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IN-DEPTH ANALYSIS OF SEMI-WELL ORDERED SETS AND SEMI-ORDINALS

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Abstract The study of anti-homogeneous topological spaces introduced the notion of semi-well ordered sets–partially ordered sets in which every non-empty subset contains either a smallest element or a largest element. Several researchers looked at and made use of the order topology defined on semi-well ordered sets. Our intention is to investigate the set-theoretic and order-theoretic properties of semi-well ordered sets in detail. We define semi-ordinals and identify a representation for them. We examine the arithmetic of semi-ordinals and their properties.

1 Introduction

In set theory, significant efforts have been undertaken to generalize the classical concepts of well ordered sets and ordinal numbers. One notable extension is the theory of partially well ordered sets (also known as well partially ordered sets) and partial ordinals, which have been extensively studied by Higman [5], Kruskal [8, 9], Jongh and Parikh [3] and Wolk [25]. Some of these authors considered quasi orderings–relations that do not require anti-symmetry-instead of partial orderings [15, 24]. However, except for technical details, the two concepts are the same. The notion of well founded posets [10, 12, 18] further generalizes partially well ordered sets and, consequently, well ordered sets. Moreover, the theory of posets and well ordered sets plays a significant role in various areas of mathematics, including graph theory [14], topology[20, 22] and group theory [19].

Here, we generalize the well order theory by investigating the concept of semi-well ordered sets. Ramachandran(1987) [16, 17] introduced semi-well ordered sets as posets in which every non-empty subset possesses either a smallest element or a largest element. He further demonstrated that such sets admit a unique representation as the linear sum of a well ordered set and the dual of a well ordered set. Moreover, he characterized anti-homogeneity in terms of semi-well ordered sets and established that a poset is semi-well ordered if and only if it is hereditarily rigid.

In 2013, Sreeja [21] established that the order topology on a semi-well ordered set is both scattered and anti-rigid. A detailed topological study of these sets was done in [23], where it was shown that the corresponding topological space is zero-dimensional, totally disconnected and strongly locally compact. That study also includes the identification of isolated points in a semi-well ordered set, along with characterizations of those that are compact or infinitely homogeneous.

In this paper, the notion of semi-well ordered sets is illustrated through examples and various structural properties are examined. A central objective of the study is to analyze the order ideals of semi-well ordered sets and to provide a complete characterization of it. Furthermore, we determine the range of a surjective order isomorphism from a semi-well ordered set onto one of its order ideals.

A semi-well ordered set can be assigned an order type called a semi-ordinal, which we represent by a pair $(\lambda, \tilde{\upsilon})$, where λ is an ordinal and υ is a limit ordinal. We develop the arithmetic of semi-ordinals and investigate its properties. Although not all aspects of classical ordinal arithmetic extend to this broader setting, several fundamental properties are preserved.

In the final section, we show that any set of semi-ordinals, under a particular ordering, forms a well ordered set. Within this framework, we can observe that $(\omega^2, \tilde{\omega^2})$ is the largest semi-ordinal that contains no sequence of distinct limit ordinals converging to a limit ordinal in the order topology. This insight leads to a characterization of the set of periodic points in dynamical systems such as self-homeomorphisms and continuous self-maps on $(\omega^2, \tilde{\omega^2})$. A notable consequence of this characterization is the following: If $K \subset L \subset [0,1]$, where L is homeomorphic to $(\omega^2, \tilde{\omega^2})$ and $\overline{L} \setminus L$ is finite, then whenever K fails to occur as the set of periodic points of any continuous self-map on L, it also fails to arise as such for any continuous map on $[0,1]$. This aspect of the work is currently ongoing and has not yet been published.

2 Preliminaries

In this section, the basic definitions and the foundational result that we will use later are presented. For an additional background not included here, see [4, 6, 13].

Consider two partially ordered sets (posets) $(\mathcal{S}, \preceq)$ and $(\mathcal{S}', \preceq')$. A function $p : \mathcal{S} \to \mathcal{S}'$ is said to be *order preserving* [25] if, for every $x_1, y_1 \in \mathcal{S}$, the relation $x_1 \preceq y_1$ implies $p(x_1) \preceq' p(y_1)$ in $\mathcal{S}'$. Two chains $\mathcal{S}$ and $\mathcal{S}'$ are considered *ordinally similar* [6] if a bijection $q : \mathcal{S} \to \mathcal{S}'$ can be defined such that both q and its inverse q^{-1} are order preserving. The function q is known as an *order isomorphism* [6] from $\mathcal{S}$ to $\mathcal{S}'$. If $\mathcal{S}$ and $\mathcal{S}'$ are ordinally similar, we write $\mathcal{S} \approx \mathcal{S}'$. Ordinal similarity forms an equivalence relation within the collection of chains. The corresponding equivalence class of a chain under ordinal similarity is referred to as its *order type* [6].

A poset is considered *well ordered* (abbreviated as *w.o*) [6] when every non-empty subset has a smallest element. For a *w.o* set, its order type is known as an *ordinal number* or simply an *ordinal* [6]. An ordinal λ is referred to as a *limit ordinal* [4] when it does not have an immediate predecessor. Otherwise, it is referred to as a *non-limit ordinal* [4], which means there exists some ordinal ν with $\lambda = \nu + 1$.

The dual of a *w.o* set is known as *co-well ordered set* (abbreviated as co-*w.o* set)[16]; that is, a poset where each non-empty subset possesses a largest element. A poset $(\mathcal{S}, \preceq)$ is called *partially well ordered* (abbreviated as *p.w.o*) [25] if, for every sequence $\{s_i\} \subseteq \mathcal{S}$, there exist indices $l < k$ with $s_l \preceq s_k$. It is straightforward to verify that a totally ordered set is *p.w.o* if and only if it is *w.o* [11].

3 Semi-Well Ordered Sets

In this section, we present the definition of semi-well ordered sets, along with illustrative examples, followed by an exploration of their fundamental properties and characterizations.

Definition 3.1. [16] *A poset $(\mathcal{S}, \preceq)$ is called semi-well ordered if each non-empty subset of $\mathcal{S}$ contains either a smallest element or a largest element.*

In the sequel, we abbreviate semi-well ordered as *s.w.o*. It is evident that the dual of a *s.w.o* set is *s.w.o*. To aid in the understanding of the definition of *s.w.o* sets, we provide the following examples:

a) Consider the following ordering of positive integers:

$$
\begin{gathered}
1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^n \prec \cdots \\
\vdots \\
\cdots \prec 2^n \cdot 7 \prec 2^n \cdot 5 \prec 2^n \cdot 3 \prec \\
\vdots \\
\cdots \prec 2^3 \cdot 7 \prec 2^3 \cdot 5 \prec 2^3 \cdot 3 \prec \\
\cdots \prec 2^2 \cdot 7 \prec 2^2 \cdot 5 \prec 2^2 \cdot 3 \prec \\
\cdots \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \\
\cdots \prec 7 \prec 5 \prec 3
\end{gathered}
$$

Under this ordering, the set of all positive integers constitutes a *s.w.o* set. This ordering, known as Sharkovsky's ordering [2], is significant within the field of discrete dynamical systems. This *s.w.o* set has an order type represented by $(\omega, \tilde{\omega^2})$.

b) The set of all positive integers, when ordered by $3 \prec 4 \prec 5 \prec \cdots \prec 1 \prec 2$, is a *s.w.o* set. This *s.w.o* set has an order type represented by $(\omega$+2, 0).

c) The set of all integers, when ordered by $0 \prec 1 \prec 2 \prec \cdots \prec \cdots \prec -3 \prec -2 \prec -1$, forms a *s.w.o* set and its order type is $(\omega, \tilde{\omega})$.

Proposition 3.2. *Let $(\mathcal{S}, \preceq)$ be a s.w.o set. Then:*

(i) Every subset A of $\mathcal{S}$ is s.w.o with the inherited order.

(ii) $\mathcal{S}$ is totally ordered by $\preceq$.

Proof. (i) Trivial.

(ii) Let $s_1, s_2 \in \mathcal{S}$. Then$\{s_1, s_2\} \subseteq \mathcal{S}$ has either a smallest element or a largest element; thus, $s_1 \preceq s_2$ or $s_2 \preceq s_1$.

