } k \text{ is odd;} \\ 1 \pmod 5, & \text{if } k \text{ is even,} \end{cases} \tag{3.8}$$

and m is odd, we obtain $8 \equiv 5, 2 \pmod 5$. This is a contradiction.

(iv) Assume $p \equiv 3 \pmod 5$ and $8 \mid \mathrm{ord}_{p+2}(2)$. It is known from (3.7) that $1 + 3^{2n} \equiv 0 \pmod 5$. That is, $4^n \equiv 4 \pmod 5$. Using (3.8), we obtain n is odd. By (3.7) again, we get

$$1 + (-2)^{2n} \equiv 0 \pmod{p+2}.$$

So, we have $2^{4n} \equiv 1 \pmod{p+2}$. Therefore, $\mathrm{ord}_{p+2}(2) \mid 4n$. By the assumption, we get $8 \mid 4n$. It follows that n must be even, which is impossible.

This completes the proof. □

Since 11, 59 and 443 are the first few prime numbers $p \equiv 11 \pmod{24}$ such that $11 \equiv 1 \pmod 5$, $59 \equiv 1 \pmod{29}$ and $443 \equiv 1 \pmod{17}$ with $29 \equiv 17 \equiv 5 \pmod{12}$, we have (i) of Theorem 3.11 and the following corollaries.

Corollary 3.12. *The Diophantine equation* $13^x - 11^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Corollary 3.13. *The Diophantine equation* $61^x - 59^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Corollary 3.14. *The Diophantine equation* $445^x - 443^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Since 419, 491 and 2099 are the first few prime numbers $p \equiv 11 \pmod{24}$ such that $419 \equiv 1 \pmod{19}$, $491 \equiv 1 \pmod 7$ and $2099 \equiv 1 \pmod{31}$ with $31 \equiv 19 \equiv 7 \equiv -5 \pmod{12}$, we have (i) of Theorem 3.11 and the following corollaries.

Corollary 3.15. *The Diophantine equation* $421^x - 419^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Corollary 3.16. *The Diophantine equation* $493^x - 491^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Corollary 3.17. *The Diophantine equation* $2101^x - 2099^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Since 131, 251 and 587 are the first few prime numbers $p \equiv 11 \pmod{24}$ such that $131 \equiv -2 \pmod 7$, $251 \equiv -2 \pmod{11}$ and $587 \equiv -2 \pmod{19}$ with $7 \equiv 11 \equiv 19 \equiv 3 \pmod 4$, we have (ii) of Theorem 3.11 and the following corollaries.

Corollary 3.18. *The Diophantine equation* $133^x - 131^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Corollary 3.19. *The Diophantine equation* $253^x - 251^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Corollary 3.20. *The Diophantine equation* $589^x - 587^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Since 107 and 227 are the first few primes $p \equiv 11 \pmod{24}$ such that $107 \equiv 227 \equiv 2 \pmod 5$, we have (iii) of Theorem 3.11 and the following corollaries.

Corollary 3.21. *The Diophantine equation* $109^x - 107^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Corollary 3.22. *The Diophantine equation* $229^x - 227^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Since 83 is the first prime numbers $p \equiv 11 \pmod{24}$ such that $83 \equiv 3 \pmod 5$ with $\mathrm{ord}_{85}(2) = 8$, we have (iv) of Theorem 3.11 and the following corollary.

Corollary 3.23. *The Diophantine equation* $85^x - 83^y = z^2$ *has the unique non-negative integer solution* $(0,0,0)$.

Finally, we present about the solutions of (1.2) when $p \equiv 23 \pmod{24}$ as follows.

Theorem 3.24. *Let* p *be a prime number such that* $p \equiv 23 \pmod{24}$. *If there exist a prime number* q *and a positive integer* k *such that* $q \equiv 3 \pmod 4$ *and* $p = q^{2k} - 2$, *then the Diophantine equation* (1.2) *has the unique non-negative integer solution* $(0,0,0)$.

