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Invariant submanifolds of C_{12}-manifolds

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Abstract In this paper, the geometry of some submanifolds of a C_{12}-manifolds are studied. Certain conditions for these submanifolds which are tangent to ξ to be φ-invariant, minimal and totally geodesic, have been investigated. Concrete examples are given.

1 Introduction

Because of its important applications in applied mathematics and physics, the geometry of submanifolds has attracted increasing attention in the modern era. For example, properties of a nonlinear autonomous system are discussed using the concept of an invariant submanifold. Moreover, the theory of relativity relies heavily on the concept of geodesics. The geodesics of the ambient manifolds stay geodesics in the submanifolds for completely geodesic submanifolds. Therefore, in the physical sciences, completely geodesic submanifolds are likewise very significant. Therefore, working with completely geodesic submanifolds is crucial.

The work of A. Bejancu and N. Papaghiuc [2, 3] marked the beginning of studies on the geometry of invariant submanifolds. More specifically, these studies focus on invariant submanifolds of Sasakian manifolds, which are a type of almost contact metric manifold. Subsequently, these submanifolds were studied on other almost contact metric manifolds by numerous geometers (see, for example, [8, 12, 14, 15, 16, 17]).

In recent years, C_{12}-manifolds have emerged as a compelling subject of ongoing investigation in the domain of almost contact metric structures (see, for instance, [1, 4, 5, 6, 7]). This is regarded as an almost contact metric manifold that is non-normal but has interesting properties.

In recent years, C_{12}-manifolds have emerged as a compelling subject of ongoing investigation in the domain of almost contact metric structures (see, for instance, [1, 4, 5, 6, 7]). This is regarded as an almost contact metric manifold that is non-normal but has interesting properties. In [9], D. Chinea and C. Gonzalez have obtained a classification of the almost contact metric manifolds, studying the space that possesses the same symmetries as the covariant derivative of the fundamental 2-form. This space is decomposed into twelve irreducible components $C_1, ..., C_{12}$.

Most of the research related to almost contact metric structures is concerned with the normal structures that contain C_1 (cosymplectic), C_5 (β-Kenmotsu), and C_6 (α-Sasaki) classes.

One can ask, what distinguishes the class of C_{12}-manifolds from others? Every answer to this question confirms the existence of a second global vector field other than ξ, which is $\nabla_\xi\xi$. Of course, without neglecting the third global vector field $\varphi\nabla_\xi\xi$, where φ is the structural tensor. For reference, the presence of these three global vector fields, which are naturally pairwise orthogonal, prompted us to suggest the name "corner" for this type of manifold. Thus, in dimension three, if $\|\nabla_\xi\xi\| = 1$, then $\{\xi, \nabla_\xi\xi, \varphi\nabla_\xi\xi\}$ becomes an orthonormal basis we call it "fundamental basis". And as long as the three vector fields for this basis are global, this manifold is parallelizable; this means that its tangent bundle is trivial (that is, isomorphic to the product, $M \times \mathbb{R}^3$).

The paper is organized as follows: In Section 2, we revisit the concept of C_{12}-manifolds and review fundamental outcomes related to invariant submanifolds. These foundational understandings serve as a basis for our subsequent investigations. Sections 3, deal with the study of invariant submanifolds of a C_{12}-manifolds, and in the last section, we provide a set of concrete examples to confirm the results obtained.

2 Basic concepts

Let $f : M^m \to \tilde{M}^{m+d}$ be an isometric immersion of an m-dimensional Riemannian manifold (M, g) into $(m + d)$-dimensional Riemannian manifold $(\tilde{M}^{m+k}, g)$. We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}$, respectively. Then the formulas of Gauss and Weingarten are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \qquad and \qquad \tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.1}$$

for tangent vector fields X, Y and a normal vector field V, where σ, A and $\nabla^{\perp}$ are the second fundamental form, the shape operator and the normal connection, respectively.

If the second fundamental form σ is identically zero then the manifold is said to be totally geodesic. The second fundamental form σ and A_V are related by

$$\tilde{g}(\sigma(X, Y), V) = g(A_V X, Y), \tag{2.2}$$

for tangent vector fields X and Y. The mean curvature vector $\mathcal{H}$ of M is defined to be

$$\mathcal{H} = \frac{1}{m} trace(\sigma). \tag{2.3}$$

A submanifold M in an Riemann manifold is called minimal if its mean curvature vector vanishes identically. From this definition it is clear that any totally geodesic submanifold is minimal. Also, a submanifold M is known to be totally umbilical if

$$\sigma(X, Y) = \tilde{g}(X, Y)\mathcal{H}, \tag{2.4}$$

for any tangent vector fields X and Y [10].

