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Equitable Total Coloring of Extended Comb graphs G $\odot$ H

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Abstract: A graph G is referred to as equitably total k-colorable if its vertex set and edge set can be partitioned into k non-empty color classes such that the coloring is a proper total coloring and the sizes of any two color classes differ by at most one. The concepts of equitable total coloring (χ_{et}) and the equitable total chromatic number (χ''_{et}) were introduced by Fu[4] in 1994. He proposed the conjecture that for any simple graph $G(V, E)$ lies $\chi_{et} \leq \Delta(G) + 2$.In this paper, we study the equitable total coloring of the extended comb graph and shows that it lies within the range of $\Delta(G) + 1 \leq (\chi_{et}) \leq \Delta(G) + 2$.

1 Introduction

In this paper, we will focus on finite simple graphs. Let $V(G), E(G), \Delta(G)$ and $\delta(G)$ represent the vertex set, edge set, maximum degree, and minimum degree of a graph, respectively. A graph G is defined as totally k-colorable if adjacent or incident elements are assigned distinct colors. An equitable total k-coloring is a proper coloring of a graph in which the sizes of any two color classes differ by at most one. [12] Meyer (1973) introduced the concept of equitable coloring and studied its basic properties.Chen and Yen [2] established the necessary conditions for a graph G, with equitably $\Delta(G)$-colorable.[4] Chunling et al.[3] proved that the equitable total colorings of cartesian product of two cycles. Additionally, Kaliraj et al. [10] successfully addressed equitable coloring for the Mycielski construction of the wheel graph and the complete bipartite graph. Girija and Veninstine Vivik [7] are working to determine the equitable total chromatic number for the double star graph $K_{1,n,n}$ and the fan graph $F_{m,n}$. Geetha and Somasundaram [6] have investigated the power graphs of paths and cycles and complements of graphs in the context of total coloring.Bau Sheng et al. [14] determined the equitable total colorings of some join graphs. [13] Gang and Zhong fu explained the equitable total coloring of multiple join-graph. In 2022, [8] Muhammad Imran et al. determined various cordial graphs related to combinations. In 2023, [11] A. Sajiya. et al. discussed the comb graph $P_n \odot K_1$ and the extended comb graph $EC(t, k)$. Looking ahead to 2024, [9] Muhammad Imran et al. proved that different families of comb graphs are mean graphs. [1] M. Barani et al. explained the equitable coloring of the extended corona graph of path, cycle and complete graphs.

2 Preliminaries

Definition 2.1. The comb graph is denoted by $P_n \odot k_1$ Here, P_n represents a path graph with $(n-1)$ edges and (n) vertices. The comb graph is formed by connecting each vertex in the path to a pendant edge of k_1. As a result, the comb graph has a total of $(2n)$ vertices and $(2n-1)$ edges.

Definition 2.2. The Extended Comb graph $G \odot H$ is created by connecting each vertex in graph G to a copy of graph H. Both G and H are extended up to n and m respectively, where n and m

are positive integers.

Conjecture 2.3. [6] For any simple graph $G(V,E), \chi_{et}^{''}(G) \leq \Delta(G)+2$

Conjecture 2.4. [5] For any graph G, G has an equitable total k-coloring for each $k \geq max(\chi_{et}^{''}(G), \Delta+2)$

3 Main Result

Here we determine an equitable total coloring of the Extended Comb graph $P_n \odot P_m, C_n \odot P_n, P_n \odot C_n, C_n \odot C_n$.

Theorem 3.1. For any positive integer n and m, $\chi_{et}(P_n \odot P_m) = 4$.

Proof. The Path graph P_n consists of $\{v_1, v_2, \cdots, v_n\}$ vertices and edges$\{v_iv_{i+1}; i = 1,2,\cdots, n-1\}$. P_m graph consists of $\{v_1, v_2, \cdots, v_m\}$ vertices and edges $\{v_jv_{j+1}; j = 1,2,\cdots, m-1\}$. By definition of extended comb graph each vertex of P_n joined to a copy of P_m, the vertex set of $V(P_n \odot P_m) = \{v_{i,j}; i = 1,2,\cdots,n; j = 1,2,\cdots,m\}$ and edge set of $E(P_n \odot P_m) = \{v_{i,j}v_{i,j+1}; i = 1,2,3,\cdots,n; j = 1,2,3,\cdots,m-1\} \cup \{v_{i,1}v_{i+1,1}; i = 1,2,3,\cdots,n-1\}$. The value of the n-th term indicates number of elements in the order of the sequence in i and j. Here we describe four colorings below as follows.