□

Corollary 3.3. *A s.w.o set, when endowed with order topology, is Hausdorff, completely normal and Sober.*

Proof. Proof easily follows from Proposition 3.2 (ii), [1] and [7]. □

Since every *s.w.o* set is totally ordered, a *s.w.o* set is *p.w.o* if and only if it is *w.o*. Therefore, both *p.w.o* sets and *s.w.o* sets generalize *w.o* sets, with their intersection precisely being the collection of all *w.o* sets.

Proposition 3.4. *A totally ordered set that is ordinally similar to a s.w.o set must also be s.w.o.*

Proof. Consider $(\mathcal{S}, \preceq)$ as a *s.w.o* set and $(\mathcal{S}', \preceq')$ as a totally ordered set that is ordinally similar to $(\mathcal{S}, \preceq)$, with order isomorphism i: $\mathcal{S} \to \mathcal{S}'$. For a subset $S' \subseteq \mathcal{S}'$ with $S' \neq \phi$, define $S = i^{-1}(S')$. Since $S \subseteq \mathcal{S}$, it must contain either a smallest element or a largest element.

Because i is an order isomorphism, if S contains a smallest element s_0, then $i(s_0)$ is the smallest element contained in S'. Similarly, if c_0 is the largest element of S, then $i(c_0)$ is the largest element contained in S'. Thus, $(\mathcal{S}', \preceq')$ is a *s.w.o* set. □

Proposition 3.5. *Let $(\mathcal{S}, \preceq)$ be a s.w.o set and i_o an order isomorphism on $\mathcal{S}$. Then, $s \leq i_o(s)$ for all $s \in \mathcal{S}$.*

Proof. Assume, for contradiction, that there exists $s \in \mathcal{S}$ with $s > i_o(s)$. Define $B = \{s \in \mathcal{S}: s > i_o(s)\}$. Since $\mathcal{S}$ is *s.w.o*, B has either a smallest or a largest element. Suppose B contains the smallest element, say s_1. Then $s_1 > i_o(s_1)$ and since i_o is an order isomorphism, $i_o(s_1) > i_o(i_o(s_1))$, so $i_o(s_1) \in B$, contradicting the minimality of s_1.

Now suppose that B contains the largest element, say s' . Then $s' \in B$ implies $i_o(s') < s'$. Since i_o is an order isomorphism, $i_o^{-1}(i_o(s')) < i_o^{-1}(s')$, that is, $i_o(i_o^{-1}(s')) < i_o^{-1}(s')$. Thus, $i_o^{-1}(s') \in B$ and $s' < i_o^{-1}(s')$, contradicting the maximality of s'.

□

Later we will see that, if $(\mathcal{S}, \preceq)$ is a *s.w.o* set, then any order isomorphism $i : \mathcal{S} \to \mathcal{S}$ must satisfy $i(s) = s$ for all $s \in \mathcal{S}$.

Proposition 3.6. *If $(\mathcal{S}, \preceq)$ and $(\mathcal{S}', \preceq')$ are s.w.o sets that are ordinally similar, then there exists a unique order isomorphism between them.*

Proof. Assume that there exist two order isomorphisms from $\mathcal{S}$ onto $\mathcal{S}'$, say i_1 and i_2. Then i_1, i_2, i_1^{-1} and i_2^{-1} are all order preserving maps. Consider the composition $i_1^{-1} \circ i_2$, which is an order isomorphism on $\mathcal{S}$. By Proposition 3.5, we have $s \leq i_1^{-1} \circ i_2(s)$ for all $s \in \mathcal{S}$. So, $s \leq i_1^{-1}(i_2(s))$ for each $s \in \mathcal{S}$, which implies $i_1(s) \leq i_2(s)$ for all $s \in \mathcal{S}$. Hence, $i_1 \leq i_2$. Similarly, $i_2^{-1} \circ i_1$ is an order isomorphism on $\mathcal{S}$ and by the same argument, $i_2(s) \leq i_1(s)$ for all $s \in \mathcal{S}$. Therefore, $i_1 = i_2$. □

Consider two disjoint totally ordered sets $\mathcal{S}$ and Y equipped with total orders $\preceq$ and $\preceq'$, respectively. Then $\mathcal{S}+Y$ denotes the union $\mathcal{S}\cup Y$, endowed with the total order $\preceq \cup \preceq' \cup (\mathcal{S}\times Y)$.

The following two theorems provide characterizations of semi well-ordered sets, as established by Ramachandran [16, 17].

Theorem 3.7. *[16] The poset $(\mathcal{S}, \preceq)$ is s.w.o if and only if $\mathcal{S}$ can be expressed as $\mathcal{W}+\mathcal{C}$, where $(\mathcal{W}, \leq)$ is a w.o set and $(\mathcal{C}, \leq)$ is a co-w.o set.*

In the proof of Theorem 3.7 [16], $\mathcal{W}$ is constructed by selecting elements of $\mathcal{S}$ that do not have an immediate predecessor, along with those elements of $\mathcal{S}$ that are the nth successor of such elements for some positive integer n. The set $\mathcal{C}$ consists of all elements in $\mathcal{S}$ that are not included in $\mathcal{W}$. From here onwards, we use $\mathcal{S}=\mathcal{W}+\mathcal{C}$ to denote a *s.w.o* set.

A poset $(\mathcal{S}, \preceq)$ is referred to as *rigid* [16] if the identity map is the only order isomorphism from $(\mathcal{S}, \preceq)$ onto itself. A poset is called *hereditarily rigid* [16] if every subset of $\mathcal{S}$ is rigid. We now present the following characterization of *s.w.o* sets.

Theorem 3.8. *[16] A poset is s.w.o if and only if it is hereditarily rigid.*

The following theorem provides an additional characterization for posets that are *s.w.o*.

Theorem 3.9. *Let $(\mathcal{S}, \preceq)$ be an arbitrary totally ordered set. Then $\mathcal{S}$ is s.w.o if and only if it contains no subsets of the form $D+I$, where D is a sequence arranged in strictly descending order and I is a strictly increasing sequence.*

Proof. Suppose that $\mathcal{S}$ is *s.w.o*. Then, by Theorem 3.7, $\mathcal{S}=\mathcal{W}+\mathcal{C}$, with $\mathcal{W}$ being *w.o* and $\mathcal{C}$ co-*w.o*. If $\mathcal{S}$ has a subset $D+I$, where D is a sequence arranged in strictly descending order and I is a strictly increasing sequence, then $D\subseteq\mathcal{C}$ as it lacks a smallest element. Consequently, $I\subseteq\mathcal{C}$, a contradiction. Hence, no such subset exists.

Conversely, assume that $\mathcal{S}$ contains no subsets of the form $D+I$, but suppose $\mathcal{S}$ is not *s.w.o*. Then $\mathcal{S}$ contains a subset $K\neq\phi$ with neither a smallest nor a largest element. Let $a_0\in K$ be arbitrary. Since a_0 cannot be the smallest element of K, we can find $a_1\in K$ with $a_1\prec a_0$; similarly, we can find $a_2\prec a_1$ and so on. This forms a strictly decreasing sequence $D=\{\ldots, a_2, a_1, a_0\}\subseteq K$. Likewise, since a_0 is not the largest element of K, there exists $b_1\in K$ with $a_0\prec b_1$, and similarly $b_1\prec b_2$, and so on, forming a sequence which is strictly increasing $I=\{b_1, b_2, \ldots\}\subseteq K$. Hence, $D+I\subseteq K\subseteq\mathcal{S}$, contradicting the assumption. So $\mathcal{S}$ must be *s.w.o*. □

Consider $(\mathcal{S}, \preceq)$ as a semi-well ordered set. For an element $s\in\mathcal{S}$, the set $\mathcal{S}s=\{s_1\in\mathcal{S}: s_1\prec s\}$ is known as the *initial segment* determined by s and $s\mathcal{S}=\{s_1\in\mathcal{S}: s\prec s_1\}$ is known as the *terminal segment* determined by s.

Indeed, a *w.o* set cannot be order isomorphic to any of its proper initial segments and similarly, a co-*w.o* set cannot be order isomorphic to any of its proper terminal segments. In contrast, a *s.w.o* set may be order isomorphic to one of its initial or terminal segments. This can be seen in the next example.

Example 3.10. Let $\mathcal{S}=\{\ldots, -3, -2, -1, 0\}$ and $\mathcal{S}'=\{0, 1, 2, \ldots\}$. Then the initial segment determined by 0 in $\mathcal{S}$ is $\mathcal{S}0=\{\ldots, -3, -2, -1\}$ and the terminal segment determined by 0 in $\mathcal{S}'$ is $0\mathcal{S}'=\{1, 2, \ldots\}$. It can be observed that $\mathcal{S}\approx\mathcal{S}0$ and $\mathcal{S}'\approx 0\mathcal{S}'$.