The covariant derivative $\tilde{\nabla}\sigma$ is defined by

$$(\tilde{\nabla}_X \sigma)(Y, Z) = \nabla^{\perp}_X\big(\sigma(Y, Z)\big) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.5}$$

where, $\tilde{\nabla}\sigma$ is a normal bundle valued tensor of type $(0, 3)$ and is called the third fundamental form of M. Here, $\tilde{\nabla}$ is called the van der Waerden-Bortolotti connection of M. If $\tilde{\nabla}\sigma = 0$, then M is called parallel [11].

For any submanifold M of a Riemannian manifold $\tilde{M}$, the equation of Gauss is given by

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X \\ &\quad +(\tilde{\nabla}_X \sigma)(Y, Z) - (\tilde{\nabla}_Y \sigma)(X, Z), \end{aligned} \tag{2.6}$$

for any $X, Y, Z \in \Gamma(TM)$, where $\tilde{R}$ and R denote the Riemannian curvature tensors of $\tilde{M}$ and M respectively.

An odd-dimensional Riemannian manifold $(\tilde{M}^{2m+1}, g)$ is said to be an almost contact metric manifold if there exist on $\tilde{M}$ a $(1, 1)$-tensor field $\tilde{\varphi}$, a vector field $\tilde{\xi}$ (called the structure vector field) and a 1-form $\tilde{\eta}$ such that

$$\begin{cases} \tilde{\eta}(\tilde{\xi}) = 1, \\ \tilde{\varphi}^2 X = -X + \tilde{\eta}(X)\tilde{\xi}, \\ \tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = \tilde{g}(X, Y) - \tilde{\eta}(X)\tilde{\eta}(Y), \end{cases} \tag{2.7}$$

for any vector fields X, Y on $\tilde{M}$. In particular, in an almost contact metric manifold we also have

$$\tilde{\varphi}\tilde{\xi} = 0, \quad \tilde{\eta}(\tilde{\varphi}X) = 0 \quad and \quad \tilde{\eta}(X) = \tilde{g}(\tilde{\xi}, X).$$

The fundamental 2-form $\tilde{\phi}$ is defined by

$$\tilde{\phi}(X,Y) = \tilde{g}(X,\tilde{\varphi}Y).$$

It is known that the almost contact structure $(\tilde{\varphi},\tilde{\xi},\tilde{\eta})$ is said to be normal if and only if

$$\tilde{N}^{(1)}(X,Y) = \tilde{N}_{\tilde{\varphi}}(X,Y) + 2d\tilde{\eta}(X,Y)\tilde{\xi} = 0, \tag{2.8}$$

for any X, Y on M, where $\tilde{N}_{\tilde{\varphi}}$ denotes the Nijenhuis torsion of $\tilde{\varphi}$, given by

$$\tilde{N}_{\tilde{\varphi}}(X,Y) = \tilde{\varphi}^2[X,Y] + [\tilde{\varphi}X,\tilde{\varphi}Y] - \tilde{\varphi}[\tilde{\varphi}X,Y] - \tilde{\varphi}[X,\tilde{\varphi}Y]. \tag{2.9}$$

It is well-known that an almost contact metric structure $(\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ on $\tilde{M}$ is:

$$\begin{cases} (1) : Cosymplectic \Leftrightarrow (\tilde{\nabla}_X\tilde{\varphi})Y = 0, \\ (2) : Sasaki \Leftrightarrow (\tilde{\nabla}_X\tilde{\varphi})Y = \tilde{g}(X,Y)\tilde{\xi} - \tilde{\eta}(Y)X, \\ (3) : Kenmotsu \Leftrightarrow (\tilde{\nabla}_X\tilde{\varphi})Y = \tilde{g}(\tilde{\varphi}X,Y)\tilde{\xi} - \tilde{\eta}(Y)\tilde{\varphi}X, \end{cases} \tag{2.10}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$.