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_1	$i = j$, $i = 1,2,3,4,\ldots,n$	$i = \underbrace{(1,5,9,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $j = \underbrace{(2,6,10,\ldots)}_{\lfloor(3m-1)/10\rfloor}$, & $i = \underbrace{(2,6,10,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $j = \underbrace{(3,7,11,\ldots)}_{\lfloor(3m-1)/10\rfloor}$, & $i = \underbrace{(3,7,11,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $j = \underbrace{(4,8,12,\ldots)}_{\lfloor(3m-1)/10\rfloor}$, & $i = \underbrace{(4,8,12,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $j = \underbrace{(1,5,9,\ldots)}_{\lfloor(3m-1)/10\rfloor}$
T_2	$i = 1, j = \underbrace{(4,8,12,\ldots)}_{\lfloor(3m-3)/10\rfloor}$, $i = 2, j = \underbrace{(1,5,9,\ldots)}_{\lfloor(m+3)/4\rfloor}$, $i = 3, j = \underbrace{(2,6,10,\ldots)}_{\lfloor(m+3)/4\rfloor}$, $i = 4, j = \underbrace{(3,7,11,\ldots)}_{\lfloor(m-2)/4\rfloor}$,	$i = j$, $i = \underbrace{(1,5,9,\ldots)}_{\lfloor(3m-1)/10\rfloor}$, & $i = j$, $j = \underbrace{(2,6,10,\ldots)}_{\lfloor(3m-1)/10\rfloor}$, & $i = j$, $j = \underbrace{(3,7,11,\ldots)}_{\lfloor(3m-1)/10\rfloor}$, & $i = j$, $j = \underbrace{(4,8,12,\ldots)}_{\lfloor(3m-1)/10\rfloor}$
T_3	$i = 1, j = \underbrace{(3,7,11,\ldots)}_{\lfloor(m-2)/4\rfloor}$, $i = 2, j = \underbrace{(4,8,12,\ldots)}_{\lfloor(m-2)/4\rfloor}$, $i = 3, j = \underbrace{(1,5,9,\ldots)}_{\lfloor(3m-3)/10\rfloor}$, $i = 4, j = \underbrace{(2,6,10,\ldots)}_{\lfloor(m+2)/4\rfloor}$,	$i = \underbrace{(1,5,9,\ldots)}_{\lfloor(n-2)/4\rfloor}$, $j = \underbrace{(4,8,12,\ldots)}_{\lfloor(m-2)/4\rfloor}$, & $i = \underbrace{(2,6,10,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $j = \underbrace{(1,5,9,\ldots)}_{\lfloor(3m-1)/10\rfloor}$, & $i = \underbrace{(3,7,11,\ldots)}_{\lfloor(n-2)/4\rfloor}$, $j = \underbrace{(2,6,10,\ldots)}_{\lfloor(m-2)/4\rfloor}$, & $i = \underbrace{(4,8,12,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $j = \underbrace{(3,7,11,\ldots)}_{\lfloor(3m-1)/10\rfloor}$

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_4	$i=1, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(m+3)/4\rfloor}$,	$i=\underbrace{(1,5,9,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $j=\underbrace{(3,7,11,\ldots)}_{\lfloor(3m-1)/10\rfloor}$, &
	$i=2, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(m+1)/4\rfloor}$,	$i=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}$,, $j=\underbrace{(4,8,12,\ldots)}_{\lfloor(m-1)/4\rfloor}$, &
	$i=3, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(3m-3)/10\rfloor}$,	$i=\underbrace{(3,7,11,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $j=\underbrace{(2,6,10,\ldots)}_{\lfloor(3m-1)/10\rfloor}$, &
	$i=4, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(m+3)/4\rfloor}$,	$i=\underbrace{(4,8,12,\ldots)}_{\lfloor(n+1)/4\rfloor}$, $j=\underbrace{(2,6,10,\ldots)}_{\lfloor(m+1)/4\rfloor}$

Clearly T_1, T_2, T_3, and T_4 are independent sets of $(P_n \odot P_m)$.