Theorem 3.11. *The statements that follow are equivalent:*

(i) *For every set, there exists a well ordering (Well ordering theorem).*

(ii) *Every set admits a semi-well ordering. (Semi-well ordering theorem).*

Proof. Suppose that the Well ordering theorem is true. Then it follows that every set $\mathcal{S}$ admits a well ordering. Since any *w.o* set is *s.w.o*, it follows that $\mathcal{S}$ is *s.w.o*. Conversely, assume that the Semi-well ordering theorem holds. So, any set $\mathcal{S}$ can be endowed with a semi-well ordering $\preceq$. By Theorem 3.7, $\mathcal{S}$ can be represented as $\mathcal{S}=\mathcal{W}+\mathcal{C}$, where $(\mathcal{W}, \preceq)$ is *w.o* and $(\mathcal{C}, \preceq)$ is co-*w.o*. Then $(\mathcal{C}, \succeq)$ is *w.o* and the relation $\preceq\cup\succeq\cup\{(w_1, c_1): w_1\in\mathcal{W}, c_1\in\mathcal{C}\}$ defines a well ordering on $\mathcal{S}$. Hence, it is possible to well order $\mathcal{S}$. □

Because the Well ordering theorem and the Axiom of choice are equivalent, Theorem 3.11 implies that the Semi-well ordering theorem is likewise equivalent to the Axiom of Choice.

Let $\mathcal{S}_1 = \mathcal{W}_1 + \mathcal{C}_1$ and $\mathcal{S}_2 = \mathcal{W}_2 + \mathcal{C}_2$ be two disjoint *s.w.o* sets. Define a relation on $\mathcal{S}_1 \cup \mathcal{S}_2$ which can be pictured as $(\mathcal{W}_1 + \mathcal{W}_2)$ + $(\mathcal{C}_2 + \mathcal{C}_1)$. Since $\mathcal{W}_1 + \mathcal{W}_2$ forms a *w.o* set and $\mathcal{C}_2 + \mathcal{C}_1$ constitutes a co-*w.o* set, $\mathcal{S}_1 \cup \mathcal{S}_2$ with this relation is *s.w.o*. Now, consider I_w as a *w.o* set and let $\{\mathcal{S}_i = \mathcal{W}_i$+$\mathcal{C}_i$: $i \in I_w\}$ denote a collection of disjoint *s.w.o* sets indexed by I_w. Consider the relation $\preceq^*$ on $\bigcup_{i \in I_w} \mathcal{S}_i$ which can be described as listing the elements of $\mathcal{W}_i + \mathcal{W}_j$ followed by the elements of $\mathcal{C}_j + \mathcal{C}_i$ for $i < j$. That is, $(\mathcal{W}_i + \mathcal{W}_j)$+$(\mathcal{C}_j + \mathcal{C}_i)$.

Theorem 3.12. *If $\{\mathcal{S}_i$: $i \in I_w\}$ forms a collection of pairwise disjoint s.w.o sets where I_w is a w.o indexing set, then $\bigcup_{i \in I_w} \mathcal{S}_i$ together with the ordering $\preceq^*$ is s.w.o.*

Proof. This definition of $\preceq^*$ on the set $\bigcup_{i \in I_w} \mathcal{S}_i$ leads directly to this conclusion. □

4 Order Ideals in Semi-Well Ordered Sets

Consider a poset $(\mathcal{S}, \preceq)$. A subset $W \subseteq \mathcal{S}$ is referred to as an *order ideal* [25] in $\mathcal{S}$ if, for every $w \in W$, whenever $x \preceq w$, it follows that $x \in W$. We write $L(\mathcal{S})$ for the collection of all order ideals of $\mathcal{S}$. Both the whole set $\mathcal{S}$ and the empty set belong to $L(\mathcal{S})$ with $\mathcal{S}$ being referred to as the trivial order ideal. Ordered by set inclusion, $L(\mathcal{S})$ forms a complete lattice. In this section, we show that the collection of all order ideals of a *s.w.o* set is itself *s.w.o* under the usual set inclusion relation.

In [25], Wolk demonstrated that if $(\mathcal{S}, \preceq)$ is a *p.w.o* set, then $L(\mathcal{S})$ does not admit a sequence that is strictly decreasing. Since every *w.o* set is *p.w.o*, $L(\mathcal{S})$ contains no strictly decreasing sequence if $\mathcal{S}$ is *w.o*.

Proposition 4.1. *If $(\mathcal{S}, \preceq)$ is a co-w.o set, then $L(\mathcal{S})$ does not admit any strictly increasing sequence under inclusion.*

Proof. Suppose that $L(\mathcal{S})$ admits an infinite strictly increasing sequence $\{V_n : n \in \mathbb{N}\}$ of order ideals. Then for any positive integer n, we can find an element $x_n \in V_{n+1} \setminus V_n$. Define $A = \{x_n : n \in \mathbb{N}\}$. Because $\mathcal{S}$ is co-*w.o* and A is non-empty, A contains a largest element $x_j \in A$. Then, for each n, $x_n \preceq x_j$; in particular, $x_{j+1} \preceq x_j$, which implies $x_{j+1} \in V_{j+1}$, as V_{j+1} is an order ideal and contains x_j. However, by construction, $x_{j+1} \in V_{j+2} \setminus V_{j+1}$, a contradiction. Hence, $L(\mathcal{S})$ contains no infinite strictly increasing sequence. □

Remark 4.2. If $(\mathcal{S}, \preceq)$ is a *s.w.o* set, then $L(\mathcal{S})$ might include a strictly increasing or strictly decreasing sequence.

Example 4.3. Consider $(\mathbb{N}, \leq)$, where $\mathbb{N}$ denotes the collection of all positive integers and $\leq$ represents the standard ordering. Then the sequence $\{\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots\}$ forms a strictly increasing sequence in $L(\mathbb{N})$.

Example 4.4. Consider $(\mathbb{N}', \leq)$, where $\mathbb{N}'$ denotes the collection of all negative integers and $\leq$ represents the standard ordering. Then the sequence {{. . ., -3, -2, -1}, {. . ., -3, -2}, {. . ., -4, -3}, . . .} forms a sequence which is strictly descending in $L(\mathbb{N}')$.

The proposition below is immediate.

Proposition 4.5. *Let $(\mathcal{S}, \leq)$ be an arbitrary w.o set. Then the set $L(\mathcal{S})$ consists precisely of the following: the entire set $\mathcal{S}$, the empty set and all initial segments of the form $Sx = \{s \in \mathcal{S} : s < x\}$ for each $x \in \mathcal{S}$.*

Proposition 4.6. *Let $(\mathcal{S}, \leq)$ be an arbitrary infinite co-w.o set. Then its order ideals include the entire set $\mathcal{S}$, the empty set and sets $\mathcal{S}z \cup \{z\}$, where $z \in \mathcal{S}$ and $Sz = \{s \in \mathcal{S} : s < z\}$.*

Proof. Clearly, the entire set $\mathcal{S}$ and the empty set are order ideals of $\mathcal{S}$. Let $W \in L(\mathcal{S})$ with $W \neq \phi$ and $W \neq \mathcal{S}$. Because $\mathcal{S}$ is co-*w.o*, W contains a largest element, namely s_1. We claim $W = \mathcal{S}s_1 \cup \{s_1\}$. If $y \in W$ and $y \neq s_1$, then $y \prec s_1$, so $y \in \mathcal{S}_{s_1}$, implying, $W \subseteq \mathcal{S}s_1 \cup \{s_1\}$. Conversely, if $y \in \mathcal{S}_{s_1}$, then $y \prec s_1$ and since W is an order ideal containing s_1, $y \in W$. Hence $\mathcal{S}s_1 \cup \{s_1\} \subseteq W$. Therefore, $W = \mathcal{S}s_1 \cup \{s_1\}$. □

Theorem 4.7. *Let $(\mathcal{S}, \preceq)$ is a s.w.o set represented by $\mathcal{S} = \mathcal{W} + \mathcal{C}$, where $\mathcal{C}$ is non-empty. Then the elements of the set $L(\mathcal{S})$ are: $\mathcal{S}$, $\emptyset$, $\mathcal{W}$, $\mathcal{S}a$ for $a \in \mathcal{W}$ and $\mathcal{S}b \cup \{b\}$ for $b \in \mathcal{C}$.*