It is proved that $(\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is a C_{12}-structure on $\tilde{M}$ if and only if

$$(\tilde{\nabla}_X\tilde{\varphi})Y = \tilde{\eta}(X)\big(\tilde{\omega}(\tilde{\varphi}Y)\tilde{\xi} + \tilde{\eta}(Y)\tilde{\varphi}\tilde{\psi}\big), \tag{2.11}$$

where $\tilde{\omega}(X) = \tilde{g}(\tilde{\psi},X)$ and $\tilde{\psi} = -\tilde{\nabla}_{\tilde{\xi}}\tilde{\xi}$. From this formula one easily obtains

$$\tilde{\nabla}_X\tilde{\xi} = -\tilde{\eta}(X)\tilde{\psi}. \tag{2.12}$$

Moreover, in [7] the $(2m+1)$-dimensional C_{12}-manifolds is also characterized by

$$\mathrm{d}\tilde{\eta} = \tilde{\omega}\wedge\tilde{\eta}, \qquad \mathrm{d}\tilde{\phi} = 0 \qquad and \qquad \tilde{N}_{\tilde{\varphi}} = 0. \tag{2.13}$$

Also, in [4, 6], it is proved that a three-dimensional almost contact metric structure $(\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is a C_{12}-structure if and only if (2.12) holds.
For more details and background, see the references [1, 4, 6, 7].

For a submanifold M of a C_{12}-manifold $\tilde{M}$ similar definitions in [13] can be retained, first we put

$$\begin{cases} \tilde{\varphi}X = TX + NX, \\ \tilde{\varphi}V = tV + nV, \end{cases} \tag{2.14}$$

where X, TX, tV are tangent to M and NX, V, nV are normal to M.

Definition 2.1. A submanifold M of a C_{12}-manifold $\tilde{M}$ is said to be:

1) $\tilde{\varphi}$-invariant if $\tilde{\varphi}X$ is tangent to M (i.e., $NX = 0$).

2) invariant if it is $\tilde{\varphi}$-invariant and ξ is always tangent to M.

for any vector field X tangent to M.

3 Invariant submanifolds of a C_{12}-manifold

Let $(\tilde{M}^{2m+1},\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ be a $(2m+1)$-dimensional C_{12}-manifold. Recall that a $(2k+1)$-dimensional submanifold M of $\tilde{M}$ with $k<m$ is said to be invariant if the structure vector field $\tilde{\xi}$ is tangent to M everywhere on M and $\tilde{\varphi}X$ is tangent to M for any vector X tangent to M at every point of M. For this, we write ξ instead $\tilde{\xi}$ and φX instead $\tilde{\varphi}X$.

One can see that

$$\begin{aligned} \tilde{\psi} = -\tilde{\nabla}_{\xi}\xi &= -\nabla_{\xi}\xi - \sigma(\xi,\xi) \\ &= \psi - \sigma(\xi,\xi). \end{aligned} \tag{3.1}$$

Also, for any vector field X tangent to M, we have

$$\begin{aligned}\tilde{\omega}(X) &= \tilde{g}(\tilde{\psi}, X)\\ &= g(\psi, X) = \omega(X).\end{aligned} \tag{3.2}$$

So, by (2.12), we have

$$\begin{aligned}\tilde{\nabla}_X\xi &= -\eta(X)\tilde{\psi}\\ &= -\eta(X)\psi + \eta(X)\sigma(\xi,\xi),\end{aligned} \tag{3.3}$$

and by (2.1) we have

$$\tilde{\nabla}_X\xi = \nabla_X\xi + \sigma(X,\xi). \tag{3.4}$$

Then, comparing (3.3) and (3.4) we obtain

$$\nabla_X\xi = -\eta(X)\psi \qquad and \qquad \sigma(X,\xi) = \eta(X)\sigma(\xi,\xi) \Leftrightarrow \sigma(\varphi^2 X,\xi) = 0. \tag{3.5}$$

Also, for any vector fields X and Y tangent to M, from (2.1) we have

$$\sigma(X,\varphi Y) = \tilde{\nabla}_X\varphi Y - \nabla_X\varphi Y. \tag{3.6}$$

Again, by (2.1) we have

$$\begin{aligned}\tilde{\varphi}\sigma(X,Y) &= \tilde{\varphi}\tilde{\nabla}_X Y - \varphi\nabla_X Y\\ &= \tilde{\nabla}_X\varphi Y - (\tilde{\nabla}_X\tilde{\varphi})Y - \nabla_X\varphi Y + (\nabla_X\varphi)Y.\end{aligned} \tag{3.7}$$

By (2.11) taking into account (3.1) and (3.2), we compute

$$\begin{aligned}(\tilde{\nabla}_X\tilde{\varphi})Y &= \eta(X)\big(\tilde{\omega}(\varphi Y)\xi + \eta(Y)\varphi\tilde{\psi}\big)\\ &= \eta(X)\Big(\omega(\varphi Y)\xi + \eta(Y)\tilde{\varphi}\big(\psi - \sigma(\xi,\xi)\big)\Big)\\ &= \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big) - \eta(X)\eta(Y)\tilde{\varphi}\sigma(\xi,\xi)\\ &= (\nabla_X\varphi)Y - \eta(X)\eta(Y)\tilde{\varphi}\sigma(\xi,\xi).\end{aligned} \tag{3.8}$$