Case 1: n is odd
$|T_1| = \left(\frac{nm+1}{2}\right) and\, |T_2| = |T_3| = |T_4| = \left(\frac{nm+1}{2}\right) - 1$ thus $|T_i - T_j| \leq 1$ for $i \neq j$.
Case 2: n is even
$|T_3| = \left(\frac{nm}{2}\right) - 1 and\, |T_1| = |T_2| = |T_4| = \left(\frac{nm}{2}\right)$ thus $|T_i - T_j| \leq 1$ for $i \neq j$.
This implies $\chi_{et}(P_n \odot P_m) = 4$. Hence, the result is established. □

Here we give Examples of the value of i and j from the n-th terms.

1. If $n = 7$ then $\lfloor\frac{3(7)-1}{10}\rfloor = 2$. So the value of $i = 3, 7$ and $\lfloor\frac{3(7)-1}{10}\rfloor = 2$ then $j = 4, 8$. So the edges are constructed given form $(v_{3,4}v_{3,5})(v_{3,8}v_{3,9})(v_{7,4}v_{7,5})(v_{7,8}v_{7,9})$.

2. If $n = 12$ then $\lfloor\frac{3(12)-1}{10}\rfloor = \lfloor 3.5\rfloor = 3$. So the value of $i = 1, 5, 9$ then $m = 10$ $\lfloor\frac{3(10)-1}{10}\rfloor = \lfloor 2.9\rfloor = 2, j = 2, 6$ $(v_{1,2}v_{1,3})(v_{1,6}v_{1,7})(v_{5,2}v_{5,3})(v_{5,6}v_{5,7})(v_{9,2}v_{9,3})(v_{9,6}v_{9,7})$. i consider only first two elements in the sequence.

Theorem 3.2. *For any positive integer n,* $\chi_{et}(C_n \odot P_n) = 4$.

Proof. The cycle graph C_n consists of $\{v_1, v_2, \cdots, v_n\}$ vertices and edges $\{v_iv_{i+1}\} \cup \{v_1v_n; i = 1, 2, \cdots, n-1\}$ P_n graph consists of $\{v_1, v_2, \cdots, v_n\}$ vertices and edges $\{v_jv_{j+1}; j = 1, 2, \cdots, n-1\}$. By definition of extended comb graph each vertex of C_n joined to a copy of P_n the vertex set of $V(C_n \odot P_n) = \{v_{i,j}; i = 1, 2, \cdots, n; j = 1, 2, \cdots, n\}$ and edge set of $(C_n \odot P_n) = \{v_1v_n\} \cup \{v_{i,j}v_{i,j+1}; i = 1, 2, \cdots, n; j = 1, 2, \cdots, n-1\}$. Here we consider the four cases as follows.

Case:1 The coloring rules of $P_n \odot P_n$ are the same as for $C_n \odot P_n$, where $n = 8, 12, 16, \cdots, 4i+4, i = 1, 2, 3, 4, \cdots$ and we have to include one color $T_3 = \{v_1v_n\}$.

Case:2 $C_n \odot P_n$, where $n = 6, 10, 14, \cdots, 4i+2, i = 1, 2, 3, 4, \cdots$.