Proof. Consider $A \in L(\mathcal{S})$ that is neither empty nor equal to the entire set. If $A \subseteq \mathcal{W}$, then by Proposition 4.5, $A = \emptyset$ or $A = \mathcal{W}$ or $A = Sa$ for some $a \in \mathcal{W}$. Otherwise, $A = \mathcal{W} \cup A'$, where $A' \subseteq \mathcal{C}$ and A' is an order ideal of $\mathcal{C}$. By Proposition 4.6, A' is either equal to $\mathcal{C}$ or $\mathcal{C}b \cup \{b\}$ for some $b \in \mathcal{C}$. Hence, $A = \mathcal{S}$ or $A = \mathcal{W} \cup \mathcal{C}b \cup \{b\} = \mathcal{S}b \cup \{b\}$. □

Theorem 4.8. *If $(\mathcal{S}, \preceq)$ is a s.w.o set, then the collection of its order ideals $L(\mathcal{S})$, ordered by set inclusion, also forms a s.w.o set.*

Proof. Consider a non-empty *s.w.o* set $(\mathcal{S}, \preceq)$. Then $\mathcal{S} = \mathcal{W} + \mathcal{C}$, where $\mathcal{W}$ is *w.o* and $\mathcal{C}$ is co-*w.o*. We examine the following cases:

Case (i): $\mathcal{C} = \emptyset$, $\mathcal{W} \neq \emptyset$.
Then $\mathcal{S} = \mathcal{W}$ and $L(\mathcal{S}) = \{\mathcal{S}, \emptyset\} \cup \{\mathcal{S}x\colon x \in \mathcal{S}\}$. If a non-empty subset $A \subseteq L(\mathcal{S})$ includes the empty set, then the empty set is the smallest element of A. If not, let a be the smallest element of the set $\{x\colon \mathcal{S}x \in A\}$. Then $\mathcal{S}a$ is the smallest element of A. Thus, $L(\mathcal{S})$ is *w.o*.

Case (ii): $\mathcal{W} = \emptyset, \mathcal{C} \neq \emptyset$.
Then $\mathcal{S} = \mathcal{C}$ and $L(\mathcal{S}) = \{\mathcal{S}, \emptyset\} \cup \{\mathcal{S}x \cup \{x\}\colon x \in \mathcal{S}\}$. If $A \subseteq L(\mathcal{S})$ includes the entire set $\mathcal{S}$, then $\mathcal{S}$ is the largest element of A. Otherwise, let b be the largest element of $\{x\colon \mathcal{S}x \cup \{x\} \in A\}$. Then $\mathcal{S}b \cup \{b\}$ becomes the largest element of A. Hence, $L(\mathcal{S})$ is co-*w.o*.

Case (iii): $\mathcal{W}, \mathcal{C} \neq \emptyset$.
Then, $L(\mathcal{S}) = \{\mathcal{S}, \emptyset, \mathcal{W}\} \cup \{\mathcal{S}x\colon x \in \mathcal{W}\} \cup \{\mathcal{S}y \cup \{y\}\colon y \in \mathcal{C}\}$. Any non-empty subset A of $L(\mathcal{S})$ that includes the empty set has a smallest element; if it contains $\mathcal{S}$, then $\mathcal{S}$ is the largest element. If A contains only $\mathcal{W}$ and elements of the form $\mathcal{S}a$ for $a \in \mathcal{W}$, then the smallest element of A is $\mathcal{S}x$, where x is the smallest element of $\{a\colon \mathcal{S}a \in A\}$. If A contains elements of the form $\mathcal{S}b \cup \{b\}$ for $b \in \mathcal{C}$, then $\mathcal{S}y \cup \{y\}$ becomes the largest element of A, where y is the largest element of the set $\{b \in \mathcal{C}\colon \mathcal{S}b \cup \{b\} \in A\}$.

Hence, $L(\mathcal{S})$, ordered by set inclusion, forms a *s.w.o* set. □

Theorem 4.9. *Let $(\mathcal{S}, \leq)$ be a s.w.o set expressed as $\mathcal{S} = \mathcal{W} + \mathcal{C}$. If i is an order isomorphism from $\mathcal{S}$ onto an order ideal $i(\mathcal{S})$ of $\mathcal{S}$, then $i(\mathcal{S}) = \mathcal{S}$ or $i(\mathcal{S}) = \mathcal{S}y \cup \{y\}$ for some $y \in \mathcal{C}$.*

Proof. Obviously, if $\mathcal{S}$ is empty, then $i(\mathcal{S}) = \mathcal{S}$. Now suppose $\mathcal{S}$ contains elements and $\mathcal{C}$ is empty, so $\mathcal{S} = \mathcal{W}$ is a *w.o* set. In this case, $L(\mathcal{S})$ contains precisely the empty set, $\mathcal{S}$ and Sa for $a \in \mathcal{S}$. As a *w.o* set is never ordinally similar to any of its proper initial segments, one can deduce that $i(\mathcal{S}) = \mathcal{S}$. Next, suppose $\mathcal{W}$ is empty, so $\mathcal{S} = \mathcal{C}$ is a co-*w.o* set. In this case, the order ideals of $\mathcal{S}$ are $\emptyset$, $\mathcal{S}$ and $\mathcal{S}y \cup \{y\}$ for $y \in \mathcal{C}$. Trivially, we get $i(\mathcal{S}) = \mathcal{S}$ or $i(\mathcal{S}) = \mathcal{S}y \cup \{y\}$, $y \in \mathcal{C}$.

Now assume that both $\mathcal{W}$ and $\mathcal{C}$ are non-empty, so $\mathcal{S} = \mathcal{W} + \mathcal{C}$. Then the order ideals of $\mathcal{S}$ are: $L(\mathcal{S}) = \{\mathcal{S}, \emptyset, \mathcal{W}\} \cup \{\mathcal{S}x\colon x \in \mathcal{W}\} \cup \{Sy \cup \{y\}\colon y \in \mathcal{C}\}$. We claim that $i(\mathcal{S}) = \mathcal{W}$ or $i(\mathcal{S}) = \mathcal{S}a$ for some $a \in \mathcal{W}$ is not possible.

Suppose, for contradiction, that $i(\mathcal{S}) = \mathcal{W}$. Consider $i(\mathcal{W})$. Let $w \in i(\mathcal{W})$ and $y \preceq w$. Since i is order preserving and $\mathcal{W}$ is an order ideal of $\mathcal{S}$, it follows that $y \in i(\mathcal{W})$. Therefore, $i(\mathcal{W}) \subseteq \mathcal{W}$ satisfies the condition of being an order ideal. Hence $i(\mathcal{W}) = \mathcal{W}$. This is a contradiction, since we assumed that $\mathcal{C}$ is non-empty.

Similarly, suppose $i(\mathcal{S}) = \mathcal{S}a$ for some $a \in \mathcal{W}$ and consider $i(\mathcal{W})$. Let $x \in i(\mathcal{W})$ and $y \preceq x$. Again, since i is order preserving and $\mathcal{W}$ is an order ideal of $\mathcal{S}$, we have $y \in i(\mathcal{W})$, so $i(\mathcal{W}) \subseteq \mathcal{S}a$ is an order ideal of $\mathcal{S}$. Thus, $i(\mathcal{W}) = \mathcal{S}a_1$ for some $a_1 \leq a$ and $a_1 \in \mathcal{W}$. But this contradicts the fact that no *w.o* set is order isomorphic to a proper initial segment of itself.

Therefore, the only possibilities are $i(\mathcal{S}) = \mathcal{S}$ or $i(\mathcal{S}) = \mathcal{S}y \cup \{y\}$ for some $y \in \mathcal{C}$. □

5 Semi-Ordinals

We define the order type of a *s.w.o* set as a *semi-ordinal number*, which we refer to as a *semi-ordinal*. Furthermore, any chain that is ordinally similar to a *s.w.o* set is itself *s.w.o*. Hence, every representative of a semi-ordinal is a *s.w.o* set. All ordinals are semi-ordinals, so the collection of semi-ordinals properly extends the collection of ordinals.