Substituting (3.6) and (3.8) in (3.7), we obtain

$$\tilde{\varphi}\sigma(X,Y) = \sigma(X,\varphi Y) + \eta(X)\eta(Y)\tilde{\varphi}\sigma(\xi,\xi). \tag{3.9}$$

Since σ is symmetric, we also have

$$\begin{aligned}\sigma(\varphi X,Y) = \sigma(X,\varphi Y) &= \tilde{\varphi}\sigma(X,Y) - \eta(X)\eta(Y)\tilde{\varphi}\sigma(\xi,\xi)\\ &= -\tilde{\varphi}\sigma(\varphi^2 X,Y).\end{aligned} \tag{3.10}$$

On the other hand, if we put $X = \xi$ in (2.2) we get

$$A_V\xi = \tilde{g}(\sigma(\xi,\xi),V)\xi = \eta(A_V\xi)\xi. \tag{3.11}$$

But, if we replace X by φX we obtain

$$\begin{aligned}g(A_V\varphi X,Y) &= \tilde{g}(\sigma(\varphi X,Y),V) = \tilde{g}(\sigma(X,\varphi Y),V)\\ &= g(A_V X,\varphi Y) = -g(\tilde{\varphi}A_V X,Y),\end{aligned}$$

and also,

$$\begin{aligned}g(A_V\varphi X,Y) &= \tilde{g}(\tilde{\varphi}\sigma(X,Y),V) - \eta(X)\eta(Y)\tilde{g}\big(\tilde{\varphi}\sigma(\xi,\xi),V\big)\\ &= -\tilde{g}(\sigma(X,Y),\tilde{\varphi}V) + \eta(X)\eta(Y)\tilde{g}\big(\sigma(\xi,\xi),\tilde{\varphi}V\big)\\ &= -\tilde{g}(A_{\tilde{\varphi}V}X,Y) + \eta(X)\eta(Y)\tilde{g}\big(A_{\tilde{\varphi}V}\xi,\xi\big).\end{aligned}$$

Hence,

$$\begin{aligned}\varphi A_V X = -\tilde{\varphi} A_V X &= -A_{\tilde{\varphi}V} X + \eta(X)\eta(A_{\tilde{\varphi}V}\xi)\xi \\ &= -A_{\tilde{\varphi}V} X + \eta(X) A_{\tilde{\varphi}V}\xi \\ &= A_{\tilde{\varphi}V}\varphi^2 X.\end{aligned}$$

Therefore, summing up the arguments above, we have the following Proposition:

Proposition 3.1. *Let M be an invariant submanifold of a C_{12}-manifold $\tilde{M}$. Then*

$$\begin{cases} \sigma(X,\xi) = \eta(X)\sigma(\xi,\xi), \\ A_V\xi = \eta(A_V\xi)\xi, \\ \sigma(\varphi X, Y) = \sigma(X, \varphi Y) = \tilde{\varphi}\sigma(X,Y) - \eta(X)\eta(Y)\tilde{\varphi}\sigma(\xi,\xi), \\ \varphi A_V X = -\tilde{\varphi} A_V X = A_{\tilde{\varphi}V}\varphi^2 X, \\ \nabla_X \xi = -\eta(X)\psi, \\ (\nabla_X \varphi)Y = \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big). \end{cases} \tag{3.12}$$

Corollary 3.2. *Let M be an invariant submanifold of a C_{12}-manifold $\tilde{M}$ tangent to $\tilde{\psi}$. Then*

$$\begin{cases} \sigma(X,\xi) = 0, \\ A_V\xi = 0, \\ \sigma(\varphi X, Y) = \sigma(X, \varphi Y) = \tilde{\varphi}\sigma(X,Y), \\ \varphi A_V X = -\tilde{\varphi} A_V X = -A_{\tilde{\varphi}V} X, \\ \nabla_X \xi = -\eta(X)\psi, \\ (\nabla_X \varphi)Y = \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big). \end{cases} \tag{3.13}$$

Proof. Since M is tangent to $\tilde{\psi}$ that is $\tilde{\psi} = \psi$ then, from (3.1) we have

$$\sigma(\xi,\xi) = 0.$$

So, from (3.12) we get (3.13). □

Corollary 3.3. *Let M be an invariant submanifold of a C_{12}-manifold $\tilde{M}$ normal to $\tilde{\psi}$. Then*

$$\begin{cases} \sigma(\varphi^2 X,\xi) = 0, \\ A_V\xi = \eta(A_V\xi)\xi, \\ \sigma(\varphi X, Y) = \sigma(X, \varphi Y) = -\tilde{\varphi}\sigma(\varphi^2 X,Y), \\ \varphi A_V X = -\tilde{\varphi} A_V X = A_{\tilde{\varphi}V}\varphi^2 X, \\ \nabla_X \xi = 0, \\ (\nabla_X \varphi)Y = 0. \end{cases} \tag{3.14}$$