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_1	$i = j$,	$i=\underbrace{(1,5,9,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $j=\underbrace{(2,6,10,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, &
	$i=\underbrace{(1,2,3,4,\ldots)}_{\lfloor(n+3)/4\rfloor}$	$i=\underbrace{(2,6,10,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $j=\underbrace{(3,7,11,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, &
	$i = n$,	$i=\underbrace{(3,7,11,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $j=\underbrace{(4,8,12,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, &
	$i=\underbrace{(4,8,12,\ldots)}_{\lfloor(3n-1)/10\rfloor}$	$i = j$, $i=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i = n$, $j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_2	$i=1, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $i=2, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=3, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=4, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $i=n, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$,	$i=j, i=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, & $i=j$, $j=\underbrace{(2,6,10,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, & $i=j, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, & $i=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n+3)/4\rfloor}$, & $i=n, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(3n-3)/10\rfloor}$
T_3	$i=1, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $i=2, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $i=3, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=4, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(m+3)/4\rfloor}$, $i=n, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$,	$i=\underbrace{(1,5,9,\ldots)}_{\lfloor(3n-3)/10\rfloor}, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(3n-3)/10\rfloor}$, & $i=\underbrace{(2,6,10,\ldots)}_{\lfloor(3n-3)/10\rfloor}, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(3n-3)/10\rfloor}$, & $i=\underbrace{(3,7,11,\ldots)}_{\lfloor(3n-3)/10\rfloor}, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(3n-3)/10\rfloor}$, & $i=\underbrace{(4,8,12,\ldots)}_{\lfloor(3n-3)/10\rfloor}, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(3n-3)/10\rfloor}$ & $i=n, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(3n-3)/10\rfloor}$ & v_1v_n
T_4	$i=1, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=2, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $i=3, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, $i=4, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=n, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n+3)/4\rfloor}$,	$i=\underbrace{(1,5,9,\ldots)}_{\lfloor(3n-1)/10\rfloor}, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, & $i=\underbrace{(2,6,10,\ldots)}_{\lfloor(3n-1)/10\rfloor}, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(3n-1)/10\rfloor}$, & $i=\underbrace{(3,7,11,\ldots)}_{\lfloor(n+3)/4\rfloor}, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, & $i=\underbrace{(4,8,12,\ldots)}_{\lfloor(3n-1)/10\rfloor}, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(3n-1)/10\rfloor}$ & $i=n, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}$

Case: 3 $C_n \odot P_n$, where $n = 7, 11, 15, \cdots, 4i+3, i = 1, 2, 3, 4, \cdots$.

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_1	$i=1, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$ $i=2, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-3)/4\rfloor}$ $i=3, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n+3)/4\rfloor}$ $i=4, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$ $i=n, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n+3)/4\rfloor}$	$i=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$, & $i=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, & $i=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-3)/4\rfloor}, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-3)/4\rfloor}$, & $i=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-3)/4\rfloor}, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-3)/4\rfloor}$ & $i=n, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_2	$i=1, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=2, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=3, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-3)/4\rfloor}$, $i=4, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=n, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n+3)/4\rfloor}$,	$i=\underbrace{(1,5,9,\ldots)}_{\lfloor(n-3)/4\rfloor}, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-3)/4\rfloor}$, & $i=j, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$, & $i=\underbrace{(3,7,11,\ldots)}_{\lfloor(n+3)/4\rfloor}, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$ & $i=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-3)/4\rfloor}, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-3)/4\rfloor}$ & $i=n, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-3)/4\rfloor}$ & v_1v_n
T_3	$i=1, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=2, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=3, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=4, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-3)/4\rfloor}$, $i=n, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$,	$i=\underbrace{(1,5,9,\ldots)}_{\lfloor(n-3)/4\rfloor}, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-3)/4\rfloor}$, & $i=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-3)/4\rfloor}, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-3)/4\rfloor}$, & $i=\underbrace{(3,7,11,\ldots)}_{\lfloor(n+3)/4\rfloor}, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$, & $i=\underbrace{(4,8,12,\ldots)}_{\lfloor(n+3)/4\rfloor}, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$ & $i=n, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-3)/4\rfloor}$
T_4	$i=1, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-3)/4\rfloor}$, $i=2, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=3, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=4, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=n, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-3)/4\rfloor}$,	$i=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}, i=j$, & $i=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-3)/4\rfloor}, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-3)/4\rfloor}$, & $i=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-3)/4\rfloor}, i=j$, & $i=\underbrace{(4,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n+3)/4\rfloor}$ & $i=n, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-3)/4\rfloor}$,

Case: 4 $C_n \odot P_n$, where $n = 5, 9, 13, \cdots, 4i+1, i = 1, 2, 3, 4, \cdots$.

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_1	$i=1, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$ $i=2, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}$ $i=3, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}$ $i=4, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}$ $i=n, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}$	$i=\underbrace{(1,5,9,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}$, & $i=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n-1)/4\rfloor}$, & $i=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}$, & $i=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}$ & $i=n, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n-1)/4\rfloor}$