Proof. Let (x,y,z) be a non-negative integer solution of (1.2). For $x = 0$, we have $(x,y,z) = (0,0,0)$. We now assume that $x \in \mathbb{N}$. Since $p \equiv q \equiv 3 \pmod 4$ and there exists $k \in \mathbb{N}$ such that $p + 2 = q^{2k}$. It is known that

$$(p+2)^x - p^y \equiv 1 - (-1)^y \equiv \begin{cases} 0 \pmod 4, & \text{if } y \text{ is even;} \\ 2 \pmod 4, & \text{if } y \text{ is odd.} \end{cases}$$

We note that z is even. Then $z^2 \equiv 0 \pmod 4$. Therefore, y is even and so $y = 2n$ where $n \in \mathbb{N}_0$. From (1.2) and $p + 2 = q^{2k}$, we obtain that $p^{2n} = (q^{kx})^2 - z^2$ and so

$$p^{2n} = \left(q^{kx} - z\right)\left(q^{kx} + z\right).$$

This implies that

$$p^\alpha = q^{kx} - z \quad \text{and} \quad p^\beta = q^{kx} + z,$$

where $\alpha, \beta \in \mathbb{N}_0$ with $\alpha \le \beta$ and $\alpha + \beta = 2n$. Consequently, we have

$$2q^x = p^\alpha(1 + p^{\beta-\alpha})$$

This implies that $\alpha = 0$. So, we get $\beta = 2n$ and

$$2q^{kx} = 1 + p^{2n}.$$

Considering modulo q, we have $1 + (-2)^{2n} \equiv 0 \pmod q$. Then $(2^n)^2 \equiv -1 \pmod q$. It follows that $\left(\frac{-1}{q}\right) = 1$, which is impossible by using Lemma 2.4 with $q \equiv 3 \pmod 4$. This completes the proof. □

Since 47, 359 and 2399 are the first few prime numbers $p \equiv 23 \pmod{24}$ such that $47 = 7^2 - 2$, $359 = 19^2 - 2$ and $2399 = 7^4 - 2$ with $7 \equiv 19 \equiv 3 \pmod 4$, we have the following.

Corollary 3.25. *The Diophantine equation* $49^x - 47^y = z^2$ *has the unique non-negative integer solution* $(0, 0, 0)$.

Corollary 3.26. *The Diophantine equation* $361^x - 359^y = z^2$ *has the unique non-negative integer solution* $(0, 0, 0)$.

Corollary 3.27. *The Diophantine equation* $2401^x - 2399^y = z^2$ *has the unique non-negative integer solution* $(0, 0, 0)$.

4 Conclusion remarks

This paper presents about non-negative integer solutions (x, y, z) of the Diophantine equation $(p + 2)^x - p^y = z^2$, where p is a prime number. By Theorems 3.1, 3.6, 3.11, 3.24 and [12], we give a summary as follows.

(i) If $p = 2$, then the equation has infinite non-negative integer solutions $(x, y, z) = (\alpha, 2\alpha, 0)$, where $\alpha \in \mathbb{N}_0$.

(ii) If $p = 3$, then the equation has exactly three non-negative integer solutions

$$(x, y, z) \in \{(0, 0, 0), (1, 0, 2), (2, 2, 4)\},$$

which is presented by Thongnak et al. [12].

(iii) Let $p > 3$ and one of the following be satisfied:

a. $p \not\equiv 11 \pmod{12}$;

b. $p \equiv 11 \pmod{24}$ and one of the following satisfied:

i. there exists a prime number q with $q \equiv \pm 5 \pmod{12}$ such that $p \equiv 1 \pmod q$;

ii. there exists a prime number q with $q \equiv 3 \pmod 4$ such that $p \equiv -2 \pmod q$;

iii. $p \equiv 2 \pmod 5$;

iv. $p \equiv 3 \pmod 5$ and $8 \mid \operatorname{ord}_{p+2}(2)$;

c. $p \equiv 23 \pmod{24}$ and there exist a prime number q and a positive integer k such that $q \equiv 3 \pmod 4$ and $p = q^{2k} - 2$.

Then the equation has the unique non-negative integer solution $(0, 0, 0)$.

Our results with simple cases to check (prime numbers q are 5 and 7 only) guarantee that the equation (1.2) has the unique non-negative integer solution $(0, 0, 0)$ for 164 fully and 6 partially residue classes from 196 residue classes modulo 840. The problem of the Diophantine equation (1.2) having the unique non-negative integer solution $(0, 0, 0)$ when p is an arbitrary prime number greater than 3 is open. Furthermore, Tadee and Wannaphan [14], as well as Macatiag and Bacani [16] studied the equation

$$(p + a)^x - p^y = z^2$$

where a is a positive integer with $a \equiv 2 \pmod 4$ and $a \equiv \pm 1 \pmod p$ or a is an odd prime number that satisfies some conditions or a is a perfect square. It is worth noting that our results are independently studied in [14] and [16].