Proof. Since M is normal to $\tilde{\psi}$ that is

$$\tilde{\psi} = \tilde{\nabla}_\xi \xi = \sigma(\xi,\xi).$$

Then, from (3.1) we get $\psi = 0$ and $\omega = 0$. So, from (3.12) we get (3.14). □

Now we can state the following statement:

Theorem 3.4. *Let M^{2k+1} be an invariant submanifold of a C_{12}-manifold $\tilde{M}^{2m+1}$ with $k < m$.*

1) *If M is not normal to $\tilde{\psi}$, then M is a C_{12}-manifold. In addition, if it is tangent to $\tilde{\psi}$, then it is minimal.*

2) *If M is normal to $\tilde{\psi}$ then, M is a cosymplectic manifold.*

Proof. Assume that M is an invariant submanifold of a C_{12}-manifold $\tilde{M}$, then from the last formula in (3.12) it is easy to see that M is a C_{12}-manifold. Since $dimM = 2k+1$, and $k < m$ then using (3.13) in the definition of the mean curvature vector $\mathcal{H}$, we get

$$\begin{aligned}\mathcal{H} &= \frac{1}{2k+1}\sum_{i=1}^{2k+1}\sigma(e_i,e_i)\\ &= \sigma(\xi,\xi)+\frac{1}{2k+1}\sum_{i=1}^{k}\big(\sigma(e_i,e_i)+\sigma(\varphi e_i,\varphi e_i)\big)\\ &= \sigma(\xi,\xi)+\frac{1}{2k+1}\sum_{i=1}^{k}\big(\sigma(e_i,e_i)+\varphi^2\sigma(e_i,e_i)\big),\end{aligned}$$

where $\{\xi, e_i, \varphi e_i\}_{1\leq i\leq k}$ is a φ-basis on M. So, by $\varphi^2\sigma(e_i,e_i) = -\sigma(e_i,e_i)$ we conclure that $\mathcal{H} = \sigma(\xi,\xi)$. Since M is tangent to $\tilde{\psi}$ then $\mathcal{H} = 0$, which means M is minimal.

For the second assertion, it is sufficient to see the last formula in (3.14).

□

Proposition 3.5. *Let M be a totally umbilical submanifold of a C_{12}-manifold $\tilde{M}$ and tangent to ξ. M is minimal if and only if it is tangent to $\tilde{\psi}$.*

Proof. Suppose that M is totally umbilical, that is

$$\sigma(X,Y) = \tilde{g}(X,Y)\mathcal{H}.$$

By (3.12), one can get $\sigma(\xi,\xi) = \mathcal{H} = 0$. which completes the proof. □

Corollary 3.6. *Any totally umbilical submanifold of a C_{12}-manifold tangent to $\tilde{\psi}$ is totally geodesic.*

Proof. the proof directly flows from Proposition 3.5 with $\sigma(\xi,\xi) = 0$. □

Theorem 3.7. *Let M be a 3-dimensional invariant submanifold of a C_{12}-manifold $\tilde{M}$ and tangent to $\tilde{\psi}$. Then, M is parallel if and only if M is totally geodesic.*

Proof. Since M is a 3-dimensional C_{12}-manifold then, we can admit $\{\xi,\psi,\varphi\psi\}$ as an orthogonal basis. Suppose that M is parallel, that is $\tilde{\nabla}\sigma = 0$. So, from (2.5) we have

$$\nabla^{\perp}_X\big(\sigma(Y,Z)\big) - \sigma(\nabla_X Y, Z) - \sigma(Y,\nabla_X Z) = 0. \tag{3.15}$$

For $Z = \xi$ taking into account $\sigma(Y,\xi) = 0$, (3.15) gives $\sigma(Y,\nabla_X\xi) = 0$ i.e., $\sigma(Y,\psi) = 0$. Also, from (3.13) we have

$$\sigma(Y,\varphi\psi) = \varphi\sigma(Y,\psi) = 0.$$

Hence $\sigma = 0$, which means M is totally geodesic. The converse statement is trivial. □