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_2	$i=1, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}$, $i=2, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=3, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}$, $i=4, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}$, $i=n, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}$,	$i=j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n-1)/4\rfloor}$, & $i=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}$, $i=j$ & $i=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}, i=j$ & $i=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(1,5,7,\ldots)}_{\lfloor(n-1)/4\rfloor}$ & $i=n, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}$
T_3	$i=1, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}$, $i=2, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}$, $i=3, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=4, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}$, $i=n, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$,	$i=\underbrace{(1,5,9,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}$, & $i=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}$, & $i=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}$, & $i=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n-1)/4\rfloor}$ & $i=n, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}$
T_4	$i=1, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}$, $i=2, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}$, $i=3, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}$, $i=4, j=\underbrace{(1,5,9,\ldots)}_{\lfloor(n+3)/4\rfloor}$, $i=n, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}$,	$i=\underbrace{(1,5,9,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}$, & $i=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(4,8,12,\ldots)}_{\lfloor(n-1)/4\rfloor}$, & $i=\underbrace{(3,7,11,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=i$, & $i=\underbrace{(4,5,9,\ldots)}_{\lfloor(n-1)/4\rfloor}, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}$ & $i=n, j=\underbrace{(2,6,10,\ldots)}_{\lfloor(n-1)/4\rfloor}$ & v_1v_n

Clearly T_1, T_2, T_3, and T_4 are independent sets of $(C_n \odot P_n)$.
Case 1: n is odd
$|T_2| = \left(\frac{nm+1}{2}\right) and |T_1| = |T_3| = |T_4| = \left(\frac{nm+1}{2}\right) - 1$thus$|T_i - T_j| \leq 1$ for $i \neq j$.
Case 2: n is even
$|T_1| = |T_3| = |T_2| = |T_4| = \left(\frac{nm}{2}\right)$ thus $|T_i - T_j| \leq 1$ for $i \neq j$.
This implies $\chi_{et}(C_n \odot P_n) = 4$ Hence, the result is established.

□

Theorem 3.3. For any positive integer n, $\chi_{et}(P_n \odot C_n) = 5; (n = 5, 10, 15, \ldots)$ and $\chi(P_n \odot C_n) = 6; n = 6, 12, 18, \ldots)$.

Proof. The P_n graph consists of $\{v_1, v_2, \cdots, v_n\}$ vertices and edges $\{v_iv_{i+1}; i = 1, 2, \cdots, n-1\}$ and C_n graph consists of $\{v_1, v_2, \cdots, v_n\}$ vertices and edges $\{v_1v_n\} \cup \{v_jv_{j+1}; j = 1, 2, \cdots, n-1\}$. By definition of extended comb graph each vertex of P_n joined to a copy of C_n the vertex set of $V(P_n \odot C_n) = \{v_{i,j}; i = 1, 2, \cdots, n; j = 1, 2, \cdots, n\}$ and $E(P_n \odot C_n) = \{v_1v_n\} \cup \{v_{i,j}v_{i,j+1}; i = 1, 2, \cdots, n; j = 1, 2, \cdots, n-1\}$. Here we consider the four cases as follows.

Case:1 If $P_n \odot C_n$ where $n = 5, 10, 15, \cdots$.

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_1	$i = 1, j = \underbrace{(1, 4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}$ $i = 2, i = \underbrace{(3, 8, 13, \ldots)}_{\lfloor (n+3)/5 \rfloor}$ $i = 3, i = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}$ $i = 4, i = \underbrace{(6, 11, 16, \ldots)}_{\lfloor (n+3)/5 \rfloor}$ $i = 5, i = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}$	$i = \underbrace{(1, 6, 9, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(1, 6, 11, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(3, 8, 13, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}$ & $i = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(3, 8, 13, \ldots)}_{\lfloor (n+3)/5 \rfloor}$ & $v_{3,1}v_{3,n}$
T_2	$i = 1, j = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 2, j = \underbrace{(1, 4, 9, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 3, j = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 4, j = \underbrace{(1, 6, 11, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 5, j = \underbrace{(4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}$,	$i = \underbrace{(1, 6, 9, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(3, 8, 13, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(3, 8, 13, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(1, 16, 11, 16 \ldots)}_{\lfloor (n+3)/5 \rfloor}$ & $i = \underbrace{(4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}$ & $i = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}$
T_3	$i = 1, j = \underbrace{(4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 2, j = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 3, j = \underbrace{(1, 4, 9, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 4, j = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 5, j = \underbrace{(4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}$,	$i = \underbrace{(1, 6, 9, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(3, 8, 13, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(3, 8, 13, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(1, 6, 11, \ldots)}_{\lfloor (n+3)/5 \rfloor}$ & $i = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}$ & $v_{5,1}v_{5,n}$
T_4	$i = 1, j = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 2, j = \underbrace{(6, 11, 16, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 3, j = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 4, j = \underbrace{(1, 4, 9, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, $i = 5, j = \underbrace{(3, 8, 13, \ldots)}_{\lfloor (n+3)/5 \rfloor}$,	$i = \underbrace{(1, 6, 9, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(3, 8, 13, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(3, 8, 13, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(4, 9, 14, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(2, 7, 12, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $i = \underbrace{(5, 10, 15, \ldots)}_{\lfloor (n+3)/5 \rfloor}, j = \underbrace{(1, 6, 11, \ldots)}_{\lfloor (n+3)/5 \rfloor}$, & $v_{1,1}v_{1,n}$