We define the order type of a co-*w.o* set as a *co-ordinal number*, which we often call a *co-ordinal*. If $(B, \preceq)$ is a co-*w.o* set, then its dual $(B, \succeq)$ is a *w.o* set. Let υ be the order type of $(B, \succeq)$; the order type of $(B, \preceq)$ is then denoted by $\tilde{\upsilon}$. A co-ordinal is thus represented as $\tilde{\lambda}$, where λ is an ordinal.

Any *s.w.o* set $(\mathcal{S}, \preceq)$ can be decomposed into the sum $\mathcal{W} + \mathcal{C}$, where $\mathcal{W}$ is a *w.o* set and $\mathcal{C}$ is a co-*w.o* set. So we denote the semi-ordinal of $\mathcal{S}$ by $(\lambda, \tilde{\upsilon})$ where λ is the ordinal of $\mathcal{W}$ and $\tilde{\upsilon}$ is the co-ordinal of $\mathcal{C}$. In this representation, υ is always a limit ordinal, so each semi-ordinal is represented in a unique way. The class of all semi-ordinals can be viewed as the Cartesian product between the class all ordinals and the class of all limit co-ordinals. This provides a canonical representation of a semi-ordinal.

Recall that if λ_1 and λ_2 are two ordinals, then $\lambda_1 < \lambda_2$ [6] if and only if there exist representatives A_1 and A_2 of λ_1 and λ_2, respectively, such that $A_1 \approx A_2 x$ for some $x \in A_2$. Similarly, if $\tilde{\upsilon_1}$ and $\tilde{\upsilon_2}$ are two co-ordinals, we define $\tilde{\upsilon_1} < \tilde{\upsilon_2}$ if and only if it is possible to find representatives $\tilde{B_1}$ and $\tilde{B_2}$ of $\tilde{\upsilon_1}$ and $\tilde{\upsilon_2}$, respectively, with $\tilde{B_1} \approx x\tilde{B_2}$ for some $x \in \tilde{B_2}$. For any ordinals λ_1 and λ_2, we have $\lambda_1 < \lambda_2$ if and only if $\tilde{\lambda}_1 < \tilde{\lambda}_2$. For semi-ordinals $(\lambda_1, \tilde{\upsilon_1})$ and $(\lambda_2, \tilde{\upsilon_2})$, we define $(\lambda_1, \tilde{\upsilon_1}) < (\lambda_2, \tilde{\upsilon_2})$ if and only if $\upsilon_1 < \upsilon_2$ or $\upsilon_1 = \upsilon_2$ and $\lambda_1 < \lambda_2$.

Consider an arbitrary set Δ of semi-ordinals. The relation $<$ on Δ is irreflexive and transitive. Consequently, $(\Delta, \leq)$ is a poset, where $\leq$ is defined by $x_1 \leq y_1$ if and only if $x_1 < y_1$ or $x_1 = y_1$ for all $x_1, y_1 \in \Delta$. Since any two ordinals are comparable, $(\Delta, \leq)$ is a chain.

Theorem 5.1. *The poset $(\Delta, \leq)$ is w.o, where Δ denotes an arbitrary set of semi-ordinals.*

Proof. For semi-ordinals $(\lambda_1, \tilde{\upsilon_1})$ and $(\lambda_2, \tilde{\upsilon_2})$, we have $(\lambda_1, \tilde{\upsilon_1}) \leq (\lambda_2, \tilde{\upsilon_2})$ if and only if $\upsilon_1 < \upsilon_2$ or $\upsilon_1 = \upsilon_2$ and $\lambda_1 \leq \lambda_2$. Let $A \subseteq \Delta$ be non-empty. Then $\{\upsilon_i\colon (\lambda_j, \tilde{\upsilon_i}) \in A\}$ has a smallest elements υ, since any set of ordinals is *w.o*. Among all $(\lambda_j, \tilde{\upsilon}) \in A$, the set $\{\lambda_j : (\lambda_j, \tilde{\upsilon}) \in A\}$ has a smallest element say λ. Thus, $(\lambda, \tilde{\upsilon})$ represents the smallest element of A, which implies that Δ is *w.o*. □

Proposition 5.2. *There always exists a semi-ordinal number that is strictly greater than any given semi-ordinal.*

Proof. Consider a semi-ordinal $(\lambda, \tilde{\upsilon})$. Because υ is a limit ordinal, there exists another limit ordinal $\upsilon' > \upsilon$. Then $(\lambda, \tilde{\upsilon}) < (\lambda, \tilde{\upsilon'})$, so there exists a semi-ordinal strictly greater than $(\lambda, \tilde{\upsilon})$. □

We define a *limit semi-ordinal* as a semi-ordinal with no immediate predecessor and a *non-limit semi-ordinal* as one that has an immediate predecessor.

5.1 Arithmetic of Semi-ordinals

This section introduces arithmetic operations on semi-ordinals, focusing on addition and multiplication and briefly outline their properties.

(1) *Sum of two semi-ordinals*: Let $(\lambda_1, \tilde{\upsilon_1})$ and $(\lambda_2, \tilde{\upsilon_2})$ be two semi-ordinals. Their sum is defined as: $(\lambda_1, \tilde{\upsilon_1}) + (\lambda_2, \tilde{\upsilon_2}) = (\lambda_1 + \lambda_2, (\upsilon_1 + \upsilon_2)^{\sim})$.

(2) *Product of two semi-ordinals*: Let $(\lambda_1, \tilde{\upsilon_1})$ and $(\lambda_2, \tilde{\upsilon_2})$ be two semi-ordinals. Their product is defined as: $(\lambda_1, \tilde{\upsilon_1}) \cdot (\lambda_2, \tilde{\upsilon_2}) = (\lambda_1 \cdot \lambda_2, (\upsilon_1 \cdot \upsilon_2)^{\sim})$.

Because ordinal addition and multiplication do not satisfy the commutative property, the same holds for semi-ordinal addition and multiplication.

Let $\{(\lambda_i, \tilde{\upsilon_i}) : i \in I_w\}$ be a collection of semi-ordinals indexed by a *w.o* set I_w. For each $i \in I_w$, let $\mathcal{S}_i$ denote a *s.w.o* set of order type $(\lambda_i, \tilde{\upsilon_i})$. Then the collection $\{\mathcal{S}_i \times \{i\}\colon i \in I_w\}$ is clearly a *w.o* collection of disjoint *s.w.o* sets. By Theorem 3.12, $\bigcup_{i \in I_w} (\mathcal{S}_i \times \{i\})$ is *s.w.o*.

Definition 5.3. Let $\{(\lambda_i, \tilde{\upsilon_i})\colon i \in I_w\}$ be a *w.o* collection of semi-ordinals, where $(\lambda_i, \tilde{\upsilon_i})$ is the order type of a *s.w.o* set $\mathcal{S}_i$. Then, the sum $\sum\limits_{i\in I_w} (\lambda_i, \tilde{\upsilon_i})$ is defined as the order type of the union $\bigcup\limits_{i\in I_w} (\mathcal{S}_i \times \{i\})$.

Example 5.4. $\sum\limits_{i\in\mathbb{N}} (\omega, \tilde{\omega}) = (\omega^2, \tilde{\omega^2})$ and $\sum\limits_{i\in\mathbb{N}} (1, 0) = (\omega, 0)$.

Proposition 5.5. *For any semi-ordinals, the following properties hold:*

(i) The addition operation satisfies the associative property.

(ii) The multiplication operation is associative.

(iii) Left distributive law.

Proof. Let $(\lambda_1, \tilde{\upsilon_1})$, $(\lambda_2, \tilde{\upsilon_2})$ and $(\lambda_3, \tilde{\upsilon_3})$ be three semi-ordinals.

(i) We know that the addition of ordinals is associative. Now,

$$\begin{aligned}[(\lambda_1, \tilde{\upsilon_1}) + (\lambda_2, \tilde{\upsilon_2})] + (\lambda_3, \tilde{\upsilon_3}) &= ((\lambda_1 + \lambda_2), (\upsilon_1 + \upsilon_2)^{\sim}) + (\lambda_3, \tilde{\upsilon_3}) \\ &= ((\lambda_1 + \lambda_2) + \lambda_3, ((\upsilon_1 + \upsilon_2) + \upsilon_3)^{\sim}) \\ &= (\lambda_1 + (\lambda_2 + \lambda_3), (\upsilon_1 + (\upsilon_2 + \upsilon_3))^{\sim}) \\ &= (\lambda_1, \tilde{\upsilon_1}) + ((\lambda_2 + \lambda_3), (\upsilon_2 + \upsilon_3)^{\sim}) \\ &= (\lambda_1, \tilde{\upsilon_1}) + [(\lambda_2, \tilde{\upsilon_2}) + (\lambda_3, \tilde{\upsilon_3})]\end{aligned}$$

Thus, the addition of semi-ordinals is associative.