4 Construction of examples

Let $\tilde{M} = \mathbb{R}^5$ be an Euclidean space Cartesian coordinates $\{x_i, z\}_{1\leq i\leq 4}$. The vector fields,

$$\tilde{e}_1 = \frac{\partial}{\partial x_1} + z\frac{\partial}{\partial z},\quad \tilde{e}_2 = \frac{\partial}{\partial x_2},\quad \tilde{e}_3 = \frac{\partial}{\partial x_3} + z\frac{\partial}{\partial z},\quad \tilde{e}_4 = \frac{\partial}{\partial x_4},\quad \tilde{e}_5 = \tilde{\xi} = \frac{\partial}{\partial z},$$

are linearly independent at each point of $\mathbb{R}^5$. Let $\tilde{g}$ be the Riemannian metric on $\mathbb{R}^5$ defined by $\tilde{g}(\tilde{e}_i,\tilde{e}_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. That is, the form of the metric becomes

$$\tilde{g} = \begin{pmatrix} 1+z^2 & 0 & z^2 & 0 & -z\\ 0 & 1 & 0 & 0 & 0\\ z^2 & 0 & 1+z^2 & 0 & -z\\ 0 & 0 & 0 & 1 & 0\\ -z & 0 & -z & 0 & 1\end{pmatrix}.$$

Let $\tilde{\eta}$ be the 1-form on $\mathbb{R}^5$ defined by $\tilde{\eta} = dz - zdx_1 - zdx_3$. Let $\tilde{\varphi}$ be the $(1,1)$-tensor field defined by,

$$\tilde{\varphi}\tilde{e}_1 = \tilde{e}_2, \quad \tilde{\varphi}\tilde{e}_2 = -\tilde{e}_1, \quad \tilde{\varphi}\tilde{e}_3 = \tilde{e}_4, \quad \tilde{\varphi}\tilde{e}_4 = -\tilde{e}_3, \quad \tilde{\varphi}\tilde{e}_5 = 0.$$

By linearity of $\tilde{\varphi}$ and $\tilde{g}$, the relations (2.7) are satisfied on $\mathbb{R}^5$. Thus, $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ defines an almost contact metric structure on $\mathbb{R}^5$. We note that

$$\mathrm{d}\tilde{\eta} = dx_1 \wedge dz + dx_3 \wedge dz = (dx_1 + dx_3) \wedge (dz - zdx_1 - zdx_3).$$

The non-zero component of the fundamental 2-form is

$$\tilde{\phi}\Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\Big) = \tilde{\phi}\Big(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\Big) = -1,$$

and we have

$$\tilde{\phi} = -2dx_1 \wedge dx_2 - 2dx_3 \wedge dx_4.$$

Its differential gives $\mathrm{d}\tilde{\phi} = 0$. By letting $\tilde{\omega} = dx_1 + dx_3$, we have $\mathrm{d}\tilde{\eta} = \tilde{\omega} \wedge \tilde{\eta}$. and the dual vector field $\tilde{\psi}$ is given by $\tilde{\psi} = \tilde{e}_1 + \tilde{e}_3$.
It is easy to see that $\mathrm{d}\tilde{\omega} = 0$ and $\tilde{\eta}(\tilde{\psi}) = \tilde{\omega}(\tilde{\xi}) = 0$.
Knowing that the components of the tensor $\tilde{N}_{\tilde{\varphi}}$, can be written as,

$$\tilde{N}_{\tilde{\varphi}}(\tilde{e}_i, \tilde{e}_j) = \tilde{\varphi}^2[\tilde{e}_i, \tilde{e}_j] + [\tilde{\varphi}\tilde{e}_i, \tilde{\varphi}\tilde{e}_j] - \tilde{\varphi}[\tilde{\varphi}\tilde{e}_i, \tilde{e}_j] - \tilde{\varphi}[\tilde{e}_i, \tilde{\varphi}\tilde{e}_j].$$

The non zero components of Lie bracket are

$$[\tilde{\xi}, \tilde{e}_1] = [\tilde{\xi}, \tilde{e}_3] = \tilde{\xi}. \tag{4.1}$$

By direct calculations, one can check that $\tilde{N}_{\tilde{\varphi}} = 0$. Therefore, in view of (2.13), we can confirm that $(\mathbb{R}^5, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a C_{12}-manifold.