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_5	$i=1, j=\underbrace{(3,8,13,\ldots)}_{\lfloor(n+3)/5\rfloor}$,	$i=\underbrace{(1,6,9,\ldots)}_{\lfloor(n+3)/5\rfloor}$, $j=\underbrace{(1,6,11,\ldots)}_{\lfloor(n+3)/5\rfloor}$, &
	$i=2, j=\underbrace{(2,7,12,\ldots)}_{\lfloor(n+3)/5\rfloor}$,	$i=\underbrace{(2,7,12,\ldots)}_{\lfloor(n+3)/5\rfloor}$, $j=\underbrace{(5,10,15,\ldots)}_{\lfloor(n+3)/5\rfloor}$, &
	$i=3, j=\underbrace{(6,11,16,\ldots)}_{\lfloor(n+3)/5\rfloor}$,	$i=\underbrace{(3,8,13,\ldots)}_{\lfloor(n+3)/5\rfloor}$, $j=\underbrace{(4,9,14,\ldots)}_{\lfloor(n+3)/5\rfloor}$, &
	$i=4, j=\underbrace{(5,10,15,\ldots)}_{\lfloor(n+3)/5\rfloor}$,	$i=\underbrace{(3,8,13,\ldots)}_{\lfloor(n+3)/5\rfloor}$, $j=\underbrace{(4,9,14,\ldots)}_{\lfloor(n+3)/5\rfloor}$, &
	$i=5, j=\underbrace{(1,4,9,\ldots)}_{\lfloor(n+3)/5\rfloor}$,	$i=\underbrace{(4,9,14,\ldots)}_{\lfloor(n+3)/5\rfloor}$, $j=\underbrace{(3,8,13,\ldots)}_{\lfloor(n+3)/5\rfloor}$, &
		$i=\underbrace{(5,10,15,\ldots)}_{\lfloor(n+3)/5\rfloor}$, $j=\underbrace{(2,7,12,\ldots)}_{\lfloor(n+3)/5\rfloor}$, &
		$v_{2,1}v_{2,n}$

Clearly T_1, T_2, T_3, T_4 and T_5 are independent sets of $(P_n \odot C_n)$.
$|T_1| = |T_2| = |T_3| = |T_4| = |T_5| = \left(\frac{nm-1}{2}\right)$ thus $|T_i - T_j| \leq 1$ for $i \neq j$.
This implies $\chi_{et}(P_n \odot C_n) = 5$ Hence, the result is established.

Case:2 If $P_n \odot C_n$ where $n = 6, 12, 18, \cdots$.

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_1	$i=1, j=\underbrace{(1,5,11,\ldots)}_{\lfloor(n+3)/6\rfloor}$	$i=\underbrace{(1,7,13,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $j=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}$, &
	$i=2, j=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$	$i=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $j=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$, &
	$i=3, j=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}$	$i=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $j=\underbrace{(1,7,13,\ldots)}_{\lfloor(n+3)/6\rfloor}$, &
	$i=4, j=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$	$i=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $j=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$ &
	$i=5, j=\underbrace{(7,13,19,\ldots)}_{\lfloor(n+3)/6\rfloor}$	$i=\underbrace{(5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $j=1$, &
	$i=6, i=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$	$i=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $j=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$, &
		$v_{2,1}v_{2,n}$