(ii) It is well established that ordinal multiplication is an associative operation. Now,

$$\begin{aligned}[(\lambda_1, \tilde{\upsilon_1}) \cdot (\lambda_2, \tilde{\upsilon_2})] \cdot (\lambda_3, \tilde{\upsilon_3}) &= ((\lambda_1 \cdot \lambda_2), (\upsilon_1 \cdot \upsilon_2)^{\sim}) \cdot (\lambda_3, \tilde{\upsilon_3}) \\ &= ((\lambda_1 \cdot \lambda_2) \cdot \lambda_3, ((\upsilon_1 \cdot \upsilon_2) \cdot \upsilon_3)^{\sim}) \\ &= (\lambda_1 \cdot (\lambda_2 \cdot \lambda_3), (\upsilon_1 \cdot (\upsilon_2 \cdot \upsilon_3))^{\sim}) \\ &= (\lambda_1, \tilde{\upsilon_1}) \cdot ((\lambda_2 \cdot \lambda_3), (\upsilon_2 \cdot \upsilon_3)^{\sim}) \\ &= (\lambda_1, \tilde{\upsilon_1}) \cdot [(\lambda_2, \tilde{\upsilon_2}) \cdot (\lambda_3, \tilde{\upsilon_3})]\end{aligned}$$

Thus, multiplication of semi-ordinals is associative.

(iii) It is a standard result that ordinal multiplication distributes over addition from the left. Now,

$$\begin{aligned}(\lambda_1, \tilde{\upsilon_1}) \cdot [(\lambda_2, \tilde{\upsilon_2}) + (\lambda_3, \tilde{\upsilon_3})] &= (\lambda_1, \tilde{\upsilon_1}) \cdot ((\lambda_2 + \lambda_3), (\upsilon_2 + \upsilon_3)^{\sim}) \\ &= (\lambda_1 \cdot (\lambda_2 + \lambda_3), (\upsilon_1 \cdot (\upsilon_2 + \upsilon_3)^{\sim})) \\ &= (\lambda_1 \cdot \lambda_2 + \lambda_1 \cdot \lambda_3, (\upsilon_1 \cdot \upsilon_2 + \upsilon_1 \cdot \upsilon_3)^{\sim}) \\ &= (\lambda_1 \cdot \lambda_2, (\upsilon_1 \cdot \upsilon_2)^{\sim}) + (\lambda_1 \cdot \lambda_3, (\upsilon_1 \cdot \upsilon_3)^{\sim}) \\ &= [(\lambda_1, \tilde{\upsilon_1}) \cdot (\lambda_2, \tilde{\upsilon_2})] + [(\lambda_1, \tilde{\upsilon_1}) \cdot (\lambda_3, \tilde{\upsilon_3})]\end{aligned}$$

Hence, the left distributive law holds for semi-ordinals.

□

Since right cancellation for addition and multiplication, as well as the right distributive law, do not hold for ordinals, they do not hold for semi-ordinals either.

Proposition 5.6. *Let* $(\lambda_1, \tilde{\upsilon_1})$ *and* $(\lambda_2, \tilde{\upsilon_2})$ *be two semi-ordinals. If* $(\lambda_2, \tilde{\upsilon_2}) > (0, 0)$*, then* $(\lambda_1, \tilde{\upsilon_1}) + (\lambda_2, \tilde{\upsilon_2}) > (\lambda_1, \tilde{\upsilon_1})$.

Proof. Suppose $(\lambda_2, \tilde{\upsilon_2}) > (0, 0)$. Then $\upsilon_2 > 0$ or $\upsilon_2 = 0$ and $\lambda_2 > 0$. If $\upsilon_2 > 0$, then $\upsilon_1 + \upsilon_2 > \upsilon_1$, so $(\lambda_1, \tilde{\upsilon_1}) + (\lambda_2, \tilde{\upsilon_2}) = (\lambda_1 + \lambda_2, (\upsilon_1 + \upsilon_2)^{\sim}) > (\lambda_1, \tilde{\upsilon_1})$. If $\upsilon_2 = 0$ and $\lambda_2 > 0$, then $\lambda_1 + \lambda_2 > \lambda_1$, and $(\lambda_1, \tilde{\upsilon_1}) + (\lambda_2, \tilde{\upsilon_2}) = (\lambda_1 + \lambda_2, \tilde{\upsilon_1}) > (\lambda_1, \tilde{\upsilon_1})$. Thus, in either case, $(\lambda_1, \tilde{\upsilon_1}) + (\lambda_2, \tilde{\upsilon_2}) > (\lambda_1, \tilde{\upsilon_1})$. □

Theorem 5.7. *Let* $(\lambda_1, \tilde{\upsilon_1})$, $(\lambda_2, \tilde{\upsilon_2})$ *and* $(\lambda_3, \tilde{\upsilon_3})$ *be semi-ordinals. Then:*

(i) If $(\lambda_1, \tilde{\upsilon_1}) < (\lambda_2, \tilde{\upsilon_2})$, *then* $(\lambda_3, \tilde{\upsilon_3}) + (\lambda_1, \tilde{\upsilon_1}) < (\lambda_3, \tilde{\upsilon_3}) + (\lambda_2, \tilde{\upsilon_2})$. *The converse also holds.*

(ii) If $(\lambda_3, \tilde{\upsilon_3}) + (\lambda_1, \tilde{\upsilon_1}) = (\lambda_3, \tilde{\upsilon_3}) + (\lambda_2, \tilde{\upsilon_2})$, *then* $(\lambda_1, \tilde{\upsilon_1}) = (\lambda_2, \tilde{\upsilon_2})$.

Proof. (i) Suppose that $(\lambda_1, \tilde{\upsilon_1})<(\lambda_2, \tilde{\upsilon_2})$. Then, either $\upsilon_1 < \upsilon_2$ or $\upsilon_1 = \upsilon_2$ and $\lambda_1 < \lambda_2$. If $\upsilon_1 < \upsilon_2$, then $\upsilon_3 + \upsilon_1 < \upsilon_3 + \upsilon_2$, so $(\lambda_3, \tilde{\upsilon_3})+(\lambda_1, \tilde{\upsilon_1})<(\lambda_3, \tilde{\upsilon_3})+(\lambda_2, \tilde{\upsilon_2})$. If $\upsilon_1 = \upsilon_2$ and $\lambda_1 < \lambda_2$, then $\upsilon_3+\upsilon_1 = \upsilon_3+\upsilon_2$ and $\lambda_3+\lambda_1 < \lambda_3+\lambda_2$, so $(\lambda_3, \tilde{\upsilon_3})+(\lambda_1, \tilde{\upsilon_1})<(\lambda_3, \tilde{\upsilon_3})+(\lambda_2, \tilde{\upsilon_2})$.

Conversely, suppose that $(\lambda_3, \tilde{\upsilon_3})+(\lambda_1, \tilde{\upsilon_1})<(\lambda_3, \tilde{\upsilon_3})+(\lambda_2, \tilde{\upsilon_2})$. Then, either $\upsilon_3 + \upsilon_1 < \upsilon_3 + \upsilon_2$ or $\upsilon_3 + \upsilon_1 = \upsilon_3 + \upsilon_2$ and $\lambda_3 + \lambda_1 < \lambda_3 + \lambda_2$. If $\upsilon_3 + \upsilon_1 < \upsilon_3 + \upsilon_2$, then $\upsilon_1 < \upsilon_2$, so $(\lambda_1, \tilde{\upsilon_1})<(\lambda_2, \tilde{\upsilon_2})$. If $\upsilon_3 + \upsilon_1 = \upsilon_3 + \upsilon_2$ and $\lambda_3 + \lambda_1 < \lambda_3 + \lambda_2$, then $\upsilon_1 = \upsilon_2$ and $\lambda_1 < \lambda_2$, so $(\lambda_1, \tilde{\upsilon_1})<(\lambda_2, \tilde{\upsilon_2})$.