To verify result in formulas (2.11), the non zero components of the Levi-Civita connection corresponding to $\tilde{g}$ are given by:

$$\tilde{\nabla}_{\tilde{\xi}}\tilde{e}_1 = \tilde{\nabla}_{\tilde{\xi}}\tilde{e}_3 = \tilde{\xi}, \quad \tilde{\nabla}_{\tilde{\xi}}\tilde{\xi} = -\tilde{\psi},$$

Then, one can easily check that for all $i, j \in \{1, 2, ..., 5\}$,

$$(\tilde{\nabla}_{\tilde{e}_i}\tilde{\varphi})\tilde{e}_j = \tilde{\eta}(\tilde{e}_i)\big(\tilde{\omega}(\tilde{\varphi}\tilde{e}_j)\tilde{\xi} + \tilde{\eta}(\tilde{e}_j)\tilde{\varphi}\tilde{\psi}\big).$$

Let f_1 be an isometric immersion from M to $\tilde{M}$ defined by

$$f_1(x, y, z) = (0, 0, x_3, x_4, z).$$

Let $M = \{(x, y, z) \in \mathbb{R}^3 \,/\, (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \xi = \frac{\partial}{\partial z},$$

are linearly independent at each point of M. Let g be the metric given by

$$g = (1 + z^2)dx^2 + dy^2 + dz^2 - 2dxdz.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3)$, for any vector field X tangent to M. Let φ be the $(1,1)$-tensor field defined by

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1 \quad and \quad \varphi e_3 = 0.$$

The linearity of φ and g yields that

$$\eta(\xi) = \eta(e_3) = 1, \quad \varphi^2 e_i = -e_i + \eta(e_i)\xi,$$

$$g(\varphi e_i, \varphi e_j) = g(e_i, e_j) - \eta(e_i)\eta(e_j) \quad for\ all \quad 1 \leq i, j \leq 3.$$

Thus, (φ, ξ, η, g) defines an almost contact metric structure on M. We note that

$$\mathrm{d}\eta = \mathrm{d}(dz - zdx) = dx \wedge dz = \omega \wedge \eta \qquad where \quad \omega = dx.$$

The non-zero component of the fundamental 2-form is

$$\phi\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big) = -1,$$

and we have $\phi = -2dx \wedge dy$ which implies $\mathrm{d}\phi = 0$.

The only non zero component of Lie bracket is

$$[e_1, e_3] = -e_3.$$

By direct calculations, one can check that $N_\varphi = 0$. Therefore, in view of (2.13), we declare that (φ, ξ, η, g) is a C_{12}-structure on the submanifold M of $\tilde{M}$.

Using Koszul's formula, the non zero components of the Levi-Civita connection corresponding to g are given by:

$$\nabla_{e_3} e_1 = e_3 \quad and \quad \nabla_{e_3} e_3 = -e_1.$$

We see that on M the formula $\nabla_{e_i}\xi = -\eta(e_i)\psi$ holds where $\psi = -\nabla_\xi \xi = e_1$.

Let us take $D =< e_1 >$, $D^\perp =< e_2 >$. Then clearly we see that

$$TM = D \oplus D^\perp \oplus < \xi >,$$

For any $X \in D$ and $Y \in D^\perp$, we can write $X = \lambda_1 e_1$ and $Y = \lambda_2 e_2$, where λ_1 and λ_2 are two scalars. Now, $\varphi X = \varphi(\lambda_1 e_1) = \lambda_1 \varphi e_1 = \lambda_1 e_2 \in D^\perp \subset TM$. Similarly, $\varphi Y = -\lambda_2 e_1 \in D \subset TM$. Hence, the submanifold is invariant.

Let compute the mean curvature vector $\mathcal{H}$,

$$\begin{aligned} \mathcal{H} &= \frac{1}{3}\sum_{i=1}^{3} \sigma(e_i, e_i) \\ &= \sigma(e_3, e_3) = \sigma(\xi, \xi) \\ &= \tilde{\nabla}_\xi \xi - \nabla_\xi \xi = \tilde{\psi} - \psi \\ &= -\tilde{e}_1 \neq 0. \end{aligned}$$

Hence, the submanifold M is not minimal because M is not tangent to $\tilde{\psi}$.

Below we will give an example of an invariant submanifold tangent to $\tilde{\psi}$. Let f_2 be an isometric immersion from $\overline{M}$ to $\tilde{M}$ defined by

$$f_2(x, y, z) = (x_1, 0, x_3, 0, z).$$

It can be easily proved that $\overline{M} = \{(x, y, z) \in \mathbb{R}^3 / (x, y, z) \neq (0, 0, 0)\}$ is standard coordinates in $\mathbb{R}^3$, is a 3-dimensional submanifold of the 5-dimensional C_{12}-manifold. We consider the vector fields

$$\overline{e}_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, \quad \overline{e}_2 = \frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad \overline{e}_3 = \xi = \frac{\partial}{\partial z},$$

are linearly independent at each point of $\overline{M}$. Let $\overline{g}$ be the metric given by

$$\overline{g} = dx^2 + dy^2 + (dz - zdx - zdy)^2.$$

Thus, we define on $\overline{M}$ an almost contact metric structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ by

$$\overline{\varphi}\overline{\xi} = \overline{\varphi}\,\overline{e}_3 = 0, \quad \overline{\varphi}\,\overline{e}_1 = \overline{e}_2 \quad and \quad \overline{\varphi}\,\overline{e}_2 = -\overline{e}_1.$$