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_2	$i=1, j=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=2, j=\underbrace{(1,5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=3, j=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=4, j=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=5, j=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=6, j=\underbrace{(7,13,19,\ldots)}_{\lfloor(n+3)/6\rfloor}$,	$i=\underbrace{(1,7,13,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(1,4,10,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(1,7,13,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor}$
T_3	$i=1, j=\underbrace{(7,13,19,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=2, j=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=3, j=\underbrace{(1,5,11,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=4, j=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=5, j=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=6, j=\underbrace{(7,13,19,\ldots)}_{\lfloor(n+3)/6\rfloor}$,	$i=\underbrace{(1,7,13,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor}$ & $v_{4,1}v_{4,n}$
T_4	$i=1, j=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=2, j=\underbrace{(7,13,19\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=3, j=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=4, j=\underbrace{(1,5,11,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=5, j=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=6, j=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}$,	$i=\underbrace{(1,7,13,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(1,7,13,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $v_{5,1}v_{5,n}$ & $v_{1,1}v_{1,n}$

Color Class	**Vertex Set** $v_{i,j}$	**Edge Set** $v_{i,j}v_{i,j+1}$
T_5	$i=1, j=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=2, j=\underbrace{(2,8,14\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=3, j=\underbrace{(7,13,19,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=4, j=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=5, j=\underbrace{(1,5,11,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=6, j=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$,	$i=\underbrace{(1,7,13,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(1,7,13,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(4,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$ & $i=\underbrace{(5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor} j=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor} j=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $v_{6,1}v_{6,n}$
T_6	$i=1, j=\underbrace{(4,11,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=2, j=\underbrace{(3,9,15\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=3, j=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=4, j=\underbrace{(7,13,19,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=5, j=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$, $i=6, j=\underbrace{(1,5,11,\ldots)}_{\lfloor(n+3)/6\rfloor}$,	$i=\underbrace{(1,7,13,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(2,8,14,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(1,7,13,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}, j=\underbrace{(5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor}$ & $i=\underbrace{(5,11,17,\ldots)}_{\lfloor(n+3)/6\rfloor} j=\underbrace{(4,10,16,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $i=\underbrace{(6,12,18,\ldots)}_{\lfloor(n+3)/6\rfloor} j=\underbrace{(3,9,15,\ldots)}_{\lfloor(n+3)/6\rfloor}$, & $v_{3,1}v_{3,n}$

Clearly T_1, T_2, T_3, T_4, T_5 and T_6 are independent sets of $(P_n \odot C_m)$.
$|T_1| = |T_6| = \left(\frac{nm}{2}\right)$ and $|T_2| = |T_3| = |T_4| = |T_5| = \left(\frac{nm}{2}\right) - 1$ thus $|T_i - T_j| \leq 1$ for $i \neq j$.
This implies $\chi_{et}(P_n \odot C_m) = 6$ Hence, the result is established. □

Theorem 3.4. *For any positive integer n and m, $\chi_{et}(C_n \odot C_n) = 5 (n = 5, 10, 15, \ldots)$.*

Proof. The cycle C_n graph consists of $\{v_1, v_2, \cdots, v_n\}$ vertices and C_n graph consists of $\{v_1v_n\} \cup \{v_jv_{j+1}; j = 1, 2, \cdots, n-1\}$ edges. By definition of extended comb graph each vertex of C_n joined to a copy of C_n, the vertex set of $V(C_n \odot C_n) = \{v_{i,j}; i = 1, 2, \cdots, n; j = 1, 2, \cdots, n\}$ and $E(C_n \odot C_n) = \{v_1v_n\} \cup \{v_{i,j}v_{i,j+1}; i = 1, 2, \cdots, n; j = 1, 2, \cdots, m-1\}$. Here we consider the cases as follows.

Case:1 If $\chi_{et}(C_n \odot C_n) = 5, 10, 15, \ldots$

The coloring rules of $\chi_{et}(P_n \odot C_n) = 5, 10, 15, \ldots$ are the same as for $\chi_{et}(C_n \odot C_n) = 5, 10, 15, \ldots$ and we have to include one color $T_2 = \{v_1v_2\}$.

□

4 Conclusion

In this work, we demonstrate the equitable total coloring of the Extended Comb graph $(G \odot H)$ lies on $\Delta(G)+1$ or $\Delta(G)+2$. Still, the problem is open for a combination of path and star graphs and the fusion of path and star graphs.

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