(ii) Suppose $(\lambda_3, \tilde{\upsilon_3}) + (\lambda_1, \tilde{\upsilon_1}) = (\lambda_3, \tilde{\upsilon_3}) + (\lambda_2, \tilde{\upsilon_2})$. Then, $(\lambda_3+\lambda_1, (\upsilon_3+\upsilon_1)\tilde{\ })=(\lambda_3+\lambda_2, (\upsilon_3+\upsilon_2)\tilde{\ })$. This implies $\lambda_3+\lambda_1 = \lambda_3+\lambda_2$ and $\upsilon_3+\upsilon_1 = \upsilon_3+\upsilon_2$. By the property of ordinals, $\lambda_1 = \lambda_2$ and $\upsilon_1 = \upsilon_2$, so $(\lambda_1, \tilde{\upsilon_1})=(\lambda_2, \tilde{\upsilon_2})$.

□

Remark 5.8. Let υ_1, υ_2 and υ_3 be three ordinals. Then:

a) If $\upsilon_1 < \upsilon_2$ and $\upsilon_3 > 0$, then $\upsilon_3 \cdot \upsilon_1 < \upsilon_3 \cdot \upsilon_2$.

b) If $\upsilon_3 \cdot \upsilon_1 < \upsilon_3 \cdot \upsilon_2$, then $\upsilon_1 < \upsilon_2$.

c) If $\upsilon_3 \cdot \upsilon_1 = \upsilon_3 \cdot \upsilon_2$ and $\upsilon_3 > 0$, then $\upsilon_1 = \upsilon_2$.

d) If $\upsilon_1 \cdot \upsilon_3 = \upsilon_2 \cdot \upsilon_3$ and υ_3 is a non-limit ordinal, then $\upsilon_1 = \upsilon_2$.

However, these properties need not be true for semi-ordinals. The following examples illustrate this:

a) $(1, \tilde{\omega})<(2, \tilde{\omega})$ and $(0, \tilde{\omega})>(0, 0)$, but $(0, \tilde{\omega})\cdot(1, \tilde{\omega})\not<(0, \tilde{\omega})\cdot(2, \tilde{\omega})$. Also, $(2, \tilde{\omega})<(1, \tilde{\omega 2})$ and $(\omega, 0)>(0, 0)$, but $(\omega, 0)\cdot(2, \tilde{\omega})\not<(\omega, 0)\cdot(1, \tilde{\omega 2})$.

b) $(2, 0)\cdot(\omega, \tilde{\omega 2})<(2, 0)\cdot(\omega 2, \tilde{\omega})$ and $(\omega, \tilde{\omega 2})\not<(\omega 2, \tilde{\omega})$.

c) $(0, \tilde{\omega})\cdot(2, \tilde{\omega})=(0, \tilde{\omega})\cdot(0, \tilde{\omega})$ and $(0, \tilde{\omega})>(0, 0)$, but $(2, \tilde{\omega})\neq(0, \tilde{\omega})$. Also, $(2, 0)\cdot(2, \tilde{\omega})=(2, 0)\cdot(2, \tilde{\omega 2})$ and $(2, 0)>(0, 0)$, but $(2, \tilde{\omega})\neq(2, \tilde{\omega 2})$.

d) $(\omega, \tilde{\omega 2})\cdot(1, 0)=(\omega, \tilde{\omega})\cdot(1, 0)$, where $(1, 0)$ is a non-limit semi-ordinal, but $(\omega, \tilde{\omega 2})\neq(\omega, \tilde{\omega})$. Also, $(\omega, \tilde{\omega 2})\cdot(1, \tilde{\omega}\ )=(\omega, \tilde{\omega})\cdot(1, \tilde{\omega})$, where $(1, \tilde{\omega})$ is a non-limit semi-ordinal, but $(\omega, \tilde{\omega 2})\neq(\omega, \tilde{\omega})$.

Theorem 5.9. *For semi-ordinals* $(\lambda_1, \tilde{\upsilon_1})$, $(\lambda_2, \tilde{\upsilon_2})$ *and* $(\lambda_3, \tilde{\upsilon_3})$, *the following hold:*

(i) If $(\lambda_1, \tilde{\upsilon_1}) < (\lambda_2, \tilde{\upsilon_2})$ *and* $\lambda_3 > 0$, $\upsilon_3 > 0$, *then* $(\lambda_3, \tilde{\upsilon_3}) \cdot (\lambda_1, \tilde{\upsilon_1}) < (\lambda_3, \tilde{\upsilon_3}) \cdot (\lambda_2, \tilde{\upsilon_2})$. *Conversely, if* $(\lambda_3, \tilde{\upsilon_3}) \cdot (\lambda_1, \tilde{\upsilon_1}) < (\lambda_3, \tilde{\upsilon_3}) \cdot (\lambda_2, \tilde{\upsilon_2})$ *and* $\upsilon_3 > 0$, *then* $(\lambda_1, \tilde{\upsilon_1}) < (\lambda_2, \tilde{\upsilon_2})$.

(ii) If $(\lambda_3, \tilde{\upsilon_3}) \cdot (\lambda_1, \tilde{\upsilon_1}) = (\lambda_3, \tilde{\upsilon_3}) \cdot (\lambda_2, \tilde{\upsilon_2})$ *and* $\lambda_3 > 0$, $\upsilon_3 > 0$, *then* $(\lambda_1, \tilde{\upsilon_1}) = (\lambda_2, \tilde{\upsilon_2})$.

(iii) If $(\lambda_1, \tilde{\upsilon_2}) \cdot (\lambda_3, 0) = (\lambda_2, \tilde{\upsilon_2}) \cdot (\lambda_3, 0)$ *and* λ_3 *is a non-limit ordinal, then* $(\lambda_1, \tilde{\upsilon_2}) = (\lambda_2, \tilde{\upsilon_2})$.

Proof. (i) Suppose that, $(\lambda_1, \tilde{\upsilon_1})<(\lambda_2, \tilde{\upsilon_2})$ and $\lambda_3 > 0$, $\upsilon_3 > 0$. Then, $\upsilon_1 < \upsilon_2$ or $\upsilon_1 = \upsilon_2$ and $\lambda_1 < \lambda_2$. If $\upsilon_1 < \upsilon_2$, then $\upsilon_3 \cdot \upsilon_1 < \upsilon_3 \cdot \upsilon_2$, so $(\lambda_3, \tilde{\upsilon_3})\cdot(\lambda_1, \tilde{\upsilon_1})<(\lambda_3, \tilde{\upsilon_3})\cdot(\lambda_2, \tilde{\upsilon_2})$. If $\upsilon_1 = \upsilon_2$ and $\lambda_1 < \lambda_2$, then $\upsilon_3 \cdot \upsilon_1 = \upsilon_3 \cdot \upsilon_2$ and $\lambda_3 \cdot \lambda_1 < \lambda_3 \cdot \lambda_2$, so $(\lambda_3, \tilde{\upsilon_3})\cdot(\lambda_1, \tilde{\upsilon_1})<(\lambda_3, \tilde{\upsilon_3})\cdot(\lambda_2, \tilde{\upsilon_2})$.

Conversely, suppose that $(\lambda_3, \tilde{\upsilon_3})\cdot(\lambda_1, \tilde{\upsilon_1})<(\lambda_3, \tilde{\upsilon_3})\cdot(\lambda_2, \tilde{\upsilon_2})$. Then, either $\upsilon_3 \cdot \upsilon_1 < \upsilon_3 \cdot \upsilon_2$ or $\upsilon_3 \cdot \upsilon_1 = \upsilon_3 \cdot \upsilon_2$ and $\lambda_3 \cdot \lambda_1 < \lambda_3 \cdot \lambda_2$. If $\upsilon_3 \cdot \upsilon_1 < \upsilon_3 \cdot \upsilon_2$, then $\upsilon_1 < \upsilon_2$, so $(\lambda_1, \tilde{\upsilon_1})<(\lambda_2, \tilde{\upsilon_2})$. Again, if $\upsilon_3 \cdot \upsilon_1 = \upsilon_3 \cdot \upsilon_2$ and $\lambda_3 \cdot \lambda_1 < \lambda_3 \cdot \lambda_2$, then $\upsilon_1 = \upsilon_2$ and $\lambda_1 < \lambda_2$, so $(\lambda_1, \tilde{\upsilon_1})<(\lambda_2, \tilde{\upsilon_2})$.