The non zero components of Lie bracket are

$$[\overline{e}_1, \overline{e}_3] = [\overline{e}_2, \overline{e}_3] = -\overline{e}_3.$$

Also, the non zero components of the Levi-Civita connection corresponding to $\overline{g}$ are given by:

$$\overline{\nabla}_{\overline{\xi}}\overline{e}_1 = \overline{\nabla}_{\overline{\xi}}\overline{e}_3 = \overline{\xi}, \quad \overline{\nabla}_{\overline{\xi}}\overline{\xi} = -\overline{e}_1 - \overline{e}_2.$$

Then, one can easily check that for all $i \in \{1,2,3\}$, $\overline{\nabla}_{\overline{e}_i}\overline{\xi} = -\overline{\eta}(\overline{e}_i)\overline{\psi}$. This means that $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a C_{12}-structure on $\overline{M}$ with $\overline{\psi} = \overline{e}_1 + \overline{e}_2$.

Let us assume

$$T\overline{M} = \overline{D} \oplus \overline{D}^{\perp} \oplus < \overline{\xi} >,$$

where $\overline{D} =< \overline{e}_1 >$ and $\overline{D}^{\perp} =< \overline{e}_2 >$. Then we see that $\overline{\varphi e}_1 = \overline{e}_2 \in \overline{D}^{\perp}$ and $\overline{\varphi e}_2 = -\overline{e}_1 \in \overline{D}$. Hence the submanifold is invariant. Now from the values of $\tilde{\nabla}_{\tilde{e}_i}\tilde{e}_j$ and $\overline{\nabla}_{\overline{e}_i}\overline{e}_j$, we see that $\overline{\sigma}(\overline{e}_i, \overline{e}_j) = 0$ for all $i, j \in \{1,2,3\}$. That is, the submanifold $\overline{M}$ is totally geodesic.

Finally, we give an example of an invariant submanifold normal to $\tilde{\psi}$. Let f_3 be an isometric immersion from $\hat{M}$ to $\tilde{M}$ defined by

$$f_2(x,y,z) = (0, x_2, 0, x_4, z).$$

It can be easily proved that $\hat{M} = \{(x,y,z) \in \mathbb{R}^3 / (x,y,z) \neq (0,0,0)\}$ is standard coordinates in $\mathbb{R}^3$, is a 3-dimensional submanifold of the 5-dimensional C_{12}-manifold. We consider the vector fields

$$\hat{e}_1 = \frac{\partial}{\partial x}, \quad \hat{e}_2 = \frac{\partial}{\partial y}, \quad \hat{e}_3 = \xi = \frac{\partial}{\partial z},$$

are linearly independent at each point of $\hat{M}$. Let $\hat{g}$ be the metric given by

$$\hat{g} = dx^2 + dy^2 + dz^2.$$

Thus, we define on $\hat{M}$ an almost contact metric structure $(\hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$ by

$$\hat{\varphi}\hat{\xi} = \hat{\varphi}\,\hat{e}_3 = 0, \quad \hat{\varphi}\,\hat{e}_1 = \hat{e}_2 \quad and \quad \hat{\varphi}\,\hat{e}_2 = -\hat{e}_1.$$

It is easy to see that

$$\hat{\nabla}_{\hat{e}_i}\hat{e}_j = 0, \qquad for\ all \quad i,j \in \{1,2,3\},$$

which implies

$$(\hat{\nabla}_{\hat{e}_i}\hat{\varphi})\hat{e}_j = 0, \qquad for\ all \quad i,j \in \{1,2,3\}.$$

Hence, $(\hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$ is a cosymplectic structure on the submanifold $\hat{M}$ and easily one can prove that $\hat{M}$ is invariant. Note that

$$\tilde{g}(\tilde{\psi}, \hat{e}_i) = \tilde{g}(\tilde{e}_1 + \tilde{e}_3, \hat{e}_i) = 0,$$

i.e., $\hat{M}$ is normal to $\tilde{\psi}$.

Hence, the above arguments tell us that the submanifolds M, $\overline{M}$ and $\hat{M}$ under consideration agree with Theorem 3.4.

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