References

[1] P. Mihăilescu, *Primary cyclotomic units and a proof of Catalan's conjecture*, J. Reine Angew. Math., **572**, 167–195, (2004).

[2] S. Thongnak, W. Chuayjan and T. Kaewong, *On the exponential Diophantine equation* $2^x - 3^y = z^2$, Southeast-Asian J. Sci., **7**, 1–4, (2019).

[3] S. Thongnak, W. Chuayjan and T. Kaewong, *The solution of the exponential Diophantine equation* $7^x - 5^y = z^2$, Math. J., **66**, 62–67, (2021).

[4] M. Buosi, A. Lemos, A. L. P. Porto and D. F. G. Santiago, *On the exponential Diophantine equation* $p^x - 2^y = z^2$ *with* $p = k^2 + 4$ *a prime number*, Palest. J. Math. **11**, 130–135, (2022).

[5] C. G. Rao, *On the exponential Diophantine equation* $23^x - 19^y = z^2$, J. Phy. Sci., **27**, 1–4, (2022).

[6] R. C. Gope, *On the exponential Diophantine equation* $27^x - 11^y = z^2$, J. Phy. Sci., **28**, 11–15, (2023).

[7] W. Chuayjan S. Thongnak and T. Kaewong, *On the exponential Diophantine equation* $10^x - 17^y = z^2$, Ann. Pure Appl. Math., **28**, 21–24, (2023).

[8] W. Chuayjan S. Thongnak and T. Kaewong, *On the exponential Diophantine equation* $3^x - 5^y = z^2$, Ann. Pure Appl. Math., **28**, 25–28, (2023).

[9] S. Tadee and N. Laomalaw, *On the Diophantine equation* $(p+2)^x - p^y = z^2$ *where* p *is prime with* $p \equiv 5 \pmod{24}$, Int. J. Math. Comput. Sci., **18**, 149–152, (2023).

[10] S. Thongnak, W. Chuayjan and T. Kaewong, *On the Diophantine equation* $15^x - 13^y = z^2$, Ann. Pure Appl. Math., **27**, 23–26, (2023).

[11] S. Thongnak, T. Kaewong and W. Chuayjan, *On the Diophantine equation* $55^x - 53^y = z^2$, Ann. Pure Appl. Math., **27**, 27–30, (2023).

[12] S. Thongnak, T. Kaewong and W. Chuayjan, *On the exponential Diophantine equation* $5^x - 3^y = z^2$, Int. J. Math. Comput. Sci., **19**, 99–102, (2024).

[13] S. Thongnak, T. Kaewong and W. Chuayjan, *On the exponential Diophantine equation* $11^x - 17^y = z^2$, Int. J. Math. Comput. Sci. **19**, 181–184, (2024).

[14] S. Tadee and C. Wannaphan, *On the Diophantine equation* $(p+a)^x - p^y = z^2$ *and* $p^x - (p+a)^y = z^2$, Int. J. Math. Comput. Sci., **19**, 459–465, (2024).

[15] A. Siraworakun and S.Tadee, *All solutions of the Diophantine equation* $25^x - 7^y = z^2$, Int. J. Math. Comput. Sci. **19**, 631–633, (2024).

[16] K. J. Macatiag and J. Bacani, *Solutions to Diophantine equations involving primes and perfect squares*, J. Phys.: Conf. Ser., **3114**, 012001, (2025).

[17] D. M. Burton, *Elementary Number Theory*, McGraw-Hill Book Company, New York, 2010.

Author information

Weerayuth Nilsrakoo, Department of Mathematics and Digital Technology, Faculty of Science, Ubon Ratchathani University, Thailand.
E-mail: weerayuth.ni@ubu.ac.th

Khwanjira Thongta, Department of Mathematics and Digital Technology, Faculty of Science, Ubon Ratchathani University, Thailand.
E-mail: khwanjira.th.61@ubu.ac.th

Achariya Nilsrakoo, Department of Mathematics, Faculty of Science, Ubon Ratchathani Rajabhat University, Thailand.
E-mail: achariya.n@ubru.ac.th

Received: 2025-10-17
Accepted: 2026-01-19