(ii) Suppose that $(\lambda_3, \tilde{\upsilon_3})\cdot(\lambda_1, \tilde{\upsilon_1})=(\lambda_3, \tilde{\upsilon_3})\cdot(\lambda_2, \tilde{\upsilon_2})$. This implies $(\lambda_3 \cdot \lambda_1, (\upsilon_3 \cdot \upsilon_1)^{\sim})= (\lambda_3 \cdot \lambda_2, (\upsilon_3 \cdot \upsilon_2)^{\sim})$. Therefore, $\lambda_3 \cdot \lambda_1 = \lambda_3 \cdot \lambda_2$, $\upsilon_3 \cdot \upsilon_1 = \upsilon_3 \cdot \upsilon_2$. Since $\lambda_3 > 0$ and $\upsilon_3 > 0$, by the property of ordinals, we must have $\lambda_1 = \lambda_2$ and $\upsilon_1 = \upsilon_2$. Hence, $(\lambda_1, \tilde{\upsilon_1})=(\lambda_2, \tilde{\upsilon_2})$.

(iii) Suppose that $(\lambda_1, \tilde{\upsilon_2})\cdot(\lambda_3, 0)=(\lambda_2, \tilde{\upsilon_2})\cdot(\lambda_3, 0)$. This implies $(\lambda_1 \cdot \lambda_3, 0)=(\lambda_2 \cdot \lambda_3, 0)$ so, $\lambda_1 \cdot \lambda_3 = \lambda_2 \cdot \lambda_3$. Since λ_3 is a non-limit ordinal, by the property of ordinals, we must have $\lambda_1 = \lambda_2$. Hence, $(\lambda_1, \tilde{\upsilon_2})=(\lambda_2, \tilde{\upsilon_2})$.

□

Remark 5.10. For any ordinals λ and υ with $\lambda < \upsilon$, there always exists an ordinal $\mu > 0$ with $\lambda + \mu = \upsilon$ and this ordinal μ is unique. However, this property does not hold for semi-ordinals. For example, $(\omega, \tilde{\omega}) < (1, \tilde{\omega 2})$, but there exists no semi-ordinal $(\lambda, \tilde{\upsilon})$ with $(\omega, \tilde{\omega}) + (\lambda, \tilde{\upsilon}) = (1, \tilde{\omega 2})$.

However, the following theorem holds:

Theorem 5.11. *If $(\lambda_1, \tilde{\upsilon_1})$ and $(\lambda_2, \tilde{\upsilon_2})$ are two semi-ordinals with $\lambda_1 < \lambda_2$ and $\upsilon_1 < \upsilon_2$, then there exists a unique semi-ordinal $(\gamma_1, \tilde{\gamma_2})$, with $\gamma_1 > 0$, $\gamma_2 > 0$ such that $(\lambda_1, \tilde{\upsilon_1}) + (\gamma_1, \tilde{\gamma_2}) = (\lambda_2, \tilde{\upsilon_2})$.*

Proof. Since $\upsilon_1 < \upsilon_2$, it is possible to find a unique ordinal $\gamma_2 > 0$ such that $\upsilon_1 + \gamma_2 = \upsilon_2$. Also, since $\lambda_1 < \lambda_2$ we can find a unique ordinal $\gamma_1 > 0$ such that $\lambda_1 + \gamma_1 = \lambda_2$. Hence, $(\gamma_1, \tilde{\nu_2})$ is a unique semi-ordinal with $(\lambda_1, \tilde{\upsilon_1}) + (\gamma_1, \tilde{\gamma_2})=(\lambda_2, \tilde{\upsilon_2})$. □

Theorem 5.12. *Let Γ be the collection of all semi-ordinals. Then any set of semi-ordinals has a least upper bound in Γ.*

Proof. Consider a set of semi-ordinals indexed by a set I, written as $A =\{(\lambda_i, \tilde{\upsilon_i}): i \in I\}$. Define $B =\{\upsilon_i: i \in I\}$. Let υ denote the least upper bound of B, which exists since B is a set of ordinals. If $\upsilon \in B$, let λ denote the least upper bound of the set $\{\lambda_i: (\lambda_i, \tilde{\upsilon}) \in A\}$. Then $(\lambda, \tilde{\upsilon})$ is the least upper bound of A. If $\upsilon \notin B$, then since υ is a limit ordinal, the semi-ordinal $(0, \tilde{\upsilon})$ serves as the least upper bound of A. Thus, in both cases, A has a least upper bound, which is a semi-ordinal. □

6 Conclusion

In this paper, we have explored the structural properties of semi-well ordered sets and conducted a detailed investigation into their order ideals. We introduced the concept of semi-ordinals and established a unique representation for them. An ongoing direction of research involves characterizing the set of periodic points in dynamical systems, such as self-homeomorphisms and continuous self-maps on $(\omega^2, \tilde{\omega^2})$. Another open question that arises from this study is whether semi-ordinals are reconstructible.

References

[1] R. A. Alo, *A proof of the complete normality of chains*, Acta Math. Acad. Sci. Hungar, **22** (3-4), 393–395, (1971).

[2] K. Burn and B. Hasselblatt, *The Sharkovsky theorem: a natural direct proof*, Amer. Math. Monthly, **118** (3), 229–244, (2011).

[3] D. H. J. de Jongh and R. Parikh, *Well-partial ordering and hierarchies*, Indag. Math., **39**, 195–207, (1979).

[4] R. C. Freiwald, *An introduction to set theory and topology*, Washington University in St. Louis, Saint Louis, Missouri, (2014).

[5] G. Higman, *Ordering divisibility in abstract algebras*, Proc. London Math. Soc., **3** (2), 326–336, (1952).

[6] T. Jech, *Set theory*, The third millennium edition, revised and expanded, Springer, (2003).

[7] R. D. Kopperman, E. H. Kronhemer and R.G. Wilson, *Topologies on totally ordered sets*, Topol. Appl., **90**, 165–185, (1998).

[8] J. B. Kruskal, *The theory of well-quasi-ordering: a frequently discovered concept*, J. Combin. Theory Ser. A, **13**, 297–305, (1972).

[9] J. B. Kruskal, *Well-quasi-ordering, the tree theorem and Vazsonyi's conjecture*, Trans. Amer. Math. Soc., **95**, 210–225, (1960).

[10] M. Malicki and A. Rutkowski, *On operations and linear extensions of well partially ordered sets*, Order, **21**, 7–17, (2004).

[11] E. Michael, *A class of partially ordered sets*, Amer. Math. Monthly, **67**, 448–449, (1960).

[12] E. C. Milner and N. Sauer, *On chains and antichains in well founded partially ordered sets*, J. London Math. Soc. (2), **24**, 15–33, (1981).

[13] Y. Moschovakis, *Notes on set theory*, second edition, Springer, (2005).

[14] S. K. Nimbhorkar and U. R. Borsarkar, *Some results on a graph of a poset*, Palest. J. Math., **8** (Special Issue: I), 482–489, (2019).

[15] M. Pouzet, *Well-quasi-ordering and embeddability of relational structures*, order, **41**, 183–278, (2024).

[16] P. T. Ramachandran, *Anti homogeneity is equivalent to hereditary rigidity*, Ganita, **35** (1), 1–9, (1984).

[17] P. T. Ramachandran, *Some problems in set topology relating group of homeomorphisms and order*, PhD thesis, Department of Mathematics, University of Cochin, (1985).

[18] D. Schmidt, *The relation between the height of a well-founded partial ordering and the order types of its chains and antichains*, J. Combin. Theory Ser. B, **31**, 183–189, (1981).

[19] S. Shaqaqha, *Characterizations of Artinian and Noetherian gamma rings in terms of homogeneous complex fuzzy ideals*, Palest. J. Math., **11**(4), 167–171, (2022).

[20] P. Sini, *The group of homeomorphisms and the cyclic groups of permutations*, Palest. J. Math., **8**(2), 53–60, (2019).

[21] M. Sreeja, *On the total negation of rigidity*, Int. J. Pure Appl. Math., **82** (3), 391–398, (2013).

[22] A. K. Sruthi and P. Sini, *Some order properties of C-spaces*, Palest. J. Math., **12**(3), 180–185, (2023).

[23] A. Sunny and P. Sini, *Topology on semi-well ordered sets*, Kyungpook Math. J., **64** (1), 161–169, (2024).

[24] I. Vialard, *On the width of the cartesian product of ordinals*, Order, **42**, 37–58, (2025).

[25] E. S. Wolk, *Partially well ordered sets and partial ordinals*, Fund. Math., **60**, 175–186, (1967).

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